

Full Paper

Integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex and other convex functions

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Abstract: Some new integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex functions and other convex functions such as the P -convex, quasi-convex, m -convex, (α, m) -convex and s -convex functions have been established.

Keywords: integral inequality, Hermite-Hadamard type inequality, strongly logarithmically convex function, Hölder inequality, P -convex function, quasi-convex function, m -convex function, (α, m) -convex function, s -convex function

INTRODUCTION

Throughout this paper we use the following notation:

$$R = (-\infty, \infty), \quad R_0 = [0, \infty) \quad \text{and} \quad R_+ = (0, \infty).$$

The following definitions are well known in the literature.

Definition 1. A function $f : I \subseteq R \rightarrow R$ is said to be convex on I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 2. A function $f : I \subseteq R \rightarrow R_+$ is said to be logarithmically convex if

$$f(\lambda x + (1-\lambda)y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 3 [1]. A function $f : I \subseteq R \rightarrow R_0$ is said to be P -convex if

$$f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 4 [1]. A function $f : I \subseteq R \rightarrow R_0$ is said to be quasi-convex if

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 5 [2]. For $f : [0, b] \rightarrow R$ and $m \in (0, 1]$, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an m -convex function on $[0, b]$.

Definition 6 [3]. For $f : [0, b] \rightarrow R$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

is valid for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m) -convex function on $[0, b]$.

Definition 7 [4]. Let $s \in (0, 1]$. A function $f : [a, b] \subseteq R_0 \rightarrow R_0$ is said to be s -convex (in the second sense) if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 8 [5]. A function $f : [a, b] \rightarrow R$ is said to be strongly convex with modulus $c \geq 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

is valid for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 9 [6]. A function $f : I \subseteq R \rightarrow R_+$ is said to be strongly logarithmically convex with modulus $c \geq 0$ if

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t} - ct(1-t)(x-y)^2$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Remark 1. If a function $f : I \subseteq R \rightarrow R_+$ is strongly logarithmically convex, then it is also a strongly convex function, and so it is integrable on I .

Remark 2. For all $x \in [0, 1]$ and $0 \leq c \leq e^{-1}$, it is easy to obtain that $1 - cx \geq e^{-x}$. For all $x, y \in [0, 1]$ and $0 \leq c \leq e^{-1}$, we acquire

$$e^{[\alpha+(1-t)y]^2 - [\alpha^2+(1-t)y^2]} \leq 1 - ct(1-t)(x-y)^2 \leq 1 - \frac{ct(1-t)(x-y)^2}{e^{\alpha x^2 + (1-t)y^2}}$$

for all $t \in [0, 1]$. As a result, when $0 \leq c \leq e^{-1}$, the function $f(x) = e^{x^2}$ is a strongly logarithmically convex functions on $[0, 1]$.

In recent decades, a number of inequalities of Hermite-Hadamard type for different kinds of convex functions have been established. Some of them may be reformulated as follows.

Theorem 1 [1]. Let $f : I^\circ \subseteq R \rightarrow R_0$ be a P -convex mapping on I , $a, b \in I$ with $a < b$, and $f \in L([a, b])$. Then

$$\frac{1}{b-a} \int_a^b f(x) dx \leq f(a) + f(b).$$

Theorem 2 [7]. Let $f, g : [a, b] \rightarrow R_0$ be a convex functions on $[a, b] \subseteq R$ with $a < b$. Then

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b),$$

where

$$M(a, b) = f(a)g(a) + f(b)g(b) \quad (1)$$

and

$$N(a, b) = f(a)g(b) + f(b)g(a). \quad (2)$$

Theorem 3 [8]. Let $f, g : [a, b] \rightarrow R_0$ and $fg \in L([a, b])$ with $0 \leq a < b < \infty$. If f is convex and nonnegative on $[a, b]$ and g is s -convex on $[a, b]$ for some fixed $s \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq \frac{1}{s+2}M(a, b) + \frac{1}{(s+1)(s+2)}N(a, b),$$

where $M(a, b)$ and $N(a, b)$ are defined by (1) and (2).

In recent years, some inequalities of Hermite-Hadamard type for other kinds of convex functions were established [9, 10, 11, 12, 13, 14, 15, 16, and closely related references therein]. It was reported that some kinds of Hermite-Hadamard type inequalities have applications in statistics and other mathematical sciences [17]. The goal of this paper is to establish some new integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex functions and other convex functions such as P -convex, quasi-convex, m -convex, (α, m) -convex and s -convex functions.

INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE

Now we start out to establish some new integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex functions and other convex functions.

Theorem 4. For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow R_+$. If f and g^q are strongly logarithmically convex functions on $[a, b]$ with modulus $c \geq 0$ for $q \geq 1$, then

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x) dx \leq & \left[f(b)G\left(\frac{f(a)}{f(b)}\right) - \frac{c(b-a)^2}{6} \right]^{1-1/q} \left\{ f(b)g^q(b)G\left(\frac{f(a)}{f(b)}\left[\frac{g(a)}{g(b)}\right]^q\right) \right. \\ & \left. - c(b-a)^2 \left[f(b)F\left(\frac{f(a)}{f(b)}\right) + g^q(b)F\left(\left[\frac{g(a)}{g(b)}\right]^q\right) \right] + \frac{c^2(b-a)^4}{30} \right\}^{1/q}, \end{aligned}$$

where

$$G(u) = \begin{cases} 1, & u = 1, \\ \frac{u-1}{\ln u}, & u \neq 1 \end{cases} \quad \text{and} \quad F(u) = \begin{cases} \frac{1}{6}, & u = 1, \\ \frac{(u+1)\ln u - 2(u-1)}{(\ln u)^3}, & u \neq 1. \end{cases} \quad (3)$$

Proof. Taking $x = ta + (1-t)b$ for $t \in [0,1]$, $v_f = \frac{f(a)}{f(b)}$ and $u = \frac{f(a)}{f(b)} \left[\frac{g(a)}{g(b)} \right]^q$, and using Hölder's inequality and the strongly logarithmic convexity of the functions f and g^q , the following is figured out:

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x)g(x)dx = \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt \\ & \leq \left[\int_0^1 f(ta + (1-t)b) dt \right]^{1-1/q} \left[\int_0^1 f(ta + (1-t)b)g^q(ta + (1-t)b) dt \right]^{1/q} \\ & \leq \left\{ \int_0^1 [f'(a)f^{1-t}(b) - ct(1-t)(b-a)^2] dt \right\}^{1-1/q} \\ & \quad \times \left\{ \int_0^1 [f'(a)f^{1-t}(b) - ct(1-t)(b-a)^2] [g^{qt}(a)g^{q(1-t)}(b) - ct(1-t)(b-a)^2] dt \right\}^{1/q} \\ & = \left[f(b) \int_0^1 v_f^t dt - c(b-a)^2 \int_0^1 t(1-t) dt \right]^{1-1/q} \left\{ f(b)g^q(b) \int_0^1 u^t dt \right. \\ & \quad \left. - c(b-a)^2 \int_0^1 [f'(a)f^{1-t}(b) + g^{qt}(a)g^{q(1-t)}(b)] t(1-t) dt + c^2(b-a)^4 \int_0^1 t^2(1-t)^2 dt \right\}^{1/q} \\ & = \left[f(b)G(v_f) - \frac{c(b-a)^2}{6} \right]^{1-1/q} \left\{ f(b)g^q(b)G(u) - c(b-a)^2 [f(b)F(v_f) + g^q(b)F(v_g^q)] + \frac{c^2(b-a)^4}{30} \right\}^{1/q}. \end{aligned}$$

Theorem 4 is thus proved.

Corollary 1. Under the conditions of Theorem 4,

1) when $c = 0$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq f(b)g(b)G^{1-1/q} \left(\frac{f(a)}{f(b)} \right) G^{1/q} \left(\frac{f(a)}{f(b)} \left[\frac{g(a)}{g(b)} \right]^q \right);$$

2) when $c = 0$ and $q = 1$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq f(b)g(b)G \left(\frac{f(a)g(a)}{f(b)g(b)} \right).$$

Theorem 5. For $a, b \in R_0$ with $a < b$, let $f, g : [a, b] \rightarrow R_+$. If f is an s -convex function on $[a, b]$ for some $s \in (0, 1]$ and g^q is a strongly logarithmically convex function on $[a, b]$ with modulus $c \geq 0$ for $q \geq 1$, then

