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Full Paper

The Fibonacci-Padovan sequences in finite groups

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Abstract: The Fibonacci-Padovan sequence modulo *m* was studied. Also, the Fibonacci-Padovan orbits of j-generator finite groups such that $2 \le j \le 5$ was examined. The Fibonacci-Padovan lengths of the groups Q_8, $Q_8 \times Z_{2m}$ and $Q_8 \times_{\varphi} Z_{2m}$ for $m \ge 3$, where Z is integer, were then obtained.

Keywords: Fibonacci-Padovan sequences, recurrence sequences, Fibonacci-Padovan orbits of finite groups, matrix

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INTRODUCTION AND PRELIMINARIES

It is well known that linear recurrence sequences appear in modern research in many fields from mathematics, physics, computer science and architecture to nature and art [e.g. 1-10]. The study of recurrence sequences in groups began with an earlier work of Wall [11], who investigated the ordinary Fibonacci sequences in cyclic groups. The concept was extended to some special linear recurrence sequences by several authors [e.g. 12-21]. In this paper, we extend the theory to the Fibonacci-Padovan sequences.

A Fibonacci-Padovan sequence $\{a_n\}$ is defined [22] recursively by the equation

$$a_n = a_{n-1} + 2a_{n-2} - 2a_{n-3} + a_{n-5} \tag{1}$$

for $n \ge 5$, where $a_0 = 1, a_1 = 1, a_2 = 3, a_3 = 3, a_4 = 7$.

Kalman [23] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding *k* terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \cdots, c_{k-1}$ are real constants. Kalman [23] derived a number of closed-form formulas for the generalised sequence by companion matrix method as follows:

$$A_k = \left[a_{ij}\right]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}. \tag{2}$$

Then by an inductive argument he obtained:

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}. \tag{3}$$

It is well known that a sequence, including that of group elements, is periodic if, after a certain point, it consists only of the repetition of a fixed sub-sequence. The number of elements in the repeating sub-sequence is the period of the sequence. For example, the sequence $a,b,c,d,e,b,c,d,e,b,c,d,e,\cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period *k* if the first *k* elements in the sequence form a repeating sub-sequence. For example, the sequence $a,b,c,d,e,f,a,b,c,d,e,f,a,b,c,d,e,f,\cdots$ is simply periodic with period 6.

Definition 1. For a finite generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, ..., a_n\}$, the sequence $x_i = a_{i+1}$, $0 \le i \le n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \ge 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted by $F_A(G)$. If $F_A(G)$ is periodic, then the length of the period of the sequence is called the Fibonacci length of G with respect to the generating set A, written $LEN_A(G)$ [24].

FIBONACCI-PADOVAN SEQUENCES MODULO *m*

By (1) and (3), we can write

$$\begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ a_{n+4} \\ a_{n+5} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ a_{n+4} \end{bmatrix} \tag{4}$$

for the Fibonacci-Padovan sequence. Let us take

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$$M=\left[m_{ij}\right]_{5\times 5}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&-2&2&1\end{bmatrix}.$$

which is said to be Fibonacci-Padovan matrix. By mathematical induction, it can be shown that, for $n\geq 0$,

$$M^{n}\begin{bmatrix}1\\1\\3\\3\\7\end{bmatrix}=\begin{bmatrix}a_n\\a_{n+1}\\a_{n+2}\\a_{n+3}\\a_{n+4}\end{bmatrix} \tag{5}$$

Reducing the Fibonacci-Padovan sequence by a modulus m, we can get the repeating sequence denoted by

$$\{a_n(m)\}=\{a_0(m),a_1(m),a_2(m),a_3(m),a_4(m),\ldots,a_i(m),\ldots\}$$

where $a_i(m)\equiv a_i\ (\mathrm{mod}\ m)$. It has the same recurrence relation as in (1).

Theorem 1. $\{a_n(m)\}$ is a simple periodic sequence.

