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Full Paper

On r-duals of some difference sequence spaces

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Abstract: In this paper we introduce and examine some properties of the sequence spaces $C(\Delta_v^m,\lambda,p)$, $C[\Delta_v^m,\lambda,p]$, $C_\infty(\Delta_v^m,\lambda,p)$, $C_\infty[\Delta_v^m,\lambda,p]$ and $V(\Delta_v^m,\lambda,p)$, and compute the *rα-, rβ-* and *rγ*-duals of the sequence spaces $\ell_\infty(v)$, *c(v)* and $c_0(v)$, and the *rα*- and *rN*-duals of the sequence spaces $C_\infty(\Delta_v^m)$ and $C_\infty[\Delta_v^m]$.

Keywords: Cesàro sequence spaces, difference sequence, dual space

INTRODUCTION

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|=\sup|x_k|$, where $k\in\mathbb{N}=\{1,2,\ldots\}$, the set of positive integers. Also, by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series respectively.

Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\le\lambda_n+1$, $\lambda_1=1$. The generalised de la Vallée-Poussin mean is defined by $t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k$, where $I_n=[n-\lambda_n+1,n]$ for $n=1,2,\ldots$. A sequence $x=(x_k)$ is said to be (V,λ)-summable to a number L if $t_n(x)\to L$ as $n\to\infty$ [1]. If $\lambda_n=n$, then (V,λ)-summability and strongly (V,λ)-summability are reduced to $(C,1)$-summability and $[C,1]$-summability respectively.

The notion of difference sequence spaces was introduced by Kızmaz [2] and it was generalised by Et and Çolak [3]. Recently, the difference spaces bv_p consisting of the sequences $x=(x_k)$ such that $(x_k-x_{k-1})\in\ell_p$ have been studied in the case of $0<p<1$ by Altay and Başar [4], and in the case of $1\le p<\infty$ by Başar and Altay [5], Çolak *et al.* [6] and Başar [7]. Since then

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Et and Esi [8] generalised these sequence spaces to the following sequence spaces. Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers and m be a non-negative integer. Then,

$$\Delta_v^m(X) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $m \in \mathrm{N}$, $\Delta_v^0 x = (v_k x_k)$ and $\Delta_v^m x = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$, and so $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$.

The sequence spaces $\Delta_v^m(X)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^m |v_i x_i| + \|\Delta_v^m x_k\|_\infty$$

for $X = \ell_\infty$, c or c_0. Recently the difference sequence spaces have been studied by different research workers [9-29]. The Cesàro sequence spaces Ces_p and Ces_∞ were introduced by Shiue [30], and Jagers [31] determined the Köthe duals of the sequence space Ces_p $(1 < p < \infty)$. It can be shown that the inclusion $\ell_p \subset Ces_p$ is strict for $1 < p < \infty$. Later on the Cesàro sequence spaces X_p and X_∞ of non-absolute type were defined by Ng and Lee [32, 33].

Let X be a sequence space. Then X is called:

i) *Solid* (or *normal*) if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \le 1$ for all $k \in \mathrm{N}$, whenever $(x_k) \in X$;

ii) *Symmetric* if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where π is a permutation of N;

iii) *Sequence algebra* if $x.y \in X$, whenever $x, y \in X$.

The determination of the dual spaces is important in the theory of sequence spaces. The concepts of α-, β- and γ-duality are well known and the topology of the sequence spaces can be defined by duality. The idea of α-, β- and γ- duality was generalised by Et [34] to $r\alpha$-, $r\beta$- and $r\gamma$-duality $(r \ge 1)$. The main purpose of this paper is to introduce the $r\alpha$-, $r\beta$-, $r\gamma$- and rN-duals of some sequence spaces.

MAIN RESULTS

In this section we prove some results involving the sequence spaces $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$.

