Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

Maejo International
Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

Levitin-Polyak well-posedness of inverse quasi-variational inequality with perturbations

Garima Virmani * **and Manjari Srivastava**

Department of Mathematics, University of Delhi, Delhi 110007, India

* Corresponding author, e-mail: garimavirmani86@gmail.com

Received: 20 November 2013/ Accepted: 13 October 2014/ Published: 30 October 2014

Abstract: Levitin-Polyak α-well-posedness for inverse quasi-variational inequality is investigated. We establish some metric characterisations of Levitin-Polyak α -well-posedness for inverse quasi-variational inequality problems having a unique solution and give some conditions under which the above problem is Levitin-Polyak α -well-posed by perturbations in the generalised sense.

Keywords: inverse quasi-variational inequality, Levitin-Polyak well-posedness by perturbations, metric characterisation

---

INTRODUCTION

Well-posedness is one of the most important concepts in the theory of optimisation and it has been studied extensively in the literature [1-8]. It plays a crucial role in the stability theory for optimisation problems, which guarantees that for an approximating solution sequence, there exists a subsequence which converges to a solution. The study of well-posedness for scalar minimisation problems was started by Tykhonov [1], according to whom the problem of minimising a function $h(x)$ over a closed convex set K is Tykhonov well-posed if it has a unique solution and every minimising sequence converges to the solution.

This type of well-posedness requires every minimising sequence to lie in the feasible set K. In many practical situations, the minimising sequence $\{x_n\}$ produced by a numerical optimisation method usually fails to be in K but $d_K(x_n) \to 0$ as $n \to \infty$. Such a sequence is called a generalised minimising sequence. Taking this into account, Levitin and Polyak [2] introduced a strengthening notion of Tykhonov well-posedness by requiring the convergence of every generalised minimising sequence to a unique solution, which is known as Levitin-Polyak (LP) well-posedness. Zolezzi [7, 8] also introduced a strengthening version of Tykhonov well-posedness by the name of extended well-posedness (also named well-posedness by perturbations) by embedding the original problem into a family of perturbed problems depending on a parameter. This form of well-posedness establishes continuous dependence of the solution upon a parameter.

In recent years, the concepts of well-posedness have been generalised to various variational inequality problems [3, 9-19] and quasi-variational inequality problems [20-24]. The main motivation originates from the fact that a minimisation problem is equivalent to a variational inequality under convexity and differentiability assumptions. There is vast literature which relates the solutions of a minimisation problem and (generalised) variational inequality problem (see [25-28] and references therein). Lucchetti and Patrone [17] were the first to introduce the notion of well-posedness for a variational inequality. Lignola and Morgan [15] generalised the notion of well-posedness by perturbations to a variational inequality problem and established the equivalence between the well-posedness by perturbations of a variational inequality and the corresponding minimisation problem. Hu and Fang [12] and Hu et al. [29] studied the LP well-posedness of variational inequalities. Huang et al. [14] studied the LP well-posedness of variational inequality problems with functional constraints.

A new class of problems known as the inverse variational inequality (IVI) has been proposed and studied [30-34]. The IVI has broad applications in the market equilibrium problems in economics and normative flow control problems, appearing in transportation and telecommunication. The primary motivation of research on IVI originates from the transportation system operation and control policies. He et al. [33] demonstrated the applicability of constrained black-box variational inequality to the class of network equilibrium control problems by taking the spatial price equilibrium problem as an example. However, in some practical situations it is necessary to consider IVI models in which the constraint set is not constant. Hence a new class of problems known as the inverse quasi-variational inequality was introduced and studied recently by Aussel et al. [35]. This class of problems finds applications in road pricing problem in which the environmental impact problem due to traffic flow is taken into account to fix road taxes. Very recently, Hu and Fang [11] introduced and studied the well-posedness of IVI, which was further extended for Levitin-Polyak well-posedness by Hu and Fang [36].

Motivated by the above-mentioned research work, we establish some metric characterisations of LP α-well-posedness by perturbations. We derive some conditions under which the LP well-posedness by perturbations of an inverse quasi-variational inequality (IQVI) is equivalent to the existence and uniqueness of its solution. We also establish metric characterisations of LP α-well-posedness by perturbations in the generalised sense.

PRELIMINARIES

Let $f, h : \mathbb{R}^n \to \mathbb{R}^n$ be two continuous maps. Let $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued map such that for each $x \in \mathbb{R}^n$, $K(x)$ is a closed convex set in $\mathbb{R}^n$. The IQVI associated with f, h and K denoted by IQVI(f, h, K) consists of:

Find $x^* \in \mathbb{R}^n$ such that $h(x^*) \in K(x^*)$ and

$$\langle f(x^*), y - h(x^*) \rangle \geq 0 \quad \forall \quad y \in K(x^*).$$

When h is the identity map on $\mathbb{R}^n$, then IQVI(f, h, K) reduces to the classical quasi-variational inequality.

When $K(x)$ is a constant set $\bar{K}$ on $\mathbb{R}^n$, then IQVI (f, h, K) reduces to inverse variational inequality IVI $(f, h, \bar{K})$.

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

Now, take f to be the identity map. We have

$$\text{IQVI}(h,K):\ \text{Find } x^* \in \mathbb{R}^n \text{ such that } h(x^*) \in K(x^*)$$
$$\text{and } \langle x^*, y - h(x^*)\rangle \geq 0 \quad \forall \quad y \in K(x^*).$$

Let L be a parametric normed space and $P \subset L$ be a closed ball with positive radius; $p^* \in P$ is a fixed point. Then the perturbed problem of IQVI (h,K) is given by

$$\text{IQVI}_p(\tilde{h},K):\ \text{Find } x^* \in \mathbb{R}^n \ \text{ such that } \ \tilde{h}(p,x^*) \in K(x^*)$$
$$\text{and } \langle x^*, y - \tilde{h}(p,x^*)\rangle \geq 0 \quad \forall \quad y \in K(x^*)$$

where $\tilde{h} : P \times \mathbb{R}^n \to \mathbb{R}^n$ is such that $\tilde{h}(p^*,\cdot) = h$.
When $K(x^*) = K$, $\text{IQVI}_p(\tilde{h},K)$ reduces to the inverse variational inequality, the well-posedness of which has already been dealt with [11, 36].
Let T be the solution set of $\text{IQVI}_p(\tilde{h},K)$.

Now, we introduce some notations:

Definition 1. *Let* $\{p_n\} \subset P$ *with* $p_n \to p^*$. *A sequence* $\{x_n\} \subset \mathbb{R}^n$ *is called α-approximating sequence corresponding to* $\{p_n\}$ *for* $\text{IQVI}_p(\tilde{h},K)$ *if there exists a sequence* $\varepsilon_n > 0$ *with* $\varepsilon_n \to 0$ *such that* $\tilde{h}(p_n,x_n) \in K(x_n)$ *and*

$$\langle x_n, \tilde{h}(p_n,x_n) - y\rangle \leq \frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n \quad \forall \quad y \in K(x_n) \ \text{ and } \ n \in \mathbb{N}.$$

When $\alpha = 0$, we say that $\{x_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for $\text{IQVI}_p(\tilde{h},K)$.