Proof. Let $U=\{(x_1,x_2,\ldots,x_5)\,|\,0\leq x_i\leq m-1\}$. Then we have $|U|=m^5$ being finite; that is, for any $j\geq 0$, there exists $i\geq j$ such that $a_{i+4}(m)\equiv a_{j+4}(m)$, $a_{i+3}(m)\equiv a_{j+3}(m)$, $a_{i+2}(m)\equiv a_{j+2}(m)$, $a_{i+1}(m)\equiv a_{j+1}(m)$ and $a_i(m)\equiv a_j(m)$. From the definition of the Fibonacci-Padovan sequence, we have $a_{n+5}=a_{n+4}+2a_{n+3}-2a_{n+2}+a_n$; that is, $a_n=a_{n+4}+2a_{n+3}-2a_{n+2}-a_{n+5}$. Then we can easily get that $a_{i-1}(m)\equiv a_{j-1}(m), a_{i-2}(m)\equiv a_{j-2}(m),\ldots,a_{i-j+1}(m)\equiv a_1(m), a_{i-j}(m)\equiv a_0(m)$, which implies that $\{a_n(m)\}$ is a simple periodic sequence.

Let $l(m)$ denote the smallest period of $\{a_n(m)\}$ and p is used for a prime number.

Example. We have $\{a_n(2)\}=\{1,1,1,1,1,0,1,0,1,0,0,1,1,0,0,0,1,0,0,0,0,1,1,1,1,1,\ldots\}$, and then repeat. So we get $l(m)=21$.

For a given matrix $A=\left[a_{ij}\right]$ with a_{ij}'s being integers, $A(\mathrm{mod}\ m)$ means that every entry of A is a reduced modulo m; that is, $A(\mathrm{mod}\ m)\equiv\left(a_{ij}\ (\mathrm{mod}\ m)\right)$. Let $\langle M\rangle_{p^u}=\left\{M^i\left(\mathrm{mod}\ p^u\right)\,\middle|\,i\geq 0\right\}$ be a cyclic group and let $\left|\langle M\rangle_{p^u}\right|$ denote the order of $\langle M\rangle_{p^u}$. It is easy to see from (5) that $l\left(p^u\right)=\left|\langle M\rangle_{p^u}\right|$.

Theorem 2. Let t be the largest positive integer such that $l(p)=l\left(p^t\right)$. Then $l\left(p^\alpha\right)=p^{\alpha-t}\cdot l(p)$, for every $\alpha\geq t$.

Proof. Let q be a positive integer. Since $M^{l\left(p^{q+1}\right)}\equiv I\left(\mathrm{mod}\ p^{q+1}\right)$; that is, $M^{l\left(p^{q+1}\right)}\equiv I\left(\mathrm{mod}\ p^{q}\right)$, we get that $l\left(p^q\right)$ divides $l\left(p^{q+1}\right)$. On the other hand, writing $M^{l\left(p^q\right)}=I+\left(m_{ij}^{(q)}\cdot p^q\right)$, we have

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$$M^{l(p^q)\cdot p}=\left(I+\left(m_{ij}^{(q)}\cdot p^q\right)\right)^p=\sum_{k=0}^{p}\binom{p}{k}\left(m_{ij}^{(q)}\cdot p^q\right)^k\equiv I\left(\mathrm{mod}\ p^{q+1}\right),$$

which yields that $l\left(p^{q+1}\right)$ divides $l\left(p^q\right)\cdot p$. Therefore, $l\left(p^{q+1}\right)=l\left(p^q\right)$ or $l\left(p^{q+1}\right)=l\left(p^q\right)\cdot p$, and the latter holds if and only if there is a $m_{ij}^{(q)}$ which is not divisible by p. Since $l\left(p^t\right)\neq l\left(p^{t+1}\right)$, there is an $m_{ij}^{(t+1)}$ which is not divisible by p; thus, $l\left(p^{t+1}\right)\neq l\left(p^{t+2}\right)$. The proof finishes by induction on t.