Definition 1. Let $m \ge 1$ and $1 \le p < \infty$. We define the following sequence spaces:

$$C\left(\Delta_v^m, \lambda, p\right) = \left\{ x = (x_k) \ : \ \sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k \in I_n} \Delta_v^m x_k \right|^p < \infty \right\},$$

$$C\left[\Delta_v^m, \lambda, p\right] = \left\{ x = (x_k) \ : \ \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} \left|\Delta_v^m x_k\right| \right)^p < \infty \right\},$$

$$C_\infty(\Delta_v^m, \lambda, p) = \left\{ x = (x_k) \ : \ \sup_n \left| \frac{1}{\lambda_n} \sum_{k \in I_n} \Delta_v^m x_k \right|^p < \infty \right\},$$

$$C_\infty\left[\Delta_v^m, \lambda, p\right] = \left\{ x = (x_k) \ : \ \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left|\Delta_v^m x_k\right|^p < \infty \right\},$$

$$V\left[\Delta_v^m, \lambda, p\right] = \left\{ x = (x_k) \ : \ \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left|\Delta_v^m x_k - \ell\right|^p = 0 \right\}.$$

We get the following sequence spaces from the above sequence spaces, giving particular values to λ, p, v, ℓ and m.

i) For $p = 1$, we write $C(\Delta_v^m, \lambda)$, $C[\Delta_v^m, \lambda]$, $C_\infty(\Delta_v^m, \lambda)$, $C_\infty[\Delta_v^m, \lambda]$ and $V[\Delta_v^m, \lambda]$ instead of $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$ respectively.

ii) For $\lambda_n = n$ for all $n \in \mathbb{N}$ and $p = 1$, we write $C(\Delta_v^m)$, $C[\Delta_v^m]$, $C_\infty(\Delta_v^m)$, $C_\infty[\Delta_v^m]$ and $V[\Delta_v^m]$ instead of $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$ respectively. If $x \in V[\Delta_v^m, \lambda, p]$, we say that x is Δ_v^m-strongly λ_p-summable to ℓ.

iii) In the case of $v = (1,1,1,...)$, we write $C(\Delta^m, \lambda, p)$, $C[\Delta^m, \lambda, p]$, $C_\infty(\Delta^m, \lambda, p)$, $C_\infty[\Delta^m, \lambda, p]$ and $V[\Delta^m, \lambda, p]$ instead of $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$ respectively.

iv) In the special case of $p = 1$, $\lambda_n = n$ for all $n \in \mathbb{N}$ and $\ell = 0$, we write $V_0[\Delta_v^m]$ instead of $V[\Delta_v^m, \lambda, p]$.

v) Also in the special case of $p = 1$, $v = (1,1,1,...)$ and $m = 0$, we write $C(\lambda)$, $C[\lambda]$, $C_\infty(\lambda)$, $C_\infty[\lambda]$ and $V[\lambda]$ instead of $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$ respectively.

Let X denote one of the sequence $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$, and let Y denote one of the sequence $C(\Delta^m, \lambda, p)$, $C[\Delta^m, \lambda, p]$, $C_\infty(\Delta^m, \lambda, p)$, $C_\infty[\Delta^m, \lambda, p]$ and $V[\Delta^m, \lambda, p]$. We note that the sequence space X is different from the sequence space Y and $X \cap Y \neq \phi$. For this, let $x = (k^m)$ and $v = (k)$; then $x \in C_\infty[\Delta^m, \lambda, p]$, but $x \notin C_\infty[\Delta_v^m, \lambda, p]$. Conversely, if we choose $x = (k^{m+1})$ and $v = (k^{-1})$, then $x \in C_\infty[\Delta_v^m, \lambda, p]$, but $x \notin C_\infty[\Delta^m, \lambda, p]$

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The above sequence spaces contain some unbounded sequences for $m \geq 1$. For example, the sequence $x = (k^m)$ is an element of $C_\infty[\Delta_v^m, \lambda, p]$ but is not an element of ℓ_∞.

The proof of the following two theorems can be established by using the known standard techniques; therefore we give them without proof.

Theorem 1. Let $m \geq 1$ and $1 \leq p < \infty$; then the sets of sequences $C(\Delta_v^m, \lambda, p)$, $C[\Delta_v^m, \lambda, p]$, $C_\infty(\Delta_v^m, \lambda, p)$, $C_\infty[\Delta_v^m, \lambda, p)]$ and $V[\Delta_v^m, \lambda, p]$ are linear spaces with the coordinate-wise additition and scalar multiplication of sequences.