Definition 2. *Let* $\{p_n\} \subset P$ *with* $p_n \to p^*$. *A sequence* $\{x_n\} \subset \mathbb{R}^n$ *is called LP α-approximating* sequence c*orresponding to* $\{p_n\}$ *for* $\text{IQVI}_p(\tilde{h},K)$ *if there exists a sequence* $\varepsilon_n > 0$ *with* $\varepsilon_n \to 0$ *and* $w_n \in \mathbb{R}^n$ *with* $w_n \to 0$ *such that* $\tilde{h}(p_n,x_n) + w_n \in K(x_n)$ *and*

$$\langle x_n, \tilde{h}(p_n,x_n) - y\rangle \leq \frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n \quad \forall y \in K(x_n) \ \text{ and } \ n \in \mathbb{N}.$$

When $\alpha = 0$, we say that $\{x_n\}$ is LP approximating sequence corresponding to $\{p_n\}$ for $\text{IQVI}_p(\tilde{h},K)$.

Clearly every α-approximating sequence (corresponding to $\{p_n\}$) is LP α-approximating.

Definition 3. $\text{IQVI}_p(\tilde{h},K)$ *is α-well-posed by perturbations if* $\text{IQVI}_p(\tilde{h},K)$ *has a unique solution and for any sequence* $\{p_n\} \subset P$ *with* $p_n \to p^*$, *every α-approximating sequence corresponding to* $\{p_n\}$ *converges to the unique solution.*

If $\alpha_1 > \alpha_2 \geq 0$, then α_1-well-posedness by perturbations implies α_2-well-posedness by perturbations.

Definition 4. $\text{IQVI}_p(\tilde{h},K)$ is LP α-well-posed by perturbations *if* $\text{IQVI}_p(\tilde{h},K)$ *has a unique solution and for any sequence* $\{p_n\} \subset P$ *with* $p_n \to p^*$, *every LP α-approximating sequence corresponding to* $\{p_n\}$ *converges to the unique solution.*

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

When $p_n = p^*$, LP α-well-posedness by perturbations is called LP α-well-posedness.

Definition 5. $\mathrm{IQVI}_p(\tilde{h},K)$ *is α-well-posed by perturbations in the generalised sense if* $\mathrm{IQVI}_p(\tilde{h},K)$ *has a non-empty solution set and for any sequence* $\{p_n\} \subset P$ *with* $p_n \to p^*$, *every α-approximating sequence corresponding to* $\{p_n\}$ *has some subsequence which converges to some solution.*

Definition 6. $\mathrm{IQVI}_p(\tilde{h},K)$ *is LP α-well-posed by perturbations in the generalised sense if* $\mathrm{IQVI}_p(\tilde{h},K)$ *has at least one solution and for any sequence* $\{p_n\} \subset P$ *with* $p_n \to p^*$, *every LP α-approximating sequence corresponding to* $\{p_n\}$ *has some subsequence which converges to some solution.*

Definition 7. *A function* $h : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be*
(i) monotone if $\langle h(x) - h(y), x - y \rangle \geq 0 \;\; \forall \;\; x, y \in \mathbb{R}^n$.
(ii) hemicontinuous if, for any $x, y \in \mathbb{R}^n$, $t \mapsto \langle hx + t(y-x), y - x \rangle$ *from* $[0,1]$ *to* $\mathbb{R}$ *is continuous at* 0^+.

Definition 8. *Let* A, B *be non-empty subsets of* $\mathbb{R}^n$. *The excess from* A *to* B *is defined by*

$$e(A,B) = \sup_{a \in A} d(a,B),$$

where $d(a,B) = \inf_{b \in B} \| a - b \|$.

Now, recall the notion of Mosco convergence. A sequence $(H_n)_n$ of subsets of E converges to a set H if

$$H = \liminf_n H_n = w\text{-}\limsup_n H_n,$$

where $\liminf H_n$ is the Painlevé-Kuratowski strong limit inferior and $w\text{-}\limsup H_n$ is the Painlevé-Kuratowski weak limit superior of a sequence $(H_n)_n$, i.e.

$$\liminf H_n = \{ y \in E : \exists \;\; y_n \in H_n, n \in \mathbb{N} \;\; \text{with} \;\; y_n \to y \}$$

and

$$w\text{-}\limsup H_n = \{ y \in E : \exists \;\; n_k \uparrow +\infty, n_k \in \mathbb{N}, \exists \;\; y_{n_k} \in H_{n_k}, k \in \mathbb{N} \;\; \text{with} \;\; y_{n_k} \rightharpoonup y \}.$$

If $H = \liminf H_n$, we call the sequence $(H_n)_n$ of subsets of E Lower Semi-Mosco which converges to a set H.

We state the following lemma which will be needed in the paper.

Lemma 1 [20]. *Let* $(H_n)_n$ *be a sequence of non-empty subsets of* E *such that*
(i) H_n *is convex* $\forall \;\; n \in \mathbb{N}$.
(ii) $H_0 \subseteq \liminf H_n$.
(iii) there exists $m \in \mathbb{N}$ *such that* $\bigcap_{n \geq m} H_n \neq \varnothing$.
Then for every $u_0 \in \operatorname{int} H_0$, *there exists a positive real number* δ *such that* $B(u_0, \delta) \subseteq H_n \;\; \forall \;\; n \geq m$.
If E *is a finite dimensional space, then assumption (iii) can be replaced by (iii)'* $\operatorname{int} H_0 \neq \varnothing$.

Now, consider the following approximating solution set of $\mathrm{IQVI}_p(\tilde{h},K)$:
For any $\varepsilon \geq 0$,

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

$$T_\alpha^p(\varepsilon) = \bigcup_{p \in B(p^*,\varepsilon)} \{x \in \mathbb{R}^n \ : \ d(\tilde{h}(p,x), K(x)) \le \varepsilon$$

$$\text{and } \langle x, \tilde{h}(p,x) - y \rangle \le \frac{\alpha}{2} \| \tilde{h}(p,x) - y \|^2 + \varepsilon \quad \forall\, y \in K(x)\},$$

where $B(p^*,\varepsilon)$ denotes the closed ball centred at p^* with radius ε.

CASE OF A UNIQUE SOLUTION

In this section we investigate some metric characterisations of LP α-well-posedness with perturbations for $\text{IQVI}_p(\tilde{h},K)$.

Theorem 1. *Let* x^* *be a solution of* $\text{IQVI}_p(\tilde{h},K)$. *Then* $\text{IQVI}_p(\tilde{h},K)$ *is LP* α*-well-posed by perturbations if and only if* $\theta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *where* $\theta(\varepsilon) = \sup\{\| x - x^* \| : x \in T_\alpha^p(\varepsilon)\}$ $\forall$ $\varepsilon \ge 0$.

Proof: Let $\text{IQVI}_p(\tilde{h},K)$ be LP α-well-posed by perturbations.

Then $x^* \in \mathbb{R}^n$ is the unique solution of $\text{IQVI}_p(\tilde{h},K)$.