Theorem 3. If $m=\prod_{i=1}^{t}p_i^{e_i},(t\geq 1)$ where p_i's are distinct primes, then $l(m)=\mathrm{lcm}\left[l\left(p_i^{e_i}\right)\right]$.

Proof. The statement '$l\left(p_i^{e_i}\right)$ is the length of the period of $\left\{a_n\left(p_i^{e_i}\right)\right\}$' implies that the sequence $\left\{a_n\left(p_i^{e_i}\right)\right\}$ repeats only after blocks of length $u\cdot l\left(p_i^{e_i}\right)$, $(u\in N)$, where N is natural number ; and the statement '$l(m)$ is the length of the period $\left\{a_n(m)\right\}$' implies that $\left\{a_n\left(p_i^{e_i}\right)\right\}$ repeats after $l(m)$ terms for all values i. Thus, $l(m)$ is of the form $u\cdot l\left(p_i^{e_i}\right)$ and since any such number gives a period of $\left\{a_n(m)\right\}$, then we get that $l(m)=\mathrm{lcm}\left[l\left(p_i^{e_i}\right)\right]$.

Let $l_{(a_1,a_2,\ldots,a_5)}(p)$ denote the smallest period of the integer-valued recurrence relation $u_n=u_{n-1}+2u_{n-2}-2u_{n-3}+u_{n-5}$, $u_1=a_1,u_2=a_2,\ldots,u_5=a_5$ where each entry is a reduced modulo p. Then we have the following theorem.

Theorem 4. If $a_1,a_2,\ldots,a_5,x_1,x_2,\ldots,x_5\in Z$, where Z is integer, such that $\gcd(a_1,a_2,\ldots,a_5,p)=1$ and $\gcd(x_1,x_2,\ldots,x_5,p)=1$, then

$$l_{(a_1,a_2,\ldots,a_5)}(p)=l_{(x_1,x_2,\ldots x_k)}(p).$$

Proof. Let $l(p)=\left|\langle M\rangle_p\right|=r$. From (4), it is clear that $\begin{bmatrix}u_{n+r}\\u_{n+r+1}\\u_{n+r+2}\\u_{n+r+3}\\u_{n+r+4}\end{bmatrix}=M^r\begin{bmatrix}u_n\\u_{n+1}\\u_{n+2}\\u_{n+3}\\u_{n+4}\end{bmatrix}$. So naturally

$\begin{bmatrix}u_{n+r}\\u_{n+r+1}\\u_{n+r+2}\\u_{n+r+3}\\u_{n+r+4}\end{bmatrix}\equiv\begin{bmatrix}u_n\\u_{n+1}\\u_{n+2}\\u_{n+3}\\u_{n+4}\end{bmatrix}\mathrm{mod}\ p$. This completes the proof.

Conjecture 1. If p is a prime, then there exists a σ with $0\leq\sigma\leq 5$ such that $\left|\langle M\rangle_p\right|$ divides $\left(p^6-p^\sigma\right)$.

Conjecture 2. If p is a prime such that $p>2$, then $\left|\langle M\rangle_p\right|$ is an even integer number. Table 1 lists some primes for which Conjectures 1 and 2 are true.

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Table 1. Orders of the cyclic group $\langle M \rangle_p$

p	$\lvert\langle M\rangle_p\rvert$	$\lvert\langle M\rangle_p\rvert \mid (p^6 - p^\sigma)$
2	21	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$
3	104	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$
5	120	$\lvert\langle M\rangle_p\rvert \mid p^6 - p^3$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$
29	12194	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^3$
31	9930	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^3$
47	72224	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$
71	357910	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^3$
83	6888	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^2$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$
101	100	$\lvert\langle M\rangle_p\rvert \mid (p^6 - p^\sigma)$ for $0 \le \sigma \le 5$
211	210	$\lvert\langle M\rangle_p\rvert \mid (p^6 - p^\sigma)$ for $0 \le \sigma \le 5$
401	160800	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^2$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$
523	91176	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^2$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$
811	59267970	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^3$
1973	5125429148	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$
2221	365194662	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^3$
4657	2710956	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^2$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$
9473	44868864	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^2$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$
30137	454119384	$\lvert\langle M\rangle_p\rvert \mid p^6 - 1$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^2$, $\lvert\langle M\rangle_p\rvert \mid p^6 - p^4$