Theorem 2. Let $m \geq 1$ and $1 \leq p < \infty$; then the following inclusions are strict.

i) $C(\Delta_v^{m-1}, \lambda, p) \subset C(\Delta_v^m, \lambda, p)$,

ii) $C[\Delta_v^{m-1}, \lambda, p] \subset C[\Delta_v^m, \lambda, p]$,

iii) $C[\Delta_v^m, \lambda, p] \subset C(\Delta_v^m, \lambda, p)$,

iv) $C(\Delta_v^m, \lambda, p) \subset C(\Delta_v^m, \lambda, q) \quad (0 < p < q)$,

v) $C_\infty(\Delta_v^{m-1}, \lambda, p) \subset C_\infty(\Delta_v^m, \lambda, p)$,

vi) $C_\infty[\Delta_v^{m-1}, \lambda, p] \subset C_\infty(\Delta_v^m, \lambda, p)$,

vii) $C_\infty[\Delta_v^m, \lambda, p] \subset C_\infty(\Delta_v^m, \lambda, p)$,

viii) $V[\Delta_v^{m-1}, \lambda, p] \subset V[\Delta_v^m, \lambda, p]$,

ix) $V[\Delta_v^m, \lambda, p] \subset C_\infty[\Delta_v^m, \lambda, p]$.

Note that $C(\Delta_v^m, \lambda, p)$ and $c(\Delta_v^m)$ overlap, but neither one contains the other. Actually the sequence $x = (k^m)$ is an element of $c(\Delta_v^m)$ but not an element of $C(\Delta_v^m, \lambda, p)$, and $x = ((-1)^k)$ belongs to $C(\Delta_v^m, \lambda, p)$ but not to $c(\Delta_v^m)$, where $c(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in c\}$.

Theorem 3. The sequence space $C[\Delta_v^m, \lambda, p]$ is a Banach-Coordinate- or *BK*-space normed by

$$\|x\|_1 = \sum_{i=1}^{m} |v_i x_i| + \left(\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta_v^m x_k| \right)^p \right)^{\frac{1}{p}}, \ (1 \leq p < \infty) . \qquad (1)$$

$C_\infty[\Delta_v^m, \lambda, p]$ and $V[\Delta_v^m, \lambda, p]$ are *BK*-spaces normed by

$$\|x\|_2 = \sum_{i=1}^{m} |v_i x_i| + \sup_n \left(\frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta_v^m x_k|^p \right)^{\frac{1}{p}}, \ (1 \leq p < \infty) \qquad (2)$$

Proof. We give the sketch of proof for $C_\infty[\Delta_v^m, \lambda, p]$. The others can be proved in the same way. Let (x^s) be a Cauchy sequence in $C_\infty[\Delta_v^m, \lambda, p]$, where $x^s = (x_i^s)_{i=1}^\infty$. Then there exists a positive integer n_0 such that $\|x^s - x^t\|_2 < \varepsilon$ for all $s, t > n_0$.

Hence (x_i^s) (for $i \leq m$) and $(\Delta_v^m(x_k^s))$ for all $k \in \mathbb{N}$ are Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, these sequences are convergent in $\mathbb{C}$. Suppose that $x_i^s \to x_i$ (for $i \leq m$) and $\Delta_v^m(x_k^s) \to y_k$ for each $k \in \mathbb{N}$ as $s \to \infty$. Then we can find a sequence (x_k) such that $y_k = \Delta_v^m x_k$ for each $k \in \mathbb{N}$. These x_k's can be written as

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$$x_k = v_k^{-1}\sum_{i=1}^{k-m}(-1)^m\binom{k-i-1}{m-1}y_i = v_k^{-1}\sum_{i=1}^{k}(-1)^m\binom{k+m-i-1}{m-1}y_{i-m},$$

for sufficiently large k, for instance $k>m$, where $y_{1-m}=y_{2-m}=\ldots=y_0=0.$