Let $\theta(\varepsilon) \not\to 0$ as $\varepsilon \to 0$. Then there exists $\delta > 0$ and a sequence $\{p_n\}$ such that $\text{IQVI}_p(\tilde{h},K)$.

By definition of θ, there exists $x_n \in T_\alpha^p(\varepsilon_n)$ such that

$$\| x_n - x^* \| > \delta . \tag{1}$$

Since $x_n \in T_\alpha^p(\varepsilon_n)$, there exists $p_n \in B(p^*,\varepsilon_n)$ such that

$$d(\tilde{h}(p_n,x_n), K(x_n)) \le \varepsilon_n < \varepsilon_n + \frac{1}{n} \tag{2}$$

and

$$\left\langle x_n, \tilde{h}(p_n,x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n \quad \forall\, y \in K(x_n). \tag{3}$$

Clearly, $p_n \to p^*$.

By (2), there exists $y_n \in K(x_n)$ such that $\| \tilde{h}(p_n,x_n) - y_n \| < \varepsilon_n + \frac{1}{n}$.

Take $w_n = y_n - \tilde{h}(p_n,x_n)$. Then

$$\tilde{h}(p_n,x_n) + w_n \in K(x_n) \text{ and } w_n \to 0 . \tag{4}$$

By (3) and (4), $\{x_n\}$ is LP α-approximating sequence corresponding to $\{p_n\}$ for $\text{IQVI}_p(\tilde{h},K)$ which is LP α-well-posed by perturbations.

Thus $\| x_n - x^* \| \to 0$, a contradiction to (1).

Conversely, let $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $x^* \in \mathbb{R}^n$ is the unique solution of $\text{IQVI}_p(\tilde{h},K)$. If $\bar{x}$ $(\ne x^*)$ is another solution of $\text{IQVI}_p(\tilde{h},K)$, then $\theta(\varepsilon) \ge \| x^* - \bar{x} \| > 0$ $\forall$ $\varepsilon \ge 0$, a contradiction. Let $\{p_n\} \subset P$ with $p_n \to p^*$ and let $\{x_n\}$ be LP α-approximating sequence corresponding to $\{p_n\}$ for $\varepsilon_n = \max\{\| p_n - p^* \|, \| w_n \|, \varepsilon_n'\}$. Then there exists $w_n \in \mathbb{R}^n$ with $w_n \to 0$ and $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$\tilde{h}(p_n,x_n) + w_n \in K(x_n)$$

and

$$\left\langle x_n, \tilde{h}(p_n,x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n \quad \forall\, y \in K(x_n) \quad \forall\, n \in \mathbb{N} .$$

Taking $\delta_n = \| p_n - p^* \|$ and $\varepsilon'_n = \max\{\delta_n, \varepsilon_n, \| w_n \|\}$, we have $x_n \in T_\alpha^p(\varepsilon'_n)$ with $\varepsilon'_n \to 0$.
Thus, $\| x_n - x^* \| \le \theta(\varepsilon'_n) \to 0$.
Hence $\text{IQVI}_p(\tilde{h}, K)$ is LP α-well-posed by perturbations.

The following example justifies the conclusion of Theorem 1.

Example 1. Let $X = \mathbb{R}$, $\tilde{h}(p,x) = x(1+|p|)$, $K(x) = \begin{cases} [0,\infty), & x \le 0 \\ (-\infty,0), & x > 0 \end{cases}$, $P = [-1,1]$, $p^* = 0$, $\alpha = 2$.
Clearly, $x^* = 0$ is the solution of $\text{IQVI}_p(\tilde{h}, K)$.
Let $A_\alpha^p(\varepsilon) = \{x \in \mathbb{R} : d(\tilde{h}(p,x), K(x)) \le \varepsilon\}$.
When $x \le 0$, $A_\alpha^p(\varepsilon) = \left[\frac{-\varepsilon}{1+|p|}, 0\right]$.
When $x > 0$, $A_\alpha^p(\varepsilon) = \left[0, \frac{\varepsilon}{1+|p|}\right]$.
Let

$$\begin{aligned} B_\alpha^p(\varepsilon) &= \left\{x \in \mathbb{R} : \langle x, \tilde{h}(p,x) - y\rangle \le \| \tilde{h}(p,x) - y \|^2 + \varepsilon \quad \forall \quad y \in K(x)\right\} \\ &= \left\{x \in \mathbb{R} : -\left[y - \left(\frac{2|p|+1}{2}\right)x\right]^2 + \frac{x^2}{4} \le \varepsilon \quad \forall \quad y \in K(x)\right\}. \end{aligned}$$

When $x \le 0$,

$$B_\alpha^p(\varepsilon) = \left\{x \le 0 : -\left[y - \left(\frac{2|p|+1}{2}\right)x\right]^2 + \frac{x^2}{4} \le \varepsilon \ \forall y \ge 0\right\}.$$

When $x > 0$,

$$B_\alpha^p(\varepsilon) = \left\{x > 0 \ : -\left[y - \left(\frac{2|p|+1}{2}\right)x\right]^2 + \frac{x^2}{4} \le \varepsilon \quad \forall \quad y < 0\right\}.$$

Now,

$$T_\alpha^p(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} [A_\alpha^p(\varepsilon) \cap B_\alpha^p(\varepsilon)].$$

When $x \le 0$,

$$\begin{aligned} T_\alpha^p(\varepsilon) &= \bigcup_{p \in B(0,\varepsilon)} \left(\left[\frac{-\varepsilon}{1+|p|}, 0\right] \cap B_\alpha^p(\varepsilon)\right) \\ &= \bigcup_{p \in B(0,\varepsilon)} \left[\frac{-\varepsilon}{1+|p|}, 0\right] = [-\varepsilon, 0]. \end{aligned}$$

When $x > 0$,

$$\begin{aligned} T_\alpha^p(\varepsilon) &= \bigcup_{p \in B(0,\varepsilon)} \left(\left[0, \frac{\varepsilon}{1+|p|}\right] \cap B_\alpha^p(\varepsilon)\right) \\ &= \bigcup_{p \in B(0,\varepsilon)} \left(0, \frac{\varepsilon}{1+|p|}\right] = (0, \varepsilon]. \end{aligned}$$

Thus,

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

$$\theta(\varepsilon) = \{\sup \| x - x^* \| : x \in T_\alpha^p(\varepsilon)\}$$
$$= \sup_{x \in T_\alpha^p(\varepsilon)} |x| = \begin{cases} \varepsilon, & x \le 0 \\ \varepsilon, & x > 0 \end{cases}$$

and so $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Hence the problem is LP α-well-posed by perturbations.

Next, we give a characterisation of LP α-well-posedness by perturbations using the approximate solution set $T_\alpha^p(\varepsilon)$.

Theorem 2. *Let K be a non-empty, convex-valued, set-valued map. Let $\tilde{h} : P \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume the following:*

(i) K is non-empty convex-valued and for each sequence $\{x_n\}$ converging to x, the sequence $K(x_n)_n$ Lower Semi-Mosco converges to $K(x)$.