FIBONACCI-PADOVAN ORBITS OF FINITE GROUPS

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times \cdots \times G}_{j}$ such that $(x_0, x_1, \ldots, x_{j-1}) \in X$ if and only if G is generated by $x_0, x_1, \ldots, x_{j-1}$. We call $(x_0, x_1, \ldots, x_{j-1})$ a generating j-tuple for G.

Definition 2. The Fibonacci-Padovan orbits of finite groups with j-generating $(2 \le j \le 5)$ are defined as follows:

i. Let G be a 2-generator group. For a generating pair $(x_0, x_1) \in X$, the Fibonacci-Padovan orbit $FP(G)_{x_0, x_1}$ is defined by the sequence $\{b_i\}$ of elements of G such that

$$b_0 = x_0,\ b_1 = x_1,\ b_2 = (b_0)^2 (b_1),\ b_3 = (b_0)^{-2} (b_1)^2 (b_2),\ b_4 = (b_1)^{-2} (b_2)^2 (b_3),$$
$$b_n = (b_{n-5})(b_{n-3})^{-2}(b_{n-2})^2(b_{n-1}) \text{ for } n \ge 5.$$

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ii. Let G be a 3-generator group. For a generating triplet $(x_0,x_1,x_2)\in X$, the Fibonacci-Padovan orbit $FP(G)_{x_0,x_1,x_2}$ is defined by the sequence $\{b_i\}$ of elements of G such that

$$b_0=x_0,\ b_1=x_1,\ b_2=x_2,\ b_3=(b_0)^{-2}(b_1)^2(b_2),\ b_4=(b_1)^{-2}(b_2)^2(b_3),$$
$$b_n=(b_{n-5})(b_{n-3})^{-2}(b_{n-2})^2(b_{n-1}) \text{ for } n\geq 5.$$

iii. Let G be a 4-generator group. For a generating quadruplet $(x_0,x_1,x_2,x_3)\in X$, the Fibonacci-Padovan orbit $FP(G)_{x_0,x_1,x_2,x_3}$ is defined by the sequence $\{b_i\}$ of elements of G such that

$$b_0=x_0,\ b_1=x_1,\ b_2=x_2,\ b_3=x_3,\ b_4=(b_1)^{-2}(b_2)^2(b_3),$$
$$b_n=(b_{n-5})(b_{n-3})^{-2}(b_{n-2})^2(b_{n-1}) \text{ for } n\geq 5.$$

iv. Let G be a 5-generator group. For a generating quintuplet $(x_0,x_1,x_2,x_3,x_4)\in X$, the Fibonacci-Padovan orbit $FP(G)_{x_0,x_1,x_2,x_3,x_4}$ is defined by the sequence $\{b_i\}$ of elements of G such that

$$b_0=x_0,\ b_1=x_1,\ b_2=x_2,\ b_3=x_3,\ b_4=x_4,$$
$$b_n=(b_{n-5})(b_{n-3})^{-2}(b_{n-2})^2(b_{n-1}) \text{ for } n\geq 5.$$

The classic Fibonacci-Padovan sequence in the integers modulo m can be written as $FP(Z_m)_{0,1}$.

Theorem 5. A Fibonacci-Padovan orbit of a finite group which is generating ($2\leq j\leq 5$) is simply periodic.