Thus, $\left(\Delta_v^m\left(x_k^s\right)\right)=\left(\left(\Delta_v^m\left(x_k^1\right)\right),\left(\Delta_v^m\left(x_k^2\right)\right),\ldots\right)$ converges to $\Delta_v^m x_k$ for each $k\in\mathbb{N}$ in $\mathbb{C}$. Hence $\left\|x^s-x\right\|_2\to 0$ as $s\to\infty$. Since $\left(x^s-x\right)$, $\left(x^s\right)\in C_\infty\left[\Delta_v^m,\lambda,p\right]$ and the space $C_\infty\left[\Delta_v^m,\lambda,p\right]$ are linear, we have $x=x^s-\left(x^s-x\right)\in C_\infty\left[\Delta_v^m,\lambda,p\right]$. Hence $C_\infty\left[\Delta_v^m,\lambda,p\right]$ is complete. Since $C_\infty\left[\Delta_v^m,\lambda,p\right]$ is a Banach space with continuous coordinates, that is, $\left\|x^n-x\right\|_2\to 0$ implies $\left|x_k^n-x_k\right|\to 0$ for each $k\in\mathbb{N}$ as $n\to\infty$, it is *BK*-space.

In the same way it can be shown that $C\left[\Delta_v^m,\lambda,p\right]$ is a *BK*-space normed by (1) and $V\left[\Delta_v^m,\lambda,p\right]$ is a *BK*-space normed by (2).

Theorem 4. The sequence space $C\left(\Delta_v^m,\lambda,p\right)$ is a *BK*-space normed by

$$\|x\|_3=\sum_{i=1}^{m}\left|v_i x_i\right|+\left(\sum_{n=1}^{\infty}\left|\frac{1}{\lambda_n}\sum_{k\in I_n}\Delta_v^m x_k\right|^p\right)^{\frac{1}{p}},\left(1\le p<\infty\right)$$

and the space $C_\infty(\Delta_v^m,\lambda)$ is a *BK*-space normed by

$$\|x\|_4=\sum_{i=1}^{m}\left|v_i x_i\right|+\sup_n\left(\left|\frac{1}{\lambda_n}\sum_{k\in I_n}\Delta_v^m x_k\right|\right).$$

Proof. The proof is similar to that of Theorem 3.

Theorem 5. The sequence spaces $C(\lambda)$, $C[\lambda]$, $C_\infty(\lambda)$ and $C_\infty[\lambda]$ are solid and hence monotone, but the sequence spaces $C\left(\Delta_v^m,\lambda,p\right)$, $C\left[\Delta_v^m,\lambda,p\right]$, $C_\infty(\Delta_v^m,\lambda,p)$, $C_\infty\left[\Delta_v^m,\lambda,p\right]$ and $V\left[\Delta_v^m,\lambda,p\right]$ are neither solid nor symmetric, nor sequence algebras for $m\ge 1$.

Proof. Let $x=\left(x_k\right)\in C_\infty[\lambda]$ and $y=\left(y_k\right)$ be sequences such that $\left|x_k\right|\le\left|y_k\right|$ for each $k\in\mathbb{N}$. Then we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left|x_k\right|\le\frac{1}{\lambda_n}\sum_{k\in I_n}\left|y_k\right|.$$

Hence $C_\infty[\lambda]$ is solid and hence monotone. Let $p=1$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $\left(x_k\right)=(k^{m-1})\in C_\infty\left[\Delta_v^m,\lambda,p\right]$ but $\left(\alpha_k x_k\right)\notin C_\infty\left[\Delta_v^m,\lambda,p\right]$ when $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$. Hence $C_\infty\left[\Delta_v^m,\lambda,p\right]$ is not solid. The other cases can be proved on considering similar examples.

DUAL SPACES

The definitions of the *rα*-, *rβ*-, *rγ*- and *rN*-duals of a sequence space were introduced by Et [34]. Since then the *rα*-duals of some sequence spaces were studied by Bektas *et al.* [35], Chandra and Tripathy [36], and Tripathy and Sarma [37]. In this section we compute the *rα*-, *rβ*- and *rγ*-duals of the sequence spaces $\ell_\infty(v), c(v), c_0(v)$, the *rN*-duals of the sequence spaces $C_\infty(\Delta_v^m)$, $C_\infty\left[\Delta_v^m\right]$ and $V_0\left[\Delta_v^m\right]$, and the *rα*-duals of the sequence spaces $C_\infty(\Delta_v^m)$ and $C_\infty\left[\Delta_v^m\right]$.

