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Second-order duality for minimax fractional programming involving generalised Type-I functions

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Abstract: A class of minimax fractional programming problem and its two types of second-order dual models are considered with an establishment of weak, strong and strict converse duality theorems from a view point of generalised convexity. Some previously known results in the framework of generalised convexity are naturally unified and extended.

Keywords: minimax fractional programming, second-order duality, Type-I functions, generalised convexity

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INTRODUCTION

Optimisation is a mathematical technique for obtaining the greatest or least possible value of a function with one or several variables. This becomes more difficult in the presence of certain constraints imposed on the variables. Optimisation techniques are needed in various disciplines of science and engineering. In fact they are being applied to every sphere of human activity which can be modelled in a mathematical form.

Optimisation problems in which both a minimisation and maximisation process of fractional objectives are performed are usually referred to in the optimisation literature as generalised minimax fractional programming problems. These problems have arisen in multi-objective programming [1], game theory [2], goal programming [3], minimum risk problems [4] and economics [5, 6]. Stancu-Minasian [7] gave a survey on fractional programming which covers applications as well as major theoretical and algorithmic developments.

In this paper, we consider the following minimax fractional programming problem:

(P) Minimise $\psi(\mathrm{x}) = \sup_{y \in Y} \dfrac{f(x, y)}{h(x, y)}$

subject to $g(x) \leq 0, \ x \in R^n$,

where Y is a compact subset of R^l, $f,\ h: R^n \times R^l \to R$ are C^2 functions on $R^n \times R^l$, and $g: R^n \to R^m$ is a C^2 function on R^n. It is assumed that for each $(x, y) \in R^n \times R^l$, $f(x, y) \geq 0$ and $h(x, y) > 0$.

In the study of optimality conditions and duality results for minimax programming problems, Yadav and Mukherjee [8] established the optimality conditions to construct two dual problems and derived duality theorems for differentiable fractional minimax programming. Chandra and Kumar [9] pointed out that the formulation of Yadav and Mukherjee [8] has some omissions and inconsistencies and constructed two modified dual problems and proved duality theorems for (convex) differentiable fractional minimax programming. To relax convexity assumptions involved in sufficient optimality conditions and duality theorems, various generalised convexity notions have been proposed. Focusing on the minimax fractional programming problem, Yang and Hou [10] established the sufficient optimality conditions and derived a number of duality results. Many other authors were involved in developing the optimality conditions and deriving the duality results for minimax programming problems [11-23].

Mangasarian [24] first formulated the second-order dual for a non-linear programming problem and established the duality results under somewhat involved assumptions. Mond [25] reproved second-order duality results involving simpler assumptions and showed that the second-order dual has computational advantages over the first-order dual. In order to generalise the notion of convexity to the second and higher orders and extend the validity of results to larger classes of optimisation problems, Ahmad and Husain [26] introduced a class of second-order (F, α, ρ, d)-convex functions and established duality theorems for a second-order Mond-Weir type multi-objective dual problem. Husain et al. [27] considered two types of second-order dual model for a minimax fractional programming problem and adopted the concept of η-bonvexity/generalised η-bonvexity to discuss appropriate duality theorems.

In this paper after some preliminaries and definitions are given, the weak, strong and strict converse duality theorems for two types of dual models to the minimax fractional programming problem (P) under the second-order Type-I assumptions are discussed.