Proof. Let us consider the group G as a 4-generator group and let (x_0,x_1,x_2,x_3) be a generating quadruplet of G. If the order of G is n, there are n^5 distinct 5-tuples of elements of G. So at least one of the 5-tuples appears twice in the Fibonacci-Padovan orbit of G for the generating quadruplet (x_0,x_1,x_2,x_3); that is, the sub-sequence following these 5-tuples repeats. Hence the Fibonacci-Padovan orbit is periodic. Since the Fibonacci-Padovan orbit for the generating quadruplet (x_0,x_1,x_2,x_3) is periodic, there are positive integers i and j, with $i>j$, such that $b_{i+1}=b_{j+1}$, $b_{i+2}=b_{j+2}$, $b_{i+3}=b_{j+3}$, $b_{i+4}=b_{j+4}$ and $b_{i+5}=b_{j+5}$. By the defined relation of a Fibonacci-Padovan orbit, we know that

$$b_i=(b_{i+5})(b_{i+4})^{-1}(b_{i+3})^{-2}(b_{i+2})^2$$

and

$$b_j=(b_{j+5})(b_{j+4})^{-1}(b_{j+3})^{-2}(b_{j+2})^2.$$

Thus, $b_i=b_j$, and it follows that

$$b_{i-j}=b_{j-j}=b_0=x_0,\ b_{i-j+1}=b_{j-(j-1)}=b_1=x_1,$$
$$b_{i-j+2}=b_{j-(j-2)}=b_2=x_2,\ b_{i-j+3}=b_{j-(j-3)}=b_3=x_3.$$

So the Fibonacci-Padovan orbit $FP(G)_{x_0,x_1,x_2,x_3}$ is simply periodic. The proof for the 2-generator groups, the 3-generator groups and the 5-generator groups is similar to the above and is omitted.

We denote the periods of the orbits $FP(G)_{x_0,\ldots,x_k}$ with $1\leq k\leq 4$ by $LFP(G)_{x_0,\ldots,x_k}$. From the definition, it is clear that the period of a Fibonacci-Padovan orbit of a finite group depends on the chosen generating set and the order for the assignments of $x_0,\ldots,x_k$ such that $1\leq k\leq 4$.

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Definition 3. Let G be a finite group. If there exists a Fibonacci-Padovan orbit of the group G such that every element of the group G appears in the sequence, then the group G is called Fibonacci-Padovan sequenceable.

We now address the periods of the Fibonacci-Padovan orbits of specific classes of finite groups. The usual notation $G_1 \times_\varphi G_2$ is used for the semidirect product of the group G_1 by G_2, where $\varphi : G_2 \to \mathrm{Aut}(G_1)$ is a homomorphism such that $b\varphi = \varphi_b$ and $\varphi_b : G_1 \to G_1$ is an element $\mathrm{Aut}(G_1)$.

The quaternion group Q_8 is defined by

$$Q_8 = \langle x, y : x^4 = e,\ y^2 = x^2,\ y^{-1}xy = x^{-1} \rangle;$$

the direct product $Q_8 \times Z_{2m}$ $(m \geq 3)$ is defined by

$$Q_8 \times Z_{2m} = \langle x, y, z : x^4 = e,\ y^2 = x^2,\ y^{-1}xyx = z^{2m} = [x,z] = [y,z] = e \rangle;$$

and the semidirect product $Q_8 \times_\varphi Z_{2m}$ $(m \geq 3)$ is defined by

$$Q_8 \times_\varphi Z_{2m} = \langle x, y, z : x^4 = e,\ y^2 = x^2,\ y^{-1}xyx = z^{2m} = e,\ z^{-1}xzx = e,\ z^{-1}yzy = e \rangle,$$

where, if $Z_{2m} = \langle z \rangle$, then $\varphi : Z_{2m} \to \mathrm{Aut}(Q_8)$ is a homomorphism such that $z\varphi = \varphi_z$; $\varphi_z : Q_8 \to Q_8$ is defined by $x\varphi_z = x$ and $y\varphi_z = y^{-1}$.