Definition 2 [34]. Let X be any sequence space with $1\le r<\infty$, and define

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$$X^{r\alpha} = \left\{ a = (a_k) \; : \; \sum_k |a_k x_k|^r < \infty, \text{ for each } x \in X \right\},$$

$$X^{r\beta} = \left\{ a = (a_k) \; : \; \sum_k (a_k x_k)^r \text{ is convergent, for each } x \in X \right\},$$

$$X^{r\gamma} = \left\{ a = (a_k) \; : \; \sup_n \left| \sum_{k=0}^{n} (a_k x_k)^r \right| < \infty, \text{ for each } x \in X \right\},$$

$$X^{rN} = \left\{ a = (a_k) \; : \; \lim_k (a_k x_k)^r = \lim_k a_k x_k = 0, \text{ for each } x \in X \right\} = X^N.$$

Then $X^{r\alpha}, X^{r\beta}, X^{r\gamma}$ and X^{rN} are called $r\alpha$-, $r\beta$-, $r\gamma$- and rN-duals of X respectively. It can be shown that $X^{r\alpha} \subset X^{r\beta} \subset X^{r\gamma}$ and if $X \subset Y$, then $Y^{r\eta} \subset X^{r\eta}$ for $\eta \in \{\alpha, \beta, \gamma, N\}$. If we take $r = 1$ in this definition, then we obtain the α-, β- and γ-duals of X. If $X = (X^{r\alpha})^{r\alpha}$, then X is called $r\alpha$- perfect.

Lemma 1. $x \in C_\infty(\Delta_v^m)$ implies $\sup_n \left(n^{-1} |\Delta_v^{m-1} x_n| \right) < \infty.$

Proof. Firstly, we have

$$\frac{1}{n} \sum_{k=1}^{n} \Delta_v^m x_k = \frac{1}{n} \left(\Delta_v^{m-1} x_1 - \Delta_v^{m-1} x_{n+1} \right).$$

If $x \in C_\infty(\Delta_v^m)$, then we have

$$\frac{1}{n+1} \left| \Delta_v^{m-1} x_{n+1} \right| \le \frac{1}{n} \left| \Delta_v^{m-1} x_{n+1} \right| \le \left| \frac{1}{n} \sum_{k=1}^{n} \Delta_v^m x_k \right| + \left| \Delta_v^{m-1} x_1 \right|$$

and this implies that $\sup_n \left(n^{-1} |\Delta_v^{m-1} x_n| \right) < \infty.$

Lemma 2. $\text{Sup}_n \left(n^{-1} |\Delta_v^{m-1} x_n| \right) < \infty$ implies $\sup_n \left(n^{-m} |v_n x_n| \right) < \infty$.

Proof. Omitted.

Lemma 3. $x \in C_\infty(\Delta_v^m)$ implies $\sup_n \left(n^{-m} |v_n x_n| \right) < \infty$.

Proof. Proof follows from Lemma 1 and Lemma 2.

Lemma 4 [35]. Let m be a positive integer. Then

$$[\ell_\infty(\Delta_v^m)]^N = [c(\Delta_v^m)]^N = \{a = (a_n) : v_n^{-1} n^m a_n \to 0, \; n \to \infty\}$$

and

$$[c_0(\Delta_v^m)]^N = \{a = (a_n) : \sup_n |\textstyle\sum_{k=0}^{n} \binom{n+m-k-1}{m-1} v_n^{-1} a_n | < \infty\},$$

where

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\} \quad \text{for } X = \ell_\infty, c \text{ and } c_0.$$

Theorem 6. Let $m \ge 1$ and $1 \le r < \infty$. Then

(*i*) $[C_\infty(\Delta_v^m)]^{r\alpha} = U_1^{(r)}$,

(*ii*) $\left[U_1^{(r)}\right]^{r\alpha} = U_2^{(r)}$.

where

$$U_1^{(r)} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} k^{rm} \left| v_k^{-1} a_k \right|^r < \infty \right\},$$

$$U_2^{(r)} = \left\{ a = (a_k) : \sup_k k^{-rm} \left| v_k a_k \right|^r < \infty \right\}.$$

Proof. (i) Let $a \in U_1^{(r)}$; then

$$\sum_{k=1}^{\infty} | a_k x_k |^r = \sum_{k=1}^{\infty} k^{rm} | v_k^{-1} a_k |^r \ k^{-rm} | v_k x_k |^r \quad \leq \sup_k k^{-rm} | v_k x_k |^r \sum_{k=1}^{\infty} k^{rm} | v_k^{-1} a_k |^r < \infty \qquad (3)$$

for each $x \in C_\infty(\Delta_v^m)$ by Lemma 3. Hence $a \in [C_\infty(\Delta_v^m)]^{r\alpha}$. Since $\ell_\infty(\Delta_v^m) \subset C_\infty(\Delta_v^m)$, we have $[C_\infty(\Delta_v^m)]^{r\alpha} \subset [\ell_\infty(\Delta_v^m)]^{r\alpha} = U_1^{(r)}$; hence $a \in U_1^{(r)}$.