(ii) For every converging sequence $(x_n)_n$, there exists $m \in \mathbb{N}$ such that $\operatorname{int} \bigcap_{n \ge m} K(x_n) \ne \varnothing$.

Then $\mathrm{IQVI}_p(\tilde{h}, K)$ is LP α-well-posed by perturbations if and only if $T_\alpha^p(\varepsilon) \ne \varnothing \ \forall \ \varepsilon > 0$ and $\operatorname{diam} T_\alpha^p(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof: Let $\mathrm{IQVI}_p(\tilde{h}, K)$ be LP α-well-posed by perturbations. Then $T_\alpha^p(\varepsilon) \ne \varnothing \ \forall \ \varepsilon > 0$.

On the contrary, assume diam $T_\alpha^p(\varepsilon) \not\to 0$ as $\varepsilon \to 0$. Then there exists a constant $\ell > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0$ and $\{x_n^{(1)}\}, \{x_n^{(2)}\} \in T_\alpha^p(\varepsilon_n)$ such that

$$\| x_n^{(1)} - x_n^{(2)} \| > \ell \quad \forall\, n \in \mathbb{N} \tag{5}$$

As $\{x_n^{(1)}\} \in T_\alpha^p(\varepsilon_n)$, we have

$$d(\tilde{h}(p_n, x_n^{(1)}), K(x_n)) \le \varepsilon_n < \varepsilon_n + \frac{1}{n} \tag{6}$$

and

$$\left\langle x_n^{(1)}, \tilde{h}(p_n, x_n^{(1)}) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n, x_n^{(1)}) - y \|^2 + \varepsilon_n \quad \forall\, y \in K(x_n) \quad \forall\, n \in \mathbb{N}. \tag{7}$$

By (6), there exists $\bar{x}_n^{(1)} \in K(x_n)$ such that

$$\| \tilde{h}(p_n, x_n^{(1)}) - \bar{x}_n^{(1)} \| < \varepsilon_n + \frac{1}{n}.$$

Let $w_n = \bar{x}_n^{(1)} - \tilde{h}(p_n, x_n^{(1)})$.

Thus, $\tilde{h}(p_n, x_n^{(1)}) + w_n = \bar{x}_n^{(1)} \in K(x_n)$ and $\| w_n \| = \| \bar{x}_n^{(1)} - \tilde{h}(p_n, x_n^{(1)}) \| \to 0$ as $n \to \infty$.

Thus, $\{x_n^{(1)}\}$ is LP α-approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IQVI}_p(\tilde{h}, K)$.

Similarly, $\{x_n^{(2)}\}$ is LP α-approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IQVI}_p(\tilde{h}, K)$.

So they both converge to the unique solution of $\mathrm{IQVI}_p(\tilde{h}, K)$, which is a contradiction to (5).

Conversely, let $T_\alpha^p(\varepsilon) \ne \varnothing \ \forall \ \varepsilon > 0$ and diam $T_\alpha^p(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let $\{p_n\} \subset P$ with $p_n \to p^*$, and let $\{x_n\}$ be LP α-approximating sequence for $\mathrm{IQVI}_p(\tilde{h}, K)$.

Then there exists $0 < \{\varepsilon_n'\} \downarrow 0$ and $w_n \in \mathbb{R}^n$ with $w_n \to 0$ such that $\tilde{h}(p_n, x_n) + w_n \in K(x_n)$ and

$$\left\langle x_n, \tilde{h}(p_n, x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n, x_n) - y \|^2 + \varepsilon_n' \quad \forall\, y \in K(x_n) \quad \forall\, n \in \mathbb{N}.$$

As $\tilde{h}(p_n, x_n) + w_n \in K(x_n)$, there exists $\bar{x}_n \in K(x_n)$ such that $\tilde{h}(p_n, x_n) + w_n = \bar{x}_n$.

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

Now, $d(\tilde{h},(p_n,x_n),K(x_n)) \leq \| \tilde{h}(p_n,x_n) - \bar{x}_n \| = \| w_n \| \to 0$.

Set $\varepsilon_n = \max\{\| p_n - p^* \|, \| w_n \|, \varepsilon'_n\}$. Thus, $\{x_n\} \in T_\alpha^p(\varepsilon_n)$.

Also, by hypothesis, $\{x_n\}$ is a Cauchy sequence and so it converges strongly to a point x. If x_1, x_2 are two distinct solutions of $\mathrm{IQVI}_p(\tilde{h},K)$, then $x_1, x_2 \in T_\alpha^p(\varepsilon)$ $\forall$ $\varepsilon > 0$ and $0 < \| x_1 - x_2 \| \leq \operatorname{diam} T_\alpha^p(\varepsilon) \to 0$, a contradiction. We claim that x solves $\mathrm{IQVI}_p(\tilde{h},K)$. We prove this in the next two steps:

(a) $\tilde{h}(p^*,x) \in K(x)$.

For each $n \in \mathbb{N}$, choose $\tilde{h}(p_n,x'_n) \in K(x_n)$ such that

$$\tilde{h}(p_n,x_n) - \tilde{h}(p_n,x'_n) < d(\tilde{h}(p_n,x_n),K(x_n)) + \varepsilon_n \leq 2\varepsilon_n .$$

As $x_n \to x$ and $h(\cdot,\cdot)$ is continuous, we get $\tilde{h}(p_n,x_n) \to \tilde{h}(p^*,x)$.

Thus, it follows from above and $\varepsilon \to 0$ that

$$\tilde{h}(p_n,x'_n) \to \tilde{h}(p^*,x).$$

Now,

$$\liminf K(x_n) = \{\tilde{h}(p^*,x) \in \mathbb{R}^n : \exists\, \tilde{h}(p_n,x'_n) \in K(x_n), n \in \mathbb{N} \text{ with } \tilde{h}(p_n,x'_n) \to \tilde{h}(p^*,x)\}.$$

Thus, $\liminf K(x_n) = K(x)$ and so $\tilde{h}(p^*,x) \in K(x)$.

(b) $\langle x, y - \tilde{h}(p^*,x)\rangle \geq 0$ $\forall$ $y \in K(x)$.

Consider any $y \in K(x)$.

By Lower Semi-Mosco convergence, we have

$$K(x) \subseteq \liminf K(x_n) .$$

By condition (ii) in the hypothesis applied to the constant sequence x $\forall$ $n \in \mathbb{N}$ implies $\operatorname{int} K(x) \neq \emptyset$.

Hence from Lemma 1, for $v \in \operatorname{int} K(x)$, there exists $\delta > 0$ and $m \in \mathbb{N}$ such that $B(v,\delta) \subseteq K(x_n)$ $\forall$ $n > m$. Thus, $v \in K(x_n)$ for n sufficiently large.

As $\{x_n\}$ is LP α-approximating sequence for $\mathrm{IQVI}_p(\tilde{h},K)$, we have

$$\begin{aligned}\left\langle x, \tilde{h}(p^*,x) - y \right\rangle &= \lim \left\langle x_n, \tilde{h}(p_n,x_n) - y \right\rangle \\ &\leq \lim \left(\frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n \right) \\ &= \frac{\alpha}{2} \| \tilde{h}(p^*,x) - y \|^2 .\end{aligned}$$

Now, if $v \in K(x) \setminus \operatorname{int} K(x)$, let $\{v_n\}$ be a sequence converging to v, whose point belongs to a segment contained in $\operatorname{int} K(x)$.