Theorem 6. $LFP(Q_8)_{x,y} = LFP(Q_8)_{y,x} = 42.$

Proof. $FP(Q_8)_{y,x}$ is

$$y,\, x, x^3,\, x^3,\, yx^3,\, y^3,\, xy,\, y,\, x^3y,\, e, y^3,\, x^3,\, yx,\, y^2,\, y^2, y^3, xy,\, y^2,\, e,\, e,\, y^3,\, x,\, x,$$
$$x,\, x,\, y^3x,\, y^3, xy^3,\, y, xy,\, y^2,\, y,\, x^3, yx, e,\, y^2,\, y,\, x^3y,\, e,\, e,\, e,\, y,\, x,\, x^3,\, x^3,\, yx^3, \ldots ,$$

which has period $LFP(Q_8)_{y,x} = 42$.

The proof for the orbit $FP(Q_8)_{x,y}$ is similar to the above and is omitted.

Remark 1. The quaternion group Q_8 is Fibonacci-Padovan sequenceable.

Theorem 7. The period of the Fibonacci-Padovan orbit of the direct product $Q_8 \times Z_{2m}$ $(m \geq 3)$ for each generating triplet is $\mathrm{lcm}[42, l(2m)]$.

Proof. Consider the Fibonacci-Padovan orbit $FP(Q_8 \times Z_{2m})_{x,y,z}$:

$$x,\, y, z^{a_0},\, z^{a_1},\, y^2z^{a_2},\, x^3z^{a_3},\, xyz^{a_4},\, yxz^{a_5},\, yxz^{a_6}, xyz^{a_7},\, yz^{a_8},\, x^3z^{a_9},\, yz^{a_{10}},$$
$$xz^{a_{11}}, yz^{a_{12}},\, y^2z^{a_{13}}, xz^{a_{14}},\, xyz^{a_{15}},\, yz^{a_{16}},\, y^2z^{a_{17}}, z^{a_{18}},\, x^3z^{a_{19}},\, y^3z^{a_{20}},\, x^2z^{a_{21}},$$
$$z^{a_{22}},\, y^2z^{a_{23}}, xz^{a_{24}},\, xyz^{a_{25}},\, xyz^{a_{26}}, xyz^{a_{27}},\, yxz^{a_{28}},\, yz^{a_{29}}, x^3z^{a_{30}},\, y^3z^{a_{31}}, xz^{a_{32}},$$
$$y^3z^{a_{33}},\, z^{a_{34}},\, x^3z^{a_{35}},\, xyz^{a_{36}},\, yz^{a_{37}},\, z^{a_{38}}, z^{a_{39}},\, xz^{a_{40}},\, yz^{a_{41}}, z^{a_{42}}, z^{a_{43}},\, y^2z^{a_{44}}, \ldots .$$

Using the above information, the sequence becomes:

$$b_0 = x, b_1 = y, b_2 = z, b_3 = z, b_4 = y^2z^3, \ldots ,$$
$$b_{42} = xz^{40}, b_{43} = yz^{41}, b_{44} = z^{42}, b_{45} = z^{43}, b_{46} = y^2z^{44}, \ldots ,$$
$$b_{42\cdot i} = xz^{42\cdot i-2}, b_{42\cdot i+1} = yz^{42\cdot i-1}, b_{42\cdot i+2} = z^{42\cdot i}, b_{42\cdot i+3} = z^{42\cdot i+1}, b_{42\cdot i+4} = y^2z^{42\cdot i+2}, \ldots .$$

The sequence can be said to form layers of length 42. So we need an i such that $b_{42i}=x, b_{42\cdot i+1}=y, b_{42\cdot i+2}=z, b_{42\cdot i+3}=z, b_{42\cdot i+4}=y^2z^3$. It is easy to see that the Fibonacci-Padovan orbit $FP(Q_8\times Z_{2m})_{x,y,z}$ has period $\text{lcm}[42, l(2m)]$.