NOTATIONS AND PRELIMINARIES

Let $S = \{x \in R^n : g(x) \leq 0\}$ denote a set of all feasible solutions of problem (P). For each $(x,\ y) \in R^n \times R^l$, we define:

$$J(x) = \{j \in M : g_j(x) = 0\} \quad \text{where } M = \{1, 2, \ldots, m\},$$

$$Y(x) = \left\{ y \in Y : f(x, y) + \left(x^T Bx\right)^{1/2} = \sup_{z \in Y} f(x, z) + \left(x^T Bx\right)^{1/2} \right\}, \text{ and}$$

$$K(x) = \Big\{(s, t, \bar{y}) \in N \times R_+^s \times R^{ls} : 1 \leq s \leq n+1,\ t = (t_1, t_2, \ldots, t_s) \in R_+^s$$
$$\text{with } \sum_{i=1}^{s} t_i = 1 \text{ and } \bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s) \text{ and } \bar{y}_i \in Y(x), i = 1, 2, \ldots, s\Big\}.$$

In the sequel the following result [9] is needed:

Theorem 1 (Necessary conditions). *If* x^* *is a solution (local or global) of problem* (P) *and* $\nabla g_j(x^*), j \in J(x^*)$ *are linearly independent, then there exist* $(s^*, t^*, \bar{y}^*) \in K(x^*)$, $\lambda^* \in R_+$ *, and* $\mu \in R_+^m$ *such that*

$$\nabla \sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) \right) + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*),$$

$$f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) = 0, \quad i = 1,2,\ldots,s^*,$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$t_i^* \geq 0, \quad \sum_{i=1}^{s^*} t_i^* = 1, \quad \bar{y}_i^* \in Y(x^*), \quad i = 1,2,\ldots,s^*.$$

In order to consider the second-order duality for problem (P), we define the following second order Type I and related functions:

Definition 1. *The pair* (f, g) *is said to be second order Type I at* $\bar{x} \in X$ *with respect to* η *if there exists a vector function* $\eta : X \times X \to R^n$ *such that for all* $x \in X, p \in R^n, y_i \in Y(x)$ *,* $i = 1,2,\ldots,s, j = 1,2,\ldots,m,$

$$f(x, y_i) - f(\bar{x}, y_i) + \frac{1}{2} p^T \nabla^2 f(\bar{x}, y_i) p \geq \eta^T(x, \bar{x}) \left[\nabla f(\bar{x}, y_i) + \nabla^2 f(\bar{x}, y_i) p \right]$$

$$-g_j(\bar{x}) + \frac{1}{2} p^T \nabla^2 g_j(\bar{x}) p \geq \eta^T(x, \bar{x}) \left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x}) p \right].$$

In the above definition, if the inequalities appear as strict inequalities, then we say that (f, g) is strictly second order Type I at $\bar{x} \in X$.

Definition 2. *The pair* (f, g) *is said to be second order pseudoquasi Type I at* $\bar{x} \in X$ *with respect to* η *if there exists a vector function* $\eta : X \times X \to R^n$ *such that for all* $x \in X, p \in R^n, y_i \in Y(x)$*,* $i = 1,2,\ldots,s, j = 1,2,\ldots,m,$

$$f(x, y_i) - f(\bar{x}, y_i) + \frac{1}{2} p^T \nabla^2 f(\bar{x}, y_i) p < 0$$
$$\Rightarrow \eta^T(x, \bar{x}) \left[\nabla f(\bar{x}, y_i) + \nabla^2 f(\bar{x}, y_i) p \right] < 0,$$

$$-g_j(\bar{x}) + \frac{1}{2} p^T \nabla^2 g_j(\bar{x}) p \leq 0$$
$$\Rightarrow \eta^T(x, \bar{x}) \left[\nabla g_j(\bar{x}) + \nabla^2 g_j(\bar{x}) p \right] \leq 0.$$

If the second inequality is strict, then (f, g) is said to be second-order strictly pseudoquasi Type I at $\bar{x} \in X$.