Since $v_n \in \operatorname{int} K(x)$ $\forall$ $n \in \mathbb{N}$, one has

$$\left\langle x_n, \tilde{h}(p_n,x_n) - v_n \right\rangle \leq \frac{\alpha}{2} \| \tilde{h}(p^*,x) - v_n \|^2 .$$

As h is continuous, this implies

$$\left\langle x, \tilde{h}(p^*,x) - v \right\rangle \leq \frac{\alpha}{2} \| \tilde{h}(p^*,x) - v \|^2, \quad v \in K(x).$$

As K is convex-valued, then $\forall$ $y \in K(x)$, $t \in [0,1]$, $v_t = ty + (1-t)\tilde{h}(p^*,x) \in K(x)$.

Thus, $\langle x, \tilde{h}(p^*,x) - v_t \rangle \leq \frac{\alpha}{2} \| \tilde{h}(p^*,x) - v_t \|^2$ $\forall$ $y \in K(x), t \in [0,1]$.

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

Taking $t \to 0$, we get that x solves $\mathrm{IQVI}_p(\tilde{h},K)$.

LP α -WELL-POSEDNESS BY PERTURBATIONS IN THE GENERALISED SENSE

In this section we investigate metric characterisations of LP α-well-posedness in the generalised sense for $\mathrm{IQVI}_p(\tilde{h},K)$.

Theorem 3. *Let x^* be a solution of $\mathrm{IQVI}_p(\tilde{h},K)$. Then $\mathrm{IQVI}_p(\tilde{h},K)$ is LP α-well-posed by perturbations in the generalised sense if and only if T is non-empty and compact, and $e(T_\alpha^p(\varepsilon),T) \to 0$ as $\varepsilon \to 0$.*

Proof: Let $\mathrm{IQVI}_p(\tilde{h},K)$ be LP α-well-posed by perturbations in the generalised sense. Then T is compact as when $\{x_n\}$ is any sequence in T and sequence $\{p_n\} \subset P$ with $p_n = p^*$, then $\{x_n\}$ is LP α-approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IQVI}_p(\tilde{h},K)$. So $\{x_n\}$ has a sub-sequence which converges to some point of T. Thus, T is compact.

If $e(T_\alpha^p(\varepsilon),T) \not\to 0$ as $\varepsilon \to 0$, then there exists $\ell > 0$, $\varepsilon_n > 0$ with $\varepsilon_n \downarrow 0$ and $x_n \in T_\alpha^p(\varepsilon_n)$ such that

$$x_n \notin T + B(0,\ell). \tag{8}$$

Since $x_n \in T_\alpha^p(\varepsilon_n)$, there exists $p_n \in B(p^*,\varepsilon_n)$ such that

$$d(\tilde{h}(p_n,x_n),K(x_n)) \le \varepsilon_n < \varepsilon_n + \frac{1}{n}$$

and

$$\left\langle x_n, \tilde{h}(p_n,x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n \quad \forall\ y \in K(x_n).$$

As proved in Theorem 2, $\{x_n\}$ is LP α-approximating sequence corresponding to $\{p_n\}$ for $\mathrm{IQVI}_p(\tilde{h},K)$, which is LP α-well-posed by perturbations in the generalised sense. Thus, there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of T. This contradicts (8).

Conversely, let T be compact and $e(T_\alpha^p(\varepsilon),T) \to 0$ as $\varepsilon \to 0$.

Let $\{p_n\} \subset P$ with $p_n \to p^*$, and let $\{x_n\}$ be LP α-approximating sequence corresponding to $\{p_n\}$.

Then there exists $w_n \in \mathbb{R}^n$ with $w_n \to 0$ and $0 < \varepsilon_n' \downarrow 0$ such that $\tilde{h}(p_n,x_n) + w_n \in K(x_n)$ and

$$\left\langle x_n, \tilde{h}(p_n,x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n,x_n) - y \|^2 + \varepsilon_n' \quad \forall\ y \in K(x_n)\ \forall\ n \in \mathbb{N}.$$

Take $\varepsilon_n = \max\{\varepsilon_n', \| w_n \|, \| p_n - p^* \|\}$. Then $\varepsilon_n \to 0$ and $x_n \in T_\alpha^p(\varepsilon_n)$.

Thus, $d(x_n,T) \le e(T_\alpha^p(\varepsilon_n),T) \to 0$.

Since T is compact, there exists $\bar{x}_n \in T$ such that $\| x_n - \bar{x}_n \| = d(x_n,T) \to 0$. Again, from the compactness of T, $\{\bar{x}_n\}$ has a sub-sequence $\{\bar{x}_{n_k}\}$ converging to $\bar{x} \in T$. Hence the corresponding sequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$, proving that $\mathrm{IQVI}_p(\tilde{h},K)$ is LP α-well-posed by perturbations in the generalised sense.

We now give the following example as an application of Theorem 3.

Example 2. Consider Example 1.

The solution set $T = \{0\}$ is compact and

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

$$T_\alpha^p(\varepsilon) = \begin{cases} [-\varepsilon, 0], & x \le 0 \\ (0, \varepsilon], & x > 0. \end{cases}$$

Thus, $e(T_\alpha^p(\varepsilon), T) \to 0$ as $\varepsilon \to 0$.

Hence the problem is LP α-well-posed by perturbations in the generalised sense.

Further, characterisation is given in terms of measure of non-compactness.

Theorem 4. *Let the assumptions be as in Theorem 2. Then* $\mathrm{IQVI}_p(\tilde{h}, K)$ *is LP α-well-posed by perturbations in the generalised sense if and only if* $T_\alpha^p(\varepsilon) \ne \varnothing \ \forall \ \varepsilon > 0$ *and* $\mu(T_\alpha^p(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

Proof: Let $\mathrm{IQVI}_p(\tilde{h}, K)$ be LP α-well-posed by perturbations in the generalised sense. Then $T_\alpha^p(\varepsilon) \ne \varnothing \ \forall \ \varepsilon > 0$.

By Theorem 3, T is non-empty and compact, and $e(T_\alpha^p(\varepsilon), T) \to 0$ as $\varepsilon \to 0$.

For any $\varepsilon > 0$, we have

$$\begin{aligned} \mathcal{H}(T_\alpha^p(\varepsilon), T) &= \max\{e(T_\alpha^p(\varepsilon), T), e(T, T_\alpha^p(\varepsilon))\} \\ &= e(T_\alpha^p(\varepsilon), T). \end{aligned}$$

As T is compact, $\mu(T) = 0$.