The proof for other generating triplets is similar to the above and is omitted.

Theorem 8. The period of the Fibonacci-Padovan orbit of the direct product $Q_8\times_\varphi Z_{2m}$ $(m\geq 3)$ for each generating triplet is $\text{lcm}[42, l(2m)]$.

Proof. Consider the Fibonacci-Padovan orbit $FP(Q_8\times_\varphi Z_{2m})_{y,x,z}$:

$$y, x, z^{a_0}, z^{a_1}, x^2z^{a_2}, y^3z^{a_3}, yxz^{a_4}, yxz^{a_5}, xyz^{a_6}, yxz^{a_7}, xz^{a_8}, yz^{a_9}, x^3z^{a_{10}},$$
$$y^3z^{a_{11}}, x^3z^{a_{12}}, z^{a_{13}}, yz^{a_{14}}, yxz^{a_{15}}, x^3z^{a_{16}}, z^{a_{17}}, y^2z^{a_{18}}, y^3z^{a_{19}}, x^3z^{a_{20}}, x^2z^{a_{21}},$$
$$x^2z^{a_{22}}, y^2z^{a_{23}}, yz^{a_{24}}, yxz^{a_{25}}, xyz^{a_{26}}, xyz^{a_{27}}, yxz^{a_{28}}, x^3z^{a_{29}}, y^3z^{a_{30}}, x^3z^{a_{31}}, y^3z^{a_{32}},$$
$$x^3z^{a_{33}}, x^2z^{a_{34}}, yz^{a_{35}}, yxz^{a_{36}}, x^3z^{a_{37}}, z^{a_{38}}, z^{a_{39}}, yz^{a_{40}}, xz^{a_{41}}, z^{a_{42}}, z^{a_{43}}, x^2z^{a_{44}}, \ldots .$$

Using the above information, the sequence becomes:

$$b_0=y, b_1=x, b_2=z, b_3=z, b_4=x^2z^3, \ldots ,$$
$$b_{42}=yz^{40}, b_{43}=xz^{41}, b_{44}=z^{42}, b_{45}=z^{43}, b_{46}=y^2z^{44}, \ldots ,$$
$$b_{42\cdot i}=xz^{42\cdot i-2}, b_{42\cdot i+1}=yz^{42\cdot i-1}, b_{42\cdot i+2}=z^{42\cdot i}, b_{42\cdot i+3}=z^{42\cdot i+1}, b_{42\cdot i+4}=x^2z^{42\cdot i+2}, \ldots .$$

The sequence can be said to form layers of length 42. So we need an i such that $b_{42i}=y, b_{42\cdot i+1}=x, b_{42\cdot i+2}=z, b_{42\cdot i+3}=z, b_{42\cdot i+4}=x^2z^3$. It is easy to see that the Fibonacci-Padovan orbit $FP(Q_8\times_\varphi Z_{2m})_{y,x,z}$ has period $\text{lcm}[42, l(2m)]$.

The proof for other generating triplets is similar to the above and is omitted.

Remark 2. If $l(2m)<2m$ and $42|l(2m)$, the groups $Q_8\times Z_{2m}$ and $Q_8\times_\varphi Z_{2m}$ such that $m\geq 3$ are not Fibonacci-Padovan sequenceable (where, by $42|l(2m)$, we mean that 42 divides $l(2m)$).

CONCLUSIONS

Examining the Fibonacci-Padovan sequence modulo m, we have defined the Fibonacci-Padovan orbits of j-generator finite groups for $2\leq j\leq 5$. Furthermore, we have obtained the Fibonacci-Padovan lengths of the groups Q_8, $Q_8\times Z_{2m}$ and $Q_8\times_\varphi Z_{2m}$ for $m\geq 3$.