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)dx & \leq \left[\frac{f(a) + f(b)}{s+1} \right]^{1-1/q} \left\{ \left[\frac{g^q(a) - g^q(b)}{s+2} + \frac{g^q(b)}{s+1} \right] f(a) \right. \\ & \quad \left. + \left[\frac{g^q(a) - g^q(b)}{(s+1)(s+2)} + \frac{g^q(b)}{s+1} \right] f(b) - \frac{c(b-a)^2}{(s+2)(s+3)} [f(a) + f(b)] \right\}^{1/q}. \end{aligned}$$

Proof. Employing the conditions that f is s -convex and g^q is strongly logarithmically convex on $[a, b]$, let $x = ta + (1-t)b$ for $t \in [0, 1]$ and $u = \left[\frac{g(a)}{g(b)} \right]^q$. Then making use of Hölder's inequality the following is generated:

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx = \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt$$

$$\begin{aligned}
&\leq \left[\int_0^1 f(ta + (1-t)b) dt \right]^{1-1/q} \left[\int_0^1 f(ta + (1-t)b) g^q(ta + (1-t)b) dt \right]^{1/q} \\
&\leq \left\{ \int_0^1 [t^s f(a) + (1-t)^s f(b)] dt \right\}^{1-1/q} \left\{ \int_0^1 [t^s f(a) + (1-t)^s f(b)] [g^{qt}(a) g^{q(1-t)}(b) \right. \\
&\quad \left. - ct(1-t)(b-a)^2] dt \right\}^{1/q} \\
&= \left[\frac{1}{s+1} [f(a) + f(b)] \right]^{1-1/q} \left\{ g^q(b) \int_0^1 [t^s u' f(a) + (1-t)^s u' f(b)] dt \right. \\
&\quad \left. - \frac{c(b-a)^2}{(s+2)(s+3)} [f(a) + f(b)] \right\}^{1/q}.
\end{aligned}$$

Since $u' \leq (u-1)t+1$ for all $0 \leq t \leq 1$, we obtain

$$\int_0^1 t^s u' dt \leq \int_0^1 t^s [(u-1)t+1] dt = \frac{u-1}{s+2} + \frac{1}{s+1}$$

and

$$\int_0^1 (1-t)^s u' dt \leq \int_0^1 (1-t)^s [(u-1)t+1] dt = \frac{u-1}{(s+1)(s+2)} + \frac{1}{s+1}.$$

Accordingly,

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f(x) g(x) dx &\leq \left[\frac{f(a) + f(b)}{s+1} \right]^{1-1/q} \left\{ g^q(b) \int_0^1 [t^s u' f(a) + (1-t)^s u' f(b)] dt \right. \\
&\quad \left. - \frac{c(b-a)^2}{(s+2)(s+3)} [f(a) + f(b)] \right\}^{1/q} \\
&\leq \left[\frac{f(a) + f(b)}{s+1} \right]^{1-1/q} \left\{ g^q(b) \left[\left(\frac{u-1}{s+2} + \frac{1}{s+1} \right) f(a) \right. \right. \\
&\quad \left. \left. + \left(\frac{u-1}{(s+1)(s+2)} + \frac{1}{s+1} \right) f(b) \right] - \frac{c(b-a)^2}{(s+2)(s+3)} [f(a) + f(b)] \right\}^{1/q} \\
&= \left[\frac{f(a) + f(b)}{s+1} \right]^{1-1/q} \left\{ \left[\frac{g^q(a) - g^q(b)}{s+2} + \frac{g^q(b)}{s+1} \right] f(a) \right. \\
&\quad \left. + \left[\frac{g^q(a) - g^q(b)}{(s+1)(s+2)} + \frac{g^q(b)}{s+1} \right] f(b) - \frac{c(b-a)^2}{(s+2)(s+3)} [f(a) + f(b)] \right\}^{1/q}.
\end{aligned}$$

The proof of Theorem 5 is thus complete.

Corollary 2. Under the conditions of Theorem 5,

1) if $c=0$, then

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f(x) g(x) dx &\leq \frac{[f(a) + f(b)]^{1-1/q}}{(s+1)(s+2)^{1/q}} \\
&\quad \times [(s+1)f(a)g^q(a) + f(a)g^q(b) + f(b)g^q(a) + (s+1)f(b)g^q(b)]^{1/q};
\end{aligned}$$

2) if $c=0$ and $q=1$, we have

$$\frac{1}{b-a} \int_a^b f(x) g(x) dx \leq \frac{(s+1)f(a)g(a) + f(a)g(b) + f(b)g(a) + (s+1)f(b)g(b)}{(s+1)(s+2)};$$