Since $\forall \ n \in \mathbb{N}$, the following relation holds [37]:

$$\begin{aligned} \mu(T_\alpha^p(\varepsilon)) &\le 2\mathcal{H}(T_\alpha^p(\varepsilon), T) + \mu(T) \\ &= 2e(T_\alpha^p(\varepsilon), T) \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Conversely, let $T_\alpha^p(\varepsilon) \ne \varnothing \ \forall \ \varepsilon > 0$ and $\mu(T_\alpha^P(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

Let $\{p_n\} \subset P$ be any sequence with $p_n \to p^*$, and let $\{x_n\}$ be LP α-approximating sequence for $\mathrm{IQVI}_p(\tilde{h}, K)$.

Then there exists a sequence $\{\varepsilon_n'\} > 0$ with $\varepsilon_n' \downarrow 0$ and sequence $w_n \in \mathbb{R}^n$ with $w_n \to 0$ such that $\tilde{h}(p_n, x_n) + w_n \in K(x_n)$ and

$$\left\langle x_n, \tilde{h}(p_n, x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n, x_n) - y \|^2 + \varepsilon_n' \quad \forall \ y \in K(x_n) \ \forall \ n \in \mathbb{N}.$$

As K is non-empty, there exists $\overline{x}_n \in K(x_n)$ such that $\tilde{h}(p_n, x_n) + w_n = \overline{x}_n$.

Now, $d(\tilde{h}(p_n, x_n), K(x_n)) \le \| \tilde{h}(p_n, x_n) - \overline{x}_n \| = \| w_n \| \to 0$.

Set $\varepsilon_n = \max\{\| p_n - p^* \|, \| w_n \|, \varepsilon_n'\}$. Thus, $x_n \in T_\alpha^p(\varepsilon_n)$.

We claim that $e(T_\alpha^p(\varepsilon_n), T) \to 0$ as $\varepsilon \to 0$.

As $T_\alpha^p(\varepsilon_n) \ne \varnothing$, $\mathrm{cl}(T_\alpha^p(\varepsilon_n))$ is non-empty and closed $\forall \ \varepsilon > 0$.

Thus, $\lim_{\varepsilon \to 0} \mu(\mathrm{cl}(T_\alpha^p(\varepsilon))) = \lim_{\varepsilon \to 0} \mu(T_\alpha^p(\varepsilon)) = 0$.

Hence $\mathcal{H}(\mathrm{cl}(T_\alpha^p(\varepsilon)), \bigcap_{\varepsilon > 0} \mathrm{cl}(T_\alpha^p(\varepsilon))) \to 0$ as $\varepsilon \to 0$.

We now show that $T = \bigcap_{\varepsilon > 0} \mathrm{cl}(T_\alpha^p(\varepsilon))$.

Obviously, $T \subset \bigcap_{\varepsilon > 0} \mathrm{cl}(T_\alpha^p(\varepsilon))$. For the reverse containment, let $x_0 \in \bigcap_{\varepsilon > 0} \mathrm{cl}(T_\alpha^p(\varepsilon))$. Then $d(x_0, T_\alpha^p(\varepsilon)) = 0 \ \forall \ \varepsilon > 0$.

Thus, given $\varepsilon_n > 0, \varepsilon_n \to 0 \ \forall \ n$, there exists $x_n \in T_\alpha^p(\varepsilon_n)$ such that $d(x_0, x_n) < \varepsilon_n$.

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

Hence $x_n \to x_0$ and $d(\tilde{h}(p_n, x_n), K(x_n)) \le \varepsilon_n$, and

$$\left\langle x_n, \tilde{h}(p_n, x_n) - y \right\rangle \le \frac{\alpha}{2} \| \tilde{h}(p_n, x_n) - y \|^2 + \varepsilon_n \quad \forall\ y \in K(x_n)\ \forall\ n \in \mathbb{N}.$$

By using the fact that $x_n \to x_0$, $\tilde{h}(\cdot,\cdot)$ is continuous and by condition (i) in the hypothesis, we obtain that $\tilde{h}(p^*, x) \in K(x)$.

Again by Theorem 2, we get $x_0 \in T$. Hence $\mathcal{H}(\text{cl}(T_\alpha^p(\varepsilon)), T) \to 0$ as $\varepsilon \to 0$ or $e(T_\alpha^p(\varepsilon), T) \to 0$ as $\varepsilon \to 0$.

Thus, we have $d(x_n, T) \le e(T_\alpha^p(\varepsilon), T) \to 0$ as $\varepsilon \to 0$.

T being compact, there exists $\bar{x}_n \in T$ such that

$$\| x_n - \bar{x}_n \| = d(x_n, T) \to 0.$$

Again, from compactness of T, $\{\bar{x}_n\}$ has a sub-sequence $\{\bar{x}_{n_k}\}$ converging to $\bar{x} \in T$. Hence the corresponding sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Thus, $\text{IQVI}_p(\tilde{h}, K)$ is LP α-well-posed by perturbations in the generalised sense.

LP WELL-POSEDNESS AND UNIQUENESS

In this section we show that under suitable conditions, LP well-posedness by perturbations of an inverse quasi-variational inequality is equivalent to the existence and uniqueness of its solution.

Theorem 5. *Let K be a non-empty, convex-valued and set-valued map. Let $\tilde{h} : P \times \mathbb{R}^n \to \mathbb{R}^n$ be such that*

(i) $\tilde{h}(p,\cdot)$ is monotone $\forall\ p \in P$ and $\tilde{h}(p^*,\cdot)$ is hemicontinuous.

(ii) $\tilde{h}(\cdot, x)$ is continuous $\forall\ x \in \mathbb{R}^n$.

Then $\text{IQVI}_p(\tilde{h}, K)$ is LP well-posed by perturbations if and only if it has a unique solution.

Proof: The necessary condition holds true. For the sufficient condition, let $\text{IQVI}_p(\tilde{h}, K)$ has a unique solution x^*. Then $\tilde{h}(p, x^*) \in K(x^*)$ and

$$\left\langle x^*, y - \tilde{h}(p, x^*) \right\rangle \ge 0 \quad \forall\ y \in K(x^*). \tag{9}$$

Let $\{p_n\} \subset P$ be any sequence with $p_n \to p^*$ and $\{x_n\} \subset \mathbb{R}^n$ be LP approximating sequence corresponding to $\{p_n\}$ for $\text{IQVI}_p(\tilde{h}, K)$.

Then there exists a sequence $\{\varepsilon_n\} > 0$ with $\varepsilon_n \to 0$ and $w_n \in \mathbb{R}^n$ with $w_n \to 0$ such that $\tilde{h}(p_n, x_n) + w_n \in K(x_n)$ and

$$\left\langle x_n, \tilde{h}(p_n, x_n) - y \right\rangle \le \varepsilon_n \quad \forall\ y \in K(x_n)\ \forall\ n \in \mathbb{N}. \tag{10}$$

Take $u^* = (x^*, h(x^*))$ and $u_n = (x_n, \tilde{h}(p_n, x_n) + w_n)\ \ \forall\ \ n \in \mathbb{N}$.

If $\{u_n\}$ is unbounded, without loss of generality, we can suppose that $\| u_n \| \to +\infty$.