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REFERENCES

1. P. G. Becker, "*k*-Regular power series and Mahler-type functional equations", *J. Number Theory*, **1994**, *49*, 269-286.
2. W. Bosma and C. Kraaikamp, "Metrical theory for optimal continued fractions", *J. Number Theory*, **1990**, *34*, 251-270.
3. M. S. El Naschie, "Deriving the essential features of the standard model from the general theory of relativity", *Chaos Solitons Fractals*, **2005**, *24*, 941-946.

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4. M. S. El Naschie, "Stability analysis of the two-slit experiment with quantum particles", *Chaos Solitons Fractals*, **2005**, *26*, 291-294.
5. S. Falcon and A. Plaza, "*k*-Fibonacci sequences modulo m", *Chaos Solitons Fractals*, **2009**, *41*, 497-504.
6. A. S. Fraenkel and S. T. Kleinb, "Robust universal complete codes for transmission and compression", *Discrete Appl. Math.*, **1996**, *64*, 31-55.
7. B. K. Kirchoof and R. Rutishauser, "The phyllotaxy of costus (Costaceae)", *Bot. Gaz.*, **1990**, *151*, 88-105.
8. D. M. Mandelbaum, "Synchronization of codes by means of Kautz's Fibonacci encoding", *IEEE Trans. Inform. Theory*, **1972**, *18*, 281-285.
9. W. Syein, "Modelling the evolution of stelar archictecture in vascular plants", *Int. J. Plant Sci.*, **1993**, *154*, 229-263.
10. R. G. E. Pinch, "Distribution of recurrent sequences modulo prime powers", Proceedings of 5th IMA Conference on Cryptography and Coding, **1995**, Cirencester, UK.
11. D. D. Wall, "Fibonacci series modulo m", *Am. Math. Monthly*, **1960**, *67*, 525-532.
12. O. Deveci and E. Karaduman, "The generalized order-*k* Lucas sequences in finite groups", *J. Appl. Math.*, **2012**, DOI: 10.1155/2012/464580.
13. O. Deveci, "The *k-nacci* sequences and the generalized order-*k* Pell sequences in the semi-direct product of finite cyclic groups", *Chiang Mai J. Sci.*, **2013**, *40*, 89-98.
14. O. Deveci and E. Karaduman, "The cyclic groups via the Pascal matrices and the generalized Pascal matrices", *Linear Algebra Appl.*, **2012**, *437*, 2538-2545.
15. O. Deveci and E. Karaduman, "The Pell sequences in finite groups", *Util. Math.*, **2015**, *96* (in press).
16. O. Deveci, "The Pell-Padovan sequences and the Jacobsthal-Padovan sequences in finite groups", *Util. Math.*, in press.
17. S. W. Knox, "Fibonacci sequences in finite groups", *Fibonacci Quart.*, **1992**, *30*, 116-120.
18. K. Lü and J. Wang, "*k*-Step Fibonacci sequence modulo *m*", *Util. Math.*, **2007**, *71*, 169-178.
19. E. Ozkan, H. Aydin and R. Dikici, "3-Step Fibonacci series modulo m", *Appl. Math. Comput.*, **2003**, *143*, 165-172.
20. V. W. de Spinadel, "The family of metallic means", *Visual Math.*, **1999**, *1*, 176-185.
21. V. W. de Spinadel, "The metallic means family and forbidden symmetries", *Int. Math. J.*, **2002**, *2*, 279-288.
22. N. D. Gogin and A. A. Myllari, "The Fibonacci-Padovan sequence and MacWilliams transform matrices", *Program. Comput. Softw.*, **2007**, *33*, 74-79.
23. D. Kalman, "Generalized Fibonacci numbers by matrix methods", *Fibonacci Quart.*, **1982**, *20*, 73-76.
24. C. M. Campbell and P. P. Campbell, "The Fibonacci length of certain centro-polyhedral groups", *J. Appl. Math. Comput.*, **2005**, *19*, 231-240.