(ii) Let $a \in U_2^{(r)}$ and $x \in U_1^{(r)}$. Then from (3) we have $a \in \left[U_1^{(r)}\right]^{r\alpha}$. Now suppose that $a \in \left[U_1^{(r)}\right]^{r\alpha}$ and $a \notin U_2^{(r)}$. Then we have $\sup_k k^{-rm} | v_k a_k |^r = \infty$. Hence there is a strictly increasing sequence $(k(i))$ of positive integers $k(i)$ such that

$$[k(i)]^{-rm} | v_{k(i)} a_{k(i)} |^r > i^m .$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} | a_{k(i)} |^{-1}, & k = k(i) \\ 0 \ , & k \neq k(i). \end{cases}$$

Then we have

$$\sum_{k=1}^{\infty} k^{rm} | v_k^{-1} x_k |^r = \sum_{i=1}^{\infty} [k(i)]^{rm} | v_{k(i)} a_{k(i)} |^{-r}$$
$$\leq \sum_{i=1}^{\infty} i^{-m} < \infty, \quad m \geq 2.$$

Hence $x \in U_1^{(r)}$ and $\sum_{k=1}^{\infty} | a_k x_k |^r = \sum_{k=1}^{\infty} 1 = \infty$. This contradicts $a \in \left[U_1^{(r)}\right]^{r\alpha}$; hence $a \in U_2^{(r)}$.

Theorem 7. Let $m \geq 1$ and $1 \leq r < \infty$. Then

(i) $\left\{C_\infty\left[\Delta_v^m\right]\right\}^{r\alpha} = U_1^{(r)}$,

(ii) $\left[U_1^{(r)}\right]^{r\alpha} = U_2^{(r)}$.

Proof. The proof is similar to that of Theorem 6.

Corollary 1. The sequence spaces $C_\infty(\Delta_v^m)$ and $C_\infty\left[\Delta_v^m\right]$ are not $r\alpha$-perfect for $m \geq 1$.

Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers and let E stand for ℓ_∞, c and c_0. Then we define $E(v) = \{x = (x_k) : (v_k x_k) \in E\}$. In the following theorem we give the $r\alpha$-, $r\beta$- and $r\gamma$-duals of $E(v)$.

Theorem 8. Let $m \geq 1$ and $1 \leq r < \infty$. Then $[E(v)]^{r\eta} = U^{(r)}$ for $\eta \in \{\alpha, \beta, \gamma\}$, where

$$U^{(r)} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} \left| v_k^{-1} a_k \right|^r < \infty \right\}$$

Proof. We give the proof for the case $E = \ell_\infty$ and $\eta = \alpha$. If $a \in U^{(r)}$, then

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$$\sum_{k=1}^{\infty} | a_k x_k |^r \le \sup_k \left| v_k x_k \right|^r \sum_{k=1}^{\infty} | \frac{a_k}{v_k} |^r < \infty$$

for each $x \in \ell_\infty(v)$; hence $a \in [\ell_\infty(v)]^{r\alpha}$. Now suppose that $a \in [\ell_\infty(v)]^{r\alpha}$ and $a \notin U^{(r)}$. Then there is a strictly increasing sequence (n_i) of positive integers n_i such that

$$\sum_{k=n_i+1}^{n_{i+1}} | v_k^{-1} a_k |^r > i^r .$$

Let $x \in \ell_\infty(v)$ be defined by

$$x_k = \begin{cases} 0, & 1 \le k \le n_1 \\ v_k^{-1}(\operatorname{sgn} a_k)/i, & n_i < k \le n_{i+1} \end{cases}.$$