Set $t_n := \frac{1}{\|u_n - u^*\|}$ and

$$\begin{aligned} w_n &:= (z_n, g_n) = u^* + t_n(u_n - u^*) \\ &= \left(x^* + t_n(x_n - x^*), h(x^*) + t_n[\tilde{h}(p_n, x_n) + w_n - h(x^*)] \right). \end{aligned}$$

Without loss of generality, suppose $t_n \in (0,1]$ and $w_n \to w = (z, g) \in \mathbb{R}^n \times K(x^*)$.

Then $w = (z, g) \ne u^*$. For any $y \in K(x^*)$ and $x \in \mathbb{R}^n$, it follows that

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

$$\left\langle \tilde{h}(p_n,x)-y,z-x\right\rangle+\langle g-y,x\rangle$$
$$=\left\langle \tilde{h}(p_n,x)-y,z-z_n\right\rangle+\left\langle \tilde{h}(p_n,x)-y,z_n-x^*\right\rangle+\left\langle \tilde{h}(p_n,x)-y,x^*-x\right\rangle$$
$$+\langle g-g_n,x\rangle+\left\langle g_n-h(x^*),x\right\rangle+\left\langle h(x^*)-y,x\right\rangle$$
$$=\{\left\langle \tilde{h}(p_n,x)-y,z-z_n\right\rangle+\langle g-g_n,x\rangle\}+t_n\{\left\langle \tilde{h}(p_n,x)-y,x_n-x\right\rangle+\left\langle \tilde{h}(p_n,x_n)+w_n-y,x\right\rangle\}$$
$$+(1-t_n)\{\left\langle \tilde{h}(p_n,x)-y,x^*-x\right\rangle+\left\langle h(x^*)-y,x\right\rangle \tag{11}$$

As $\tilde{h}(p,\cdot)$ is monotone, it follows from (10) and (11) that

$$\left\langle \tilde{h}(p_n,x)-y,z-x\right\rangle+\langle g-y,x\rangle$$
$$\le\left\langle \tilde{h}(p_n,x)-y,z-z_n\right\rangle+\langle g-g_n,x\rangle+t_n\{\left\langle \tilde{h}(p_n,x_n)-y,x_n-x\right\rangle+\left\langle \tilde{h}(p_n,x_n)+w_n-y,x\right\rangle\}$$
$$+(1-t_n)\{\left\langle \tilde{h}(p_n,x)-y,x^*-x\right\rangle+\left\langle h(x^*)-y,x\right\rangle\}$$
$$=\{\left\langle \tilde{h}(p_n,x)-y,z-z_n\right\rangle+\langle g-g_n,x\rangle\}+t_n\{\left\langle \tilde{h}(p_n,x_n)-y,x_n\right\rangle+\langle w_n,x\rangle\}$$
$$+(1-t_n)\{\left\langle \tilde{h}(p_n,x)-y,x^*-x\right\rangle+\left\langle \tilde{h}(x^*)-y,x\right\rangle\}$$
$$\le\{\left\langle \tilde{h}(p_n,x)-y,z-z_n\right\rangle+\langle g-g_n,x\rangle\}+(1-t_n)\{\left\langle \tilde{h}(p_n,x)-y,x^*-x\right\rangle+\left\langle h(x^*)-y,x\right\rangle\}$$
$$+t_n\varepsilon_n+t_n\langle w_n,x\rangle \quad \forall\ x\in\mathbb{R}^n \quad \forall\ y\in K(x^*).$$

Let $n\to\infty$; we get that

$$\langle h(x)-y,z-x\rangle+\langle g-y,x\rangle=\left\langle \tilde{h}(p^*,x)-y,z-x\right\rangle+\langle g-y,x\rangle$$
$$\le\left\langle h(x)-y,x^*-x\right\rangle+\left\langle h(x^*)-y,x\right\rangle$$
$$\le\left\langle h(x^*)-y,x^*-x\right\rangle+\left\langle h(x^*)-y,x\right\rangle$$
$$=\left\langle h(x^*)-y,x^*\right\rangle \quad \forall\ x\in\mathbb{R}^n \quad \forall\ y\in K(x^*).$$

This together with (9) implies that

$$\langle h(x)-y,z-x\rangle+\langle g-y,x\rangle\le 0 \quad \forall\ x\in\mathbb{R}^n \quad \forall\ y\in K(x^*). \tag{12}$$

For any $x'\in\mathbb{R}^n$ and $y'\in K(x^*)$, define $z(t)=z+t(x'-z)$ and

$$g(t)=g+t(y'-g) \quad \forall\ t\in[0,1].$$

Thus, (12) implies $\langle h(z(t))-g(t),z-z(t)\rangle+\langle g-g(t),z(t)\rangle\le 0$, i.e.

$$\langle h(z(t))-g(t),z-x'\rangle+\langle g-y',z(t)\rangle\le 0.$$

Let $t\to 0^+$, we get that

$$\langle h(z)-g,z-x'\rangle+\langle g-y',z\rangle\le 0 \quad \forall x'\in\mathbb{R}^n \quad \forall y'\in K(x^*). \tag{13}$$

Letting $y'=g$ in (13) and $x'=z-\lambda r$, we get that $\lambda\langle h(z)-g,r\rangle\le 0$, where λ is arbitrary and $r\in\mathbb{R}^n$ is also arbitrary. Thus,

$$h(z)=g\in K(x^*). \tag{14}$$

From (13) and (14), we get that $\langle h(z)-y',z\rangle\le 0 \ \forall\ y'\in K(x^*)$, and so z solves IQVI$(\tilde{h},K)$. Thus, $z=x^*$, which contradicts $(z,h(z))\ne(x^*,h(x^*))$. So we can suppose that $\{u_n\}$ is bounded. Let $\{u_{n_k}\}$ be any sub-sequence of $\{u_n\}$ such that $u_{n_k}\to(\bar{x},\bar{g})$ as $k\to\infty$. It is sufficient to show that $\bar{x}$ is a solution of IQVI$(\tilde{h},K)$.

It follows from (10) that $\tilde{h}(p_{n_k},x_{n_k})+w_{n_k}\in K(x_{n_k})$ and

$$\langle x_{n_k},\tilde{h}(p_{n_k},x_{n_k})-y_{n_k}\rangle\le\varepsilon_{n_k} \quad \forall y\in K(x_{n_k}) \ \forall k\in\mathbb{N}. \tag{15}$$

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

Let $k \to \infty$ in (15); we get that

$$\tilde{h}(p,\bar{x}) \in K(\bar{x})$$

and

$$\langle \bar{x}, \tilde{h}(p,\bar{x}) - y \rangle \leq 0 \quad \forall\, y \in K(\bar{x}) .$$

Thus, $\bar{x}$ is a solution of $\mathrm{IQVI}(\tilde{h}, K)$.

Remark 1. Theorem 5 generalises Theorem 4.1 of Hu and Fang [36], in which the equivalence of LP well-posedness and uniqueness of solution for the inverse variational inequality is derived.

CONCLUSIONS

We have established the Levitin-Polyak well-posedness for an inverse quasi-variational inequality and some metric characterisations have been given.

ACKNOWLEDGEMENTS

This research was funded by the Council of Scientific and Industrial Research, India (file number 09/045(1036)/2010-EMR-1). We are grateful to the anonymous referees and the editor for improving the paper.