3) if $c=0, q=1$ and $s=1$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{2f(a)g(a) + f(a)g(b) + f(b)g(a) + 2f(b)g(b)}{6}.$$

Theorem 6. For $a, b \in R_0$ with $a < b$, let $f, g: R_0 \rightarrow R_+$. If f is an (α, m) -convex function on $[0, b/m]$ for $(\alpha, m) \in (0, 1]^2$ and g^q is a strongly logarithmically convex function on $[a, b]$ with modulus $c \geq 0$ for $q \geq 1$, then

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)dx &\leq \left\{ \frac{1}{\alpha+1} \left[f(a) + m\alpha f\left(\frac{b}{m}\right) \right] \right\}^{1-1/q} \left\{ \left[\frac{g^q(a) - g^q(b)}{\alpha+2} + \frac{g^q(b)}{\alpha+1} \right] f(a) \right. \\ &\quad \left. + m \left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} \left(3 + \alpha + (1+\alpha) \left[\frac{g(a)}{g(b)} \right]^q \right) \right] f\left(\frac{b}{m}\right) g^q(b) \right. \\ &\quad \left. - \frac{c(b-a)^2}{(\alpha+2)(\alpha+3)} \left[f(a) + \frac{m\alpha(\alpha+5)}{6} f\left(\frac{b}{m}\right) \right] \right\}^{1/q}. \end{aligned}$$

Proof. Let $x = ta + (1-t)b$ for $t \in [0, 1]$ and $u = \left[\frac{g(a)}{g(b)} \right]^q$. By making use of the (α, m) -convexity of f and the strongly logarithmic convexity of g^q , and applying Hölder's inequality, the following is brought out:

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)dx &= \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt \\ &\leq \left[\int_0^1 f(ta + (1-t)b) dt \right]^{1-1/q} \left[\int_0^1 f(ta + (1-t)b)g^q(ta + (1-t)b) dt \right]^{1/q} \\ &\leq \left\{ \int_0^1 \left[t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right] dt \right\}^{1-1/q} \\ &\quad \times \left\{ \int_0^1 \left[t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right] \left[g^{qt}(a)g^{q(1-t)}(b) - ct(1-t)(b-a)^2 \right] dt \right\}^{1/q} \\ &= \left\{ \frac{1}{\alpha+1} \left[f(a) + m\alpha f\left(\frac{b}{m}\right) \right] \right\}^{1-1/q} \left\{ g^q(b) \int_0^1 \left[t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right] u^t dt \right. \\ &\quad \left. - \frac{c(b-a)^2}{(\alpha+2)(\alpha+3)} \left[f(a) + \frac{m\alpha(\alpha+5)}{6} f\left(\frac{b}{m}\right) \right] \right\}^{1/q}. \end{aligned}$$

By virtue of the inequality $u^t \leq (u-1)t + 1$ for all $0 \leq t \leq 1$, we have

$$\begin{aligned} g^q(b) \int_0^1 \left[t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right] u^t dt &\leq g^q(b) \int_0^1 \left[t^\alpha f(a) + m(1-t^\alpha) f\left(\frac{b}{m}\right) \right] [(u-1)t + 1] dt \\ &= g^q(b) \left\{ \left(\frac{u-1}{\alpha+2} + \frac{1}{\alpha+1} \right) f(a) + m \left[\frac{\alpha(3+u+\alpha+\alpha u)}{2(\alpha+1)(\alpha+2)} \right] f\left(\frac{b}{m}\right) \right\} \\ &= \left[\frac{g^q(a) - g^q(b)}{\alpha+2} + \frac{g^q(b)}{\alpha+1} \right] f(a) + m \left[\frac{\alpha(3+u+\alpha+\alpha u)}{2(\alpha+1)(\alpha+2)} \right] f\left(\frac{b}{m}\right) g^q(b). \end{aligned}$$

Theorem 6 is thus proved.