Then we may write

$$\begin{aligned} \sum_{k=1}^{\infty} | a_k x_k |^r &= \sum_{k=n_1+1}^{n_2} | a_k x_k |^r + \ldots + \sum_{k=n_i+1}^{n_{i+1}} | a_k x_k |^r + \ldots \\ &= \sum_{k=n_1+1}^{n_2} | v_k^{-1} a_k |^r + \ldots + \frac{1}{i^r} \sum_{k=n_i+1}^{n_{i+1}} | v_k^{-1} a_k |^r + \ldots \\ &> 1 + 1 + \ldots = \sum_{i=1}^{\infty} 1 = \infty. \end{aligned}$$

This contradicts $a \in (\ell_\infty(v))^{r\alpha}$; hence $a \in U^{(r)}$. The proofs for the cases $X = c_0$ or c and $\eta \in \{\beta, \gamma\}$ are similar.

Corollary 2. *i*) Let $v_k = 1$ for all $k \in \mathbb{N}$. Then we have

$$[C_\infty(\Delta_v^m)]^{r\alpha} = \{C_\infty[\Delta_v^m]\}^{r\alpha} = G_1^{(r)} \text{ and } \left[G_1^{(r)}\right]^{r\alpha} = G_2^{(r)}.$$

where

$$G_1^{(r)} = \{a = (a_k) : \sum_{k=1}^{\infty} k^{rm} | a_k |^r < \infty\},$$

$$G_2^{(r)} = \{a = (a_k) : \sup_k k^{-rm} | a_k |^r < \infty\},$$

(*ii*) Let $v_k = 1$ for all $k \in \mathbb{N}$ and $m = 0$. Then we have

$[C_\infty(\Delta_v^m)]^{r\alpha} = \{C_\infty[\Delta_v^m]\}^{r\alpha} = \ell_r = \{a = (a_k) : \sum_{k=1}^{\infty} | a_k |^r < \infty\}$,

(*iii*) Let $v_k = 1$ for all $k \in \mathbb{N}$. Then we have $U^{(r)} = \ell_r$.

Lemma 5 [35]. Let m be a positive integer. Then

i) There exist positive constants, M_1 and M_2, such that $M_1 k^m \le \binom{m+k}{k} \le M_2 k^m, k = 0,1,2...$

ii) $\sum_{k=0}^{n} \binom{n+m-k-1}{m-1} = \binom{n+m}{m} = \binom{n+m}{n}$,

iii) If $x \in c_0(\Delta_v^m)$, then $\binom{m+k}{k}^{-1} v_k x_k \to 0, (k \to \infty)$.

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Theorem 9. Let $1 \leq r < \infty$, and m be a positive integer. Then

$[C_\infty(\Delta_v^m)]^{rN} = \{C_\infty[\Delta_v^m]\}^{rN} = [C_\infty(\Delta_v^m)]^N = \{C_\infty[\Delta_v^m]\}^N = U_1(v)$ and

$\{V_0[\Delta_v^m]\}^{rN} = \{V_0[\Delta_v^m]\}^N = U_2(v)$

where

$$U_1(v) = \{a = (a_n) : v_n^{-1} n^m a_n \to 0, \ n \to \infty\}$$

$$U_2(v) = \{a = (a_n) : \sup_n |\sum_{k=0}^{n} \binom{n+m-k-1}{m-1} v_n^{-1} a_n | < \infty\}.$$

Proof. The proof of the part $[C_\infty(\Delta_v^m)]^N = [C_\infty(\Delta_v^m)]^N = U_1(v)$ is easy. We only show that $\{V_0[\Delta_v^m]\}^N = U_2(v)$. Since $[c_0(\Delta_v^m)]^N = U_2(v)$ and $c_0(\Delta_v^m) \subset V_0[\Delta_v^m]$, we have $[c_0(\Delta_v^m)]^N \subset U_2$. Let $a \in U_2(v)$ and $x \in V_0[\Delta_v^m]$. Then by Lemma 5 *i*), *ii*) and *iii*), we obtain

$$\lim_n a_n x_n = \lim_n (\sum_{k=0}^{n} \binom{n+m-k-1}{m-1}) v_n^{-1} a_n (\sum_{k=0}^{n} \binom{n+m-k-1}{m-1})^{-1} v_n x_n = 0.$$

Hence $a \in \{V_0[\Delta_v^m]\}^N$.

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