REFERENCES

1. A. N. Tykhonov, "On the stability of the functional optimization problem", *USSR Comput. Math. Math. Phys*., **1966**, *6*, 28-33.
2. E. S Levitin and B. T. Poljak, "Convergence of minimizing sequences in conditional extremum problems", *Soviet Math. Dokl.*, **1966**, *7*, 764-767.
3. I. D. Prete, M. B. Lignola and J. Morgan, "New concepts of well-posedness for optimization problems with variational inequality constraints", *J. Inequal. Pure Appl. Math.*, **2003**, *4*, Article 5.
4. A. L. Dontchev and T. Zolezzi, "Well-posed Optimization Problems", Springer, Berlin, **1993**.
5. X. X. Huang and X. Q. Yang, "Generalized Levitin-Polyak well-posedness in constrained optimization", *SIAM J. Optim.*, **2006**, *17*, 243-258.
6. X. X. Huang and X. Q. Yang, "Levitin-Polyak well-posedness of constrained vector optimization problems", *J. Global Optim.*, **2007**, *37*, 287-304.
7. T. Zolezzi, "Well-posedness criteria in optimization with applications to the calculus of variations", *Nonlinear Anal. Theory Meth. Appl.*, **1995**, *25*, 437-453.
8. T. Zolezzi, "Extended well-posedness of optimization problems", *J. Optim. Theory Appl.*, **1996**, *91*, 257-266.
9. Y. P. Fang, N. J. Huang and J. C. Yao, "Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems", *J. Global Optim.*, **2008**, *41*, 117-133.
10. Y. P. Fang, N. J. Huang and J. C. Yao, "Well-posedness by perturbations of mixed variational inequalities in Banach spaces", *Euro. J. Oper. Res.*, **2010**, *201*, 682-692.
11. R. Hu and Y. P. Fang, "Well-posedness of inverse variational inequalities", *J. Convex Anal.*, **2008**, *15*, 427-437.
12. R. Hu and Y. P. Fang, "Levitin-Polyak well-posedness of variational inequalities", *Nonlinear Anal. Theory Meth. Appl.*, **2010**, *72*, 373-381.
13. X. X. Huang and X. Q. Yang, "Levitin-Polyak well-posedness in generalized variational inequality problems with functional constraints", *J. Indus. Manag. Optim.*, **2007,** *3*, 671-684.

14. X. X. Huang, X. Q. Yang and D. L. Zhu, "Levitin-Polyak well-posedness of variational inequality problems with functional constraints", *J. Global Optim.*, **2009**, *44*, 159-174.
15. M. B. Lignola and J. Morgan, "Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution", *J. Global Optim.*, **2000**, *16*, 57-67.
16. M. B. Lignola, "Well-posedness and L-well-posedness for quasivariational inequalities", *J. Optim. Theory Appl.*, **2006**, *28*, 119-138.
17. R. Lucchetti and F. Patrone, "A characterization of Tykhonov well-posedness for minimum problems, with applications to variational inequalities", *Numer. Funct. Anal. Optim.*, **1981**, *3*, 461-476.
18. X. X. Huang and X. Q. Yang, "Levitin-Polyak well-posedness of vector variational inequality problems with functional constraints", *Numer. Funct. Anal. Optim.*, **2010**, *31*, 440-459.
19. X. B. Li and F. Q. Xia, "Levitin-Polyak well-posedness of generalized mixed variational inequality in Banach spaces", *Nonlinear Anal. Theory Meth. Appl.*, **2012**, *75*, 2139-2153.
20. J. W. Peng and J. Tang, "α-Well-posedness for quasivariational inequality problems", *Abstr. Appl. Anal.*, **2012**, *2012*, Art. ID 157532 (13 pages).
21. R. Hu, Y. P. Fang and N. J. Huang, "Characterizations of α-well-posedness for parametric quasivariational inequalities defined by bifunctions", *Math. Commun.*, **2010**, *15*, 37-55.
22. C. S. Lalitha and G. Bhatia, "Well-posedness for parametric quasivariational inequality problems and for optimization problems with quasivariational inequality constraints", *Optimization*, **2010**, *59*, 997-1011.
23. B. Jaing, J. Zhang and X. X. Huang, "Levitin-Polyak well-posedness of generalized quasivariational inequalities with functional constraints", *Nonlinear Anal. Theory Meth. Appl.*, **2009**, *70*, 1492-1503.
24. C. S. Lalitha and G. Bhatia, "Levitin-polyak well-posedness for parametric quasivariational inequality problem of the minty type", *Positivity*, **2012**, *16*, 527-541.
25. A. Jayswal, S. Choudhury and R. U. Verma, "Exponential type vector variational-like inequalities and vector optimization problems with exponential type invexities", *J. Appl. Math. Comput.*, **2014**, *45*, 87-97.
26. Q. H. Ansari, M. Rezaie and J. Zafarani, "Generalized vector variational-like inequalities and vector optimization", *J. Glob. Optim.*, **2012**, *53*, 271-284.
27. S. K. Mishra, S. Wang and K. K. Lai, "Role of α-pseudounivex functions in vector variational-like inequality problems", *J. Syst. Sci Complexity*, **2007**, *20*, 344-349.
28. R. Ferrentino, "Variational inequalities and optimization problems", *Appl. Math. Sci.*, **2007**, *47*, 2327-2343.
29. R. Hu, Y. P. Fang and N. J. Huang, "Levitin-Polyak well-posedness for variational inequalities and for optimization problems with variational inequality constraints", *J. Indus. Manag. Optim.*, **2010**, *6*, 465-481.
30. Q. M. Han and B. S. He, "A predict-correct projection method for monotone variant variational inequalities", *Chin. Sci. Bull.*, **1998**, *43*, 1264-1267.
31. B. S. He, "A Goldstein's type projection method for a class of variant variational inequalities", *J. Comput. Math.*, **1999**, *17*, 425-434.
32. B. S. He, "Inexact implicit methods for monotone general variational inequalities", *Math. Program.*, **1999**, *86*, 199-217.

Maejo Int. J. Sci. Technol. **2014**, *8*(03), 264-278

33. B. S. He, X. Z. He and H. X. Liu, "Solving a class of constrained black-box inverse variational inequalities", *Euro. J. Oper. Res.*, **2010,** *204*, 391-401.
34. X. He and H. X. Liu, "Inverse variational inequalities with projection-based solution methods", *Euro. J. Oper. Res.*, **2011**, *208*, 12-18.
35. D. Aussel, R. Gupta and A. Mehra, "Gap functions and error bounds for inverse quasivariational inequality problems", *J. Math. Anal. Appl.*, **2013**, *407*, 270-280.
36. R. Hu and Y. P. Fang, "Levitin-Polyak well-posedness by perturbations of inverse variational inequalities", *Optim. Lett.*, **2013**, *7*, 343-359.
37. K. Kuratowski, "Topology", Vol. 1-2, Academic Press, New York, **1968**.
38. D. Kinderlehrer and G. Stampacchia, "An Introduction to Variational Inequalities and Their Applications", Academic Publishers, New York, **1980**.