Corollary 3. Under the conditions of Theorem 6,

1) when $c = 0$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{2^{1/q}(\alpha+1)(\alpha+2)^{1/q}} \left[f(a) + \alpha m f\left(\frac{b}{m}\right) \right]^{1-1/q} \\ \times \left\{ 2[(\alpha+1)g^q(a) + g^q(b)]f(a) + \alpha m[(\alpha+3)g^q(b) + (\alpha+1)g^q(a)]f\left(\frac{b}{m}\right) \right\}^{1/q};$$

2) when $c=0$ and $q=1$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{2[(\alpha+1)g(a) + g(b)]f(a) + \alpha m[(\alpha+3)g(b) + (\alpha+1)g(a)]f(b/m)}{2(\alpha+1)(\alpha+2)};$$

3) when $c=0$, $q=1$ and $m=1$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{2[(\alpha+1)g(a) + g(b)]f(a) + \alpha[(\alpha+3)g(b) + (\alpha+1)g(a)]f(b)}{2(\alpha+1)(\alpha+2)};$$

4) when $c=0$, $q=1$ and $m=\alpha=1$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{[2g(a) + g(b)]f(a) + [2g(b) + g(a)]f(b)}{6}.$$

Theorem 7. For $a, b \in R$ with $a < b$, let $f, g: R \rightarrow R_+$. If f is a P -convex function on $[a, b]$ and g^q is a logarithmically convex function on $[a, b]$ for $q \geq 1$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq [f(a) + f(b)] \left[g^q(b)G\left(\left[\frac{g(a)}{g(b)}\right]^q\right) - \frac{c(b-a)^2}{6} \right]^{1/q},$$

where $G(u)$ is defined by (3).

Proof. Taking $x = ta + (1-t)b$ for $t \in [0, 1]$ and denoting $u = \left[\frac{g(a)}{g(b)}\right]^q$ lead to

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx = \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt \\ \leq \left[\int_0^1 f(ta + (1-t)b)dt \right]^{1-1/q} \left[\int_0^1 f(ta + (1-t)b)g^q(ta + (1-t)b)dt \right]^{1/q} \\ \leq [f(a) + f(b)]^{1-1/q} \left\{ \int_0^1 [f(a) + f(b)][g^{qt}(a)g^{q(1-t)}(b) - ct(1-t)(b-a)^2]dt \right\}^{1/q} \\ \leq [f(a) + f(b)] \left[g^q(b) \int_0^1 u^t dt - \frac{c(b-a)^2}{6} \right]^{1/q} \\ = [f(a) + f(b)] \left[g^q(b)G(u) - \frac{c(b-a)^2}{6} \right]^{1/q}.$$

Theorem 7 is thus proved.

Corollary 4. Under the conditions of Theorem 7,

1) when $c=0$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq [f(a) + f(b)]g(b)G^{1/q}\left(\left[\frac{g(a)}{g(b)}\right]^q\right);$$

2) when $c=0$ and $q=1$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq [f(a) + f(b)]g(b)G\left(\frac{g(a)}{g(b)}\right).$$

Theorem 8. For $a, b \in R$ with $a < b$, let $f, g: R \rightarrow R_+$. If f is a quasi-convex function on $[a, b]$ and g^q is a logarithmically convex function on $[a, b]$ for $q \geq 1$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \max \{f(a), f(b)\} \left[g^q(b)G \left(\left[\frac{g(a)}{g(b)} \right]^q \right) - \frac{c(b-a)^2}{6} \right]^{1/q},$$

where $G(u)$ is defined by (3).

Proof. Let $x = ta + (1-t)b$ for $t \in [0, 1]$ and $u = \left[\frac{g(a)}{g(b)} \right]^q$. This gives

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)dx &= \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b) dt \\ &\leq \left[\int_0^1 f(ta + (1-t)b) dt \right]^{1-1/q} \left[\int_0^1 f(ta + (1-t)b)g^q(ta + (1-t)b) dt \right]^{1/q} \\ &\leq [\max \{f(a), f(b)\}]^{1-1/q} \left[\int_0^1 \max \{f(a), f(b)\} [g^{qt}(a)g^{q(1-t)}(b) - ct(1-t)(b-a)^2] dt \right]^{1/q} \\ &\leq \max \{f(a), f(b)\} \left[g^q(b)G(u) - \frac{c(b-a)^2}{6} \right]^{1/q}. \end{aligned}$$

Theorem 8 is thus proved.

Corollary 5. Under conditions of Theorem 8,

1) when $c = 0$, we have

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \max \{f(a), f(b)\} g(b)G^{1/q} \left(\left[\frac{g(a)}{g(b)} \right]^q \right);$$

2) when $c = 0$ and $q = 1$,

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \max \{f(a), f(b)\} g(b)G \left(\frac{g(a)}{g(b)} \right).$$

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