FIRST DUALITY MODEL

In relation to (P), we consider the following dual problem:

$$\textbf{(MD)} \quad \max_{(s,t,\bar{y}) \in K(z)} \quad \sup_{(z,\mu,\lambda,p) \in H_1(s,t,\bar{y})} \lambda,$$

where $H_1(s,t,\bar{y})$ denotes the set of all $(z,\mu,\lambda,p)\in R^n\times R_+^m\times R_+\times R^n$ satisfying

$$\nabla\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p$$
$$+\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p=0, \tag{1}$$

$$\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)-\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p\geq 0, \tag{2}$$

$$\sum_{j=1}^{m}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\geq 0. \tag{3}$$

If, for a triplet $(s,t,\bar{y})\in K(z)$, the set $H_1(s,t,\bar{y})=\phi$, then we define the supremum over it to be -∞.

Remark 1. If $p=0$, then (MD) becomes the dual given in Liu and Wu [28].

Theorem 2 (Weak duality). *Let x and* $(z,\mu,\lambda,s,t,\bar{y},p)$ *be the feasible solutions of* (P) *and* (MD) *respectively. Assume that* $\left[\sum_{i=1}^{s}t_i\left(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i)\right),\sum_{j=1}^{m}\mu_j g_j(\cdot)\right]$ *is second order Type I at z with* $\eta(x,z)>0$. *Then* $\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}\geq\lambda$.

Proof. Suppose it is contrary to the result that $\sup_{y\in Y}\frac{f(x,y)}{h(x,y)}<\lambda$.

Thus, we have $f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)<0$ for all $\bar{y}_i\in Y(x),\ i=1,2,...,s.$

It follows from $t_i\geq 0,\ i=1,2,...,s$ that $t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)\leq 0$, with at least one strict inequality since $t=(t_1,t_2,...,t_s)\neq 0.$ Taking summation over *i* and using $\sum_{i=1}^{s}t_i=1$, we have by (2):

$$\sum_{i=1}^{s}t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)<0\leq\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)-\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p.$$

The above inequality, together with (3), implies:

$$\sum_{i=1}^{s}t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)-\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)-\sum_{j=1}^{m}\mu_j g_j(z)$$
$$+\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p+\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p<0. \tag{4}$$

Now the second-order Type-I assumption on $\left[\sum_{i=1}^{s}t_i\left(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i)\right),\sum_{j=1}^{m}\mu_j g_j(\cdot)\right]$ at z gives:

$$\sum_{i=1}^{s}t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)-\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p$$
$$\geq\eta^T(x,z)\left[\nabla\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s}t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p\right],$$

$$-\sum_{j=1}^{m}\mu_j g_j(z)+\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\geq\eta^T(x,z)\left[\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\right].$$

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Combining the above two inequalities, we get:

$$\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)-\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)-\sum_{j=1}^{m}\mu_j g_j(z)$$

$$+\frac{1}{2}p^T\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p+\frac{1}{2}p^T\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p$$

$$\geq \eta^T(x,z)\Bigg[\nabla\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p$$

$$+\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p\Bigg],$$

which, along with (4) and $\eta(x,z)>0$, implies:

$$\nabla\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p$$

$$+\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p<0,$$

which contradicts (1). This completes the proof.

Theorem 3 (Strong duality). *Assume that* x^* *is an optimal solution of* (P) *and* $\nabla g_j(x^*)$, $j\in J(x^*)$ *are linearly independent. Then there exist* $(s^*,t^*,\bar{y}^*)\in K(x^*)$ *and* $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ *such that* $(x^*,\ \mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ *is a feasible solution of* (MD) *and the two objectives have the same values. Further, if the hypothesis of Theorem 2 (weak duality) holds for all feasible solutions* $(z,\mu,\lambda,s,t,\bar{y},p)$ *of* (MD)*, then* $(x^*,\ \mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ *is an optimal solution of* (MD).

Proof. Since x^* is an optimal solution of (P) and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, then by Theorem 1, there exist $(s^*,t^*,\bar{y}^*)\in K(x^*)$ and $(x^*,\mu^*,\lambda^*,p^*=0)\in H_1(s^*,t^*,\bar{y}^*)$ such that $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ is a feasible solution of (MD) and the two objectives have the same values. The optimality of $(x^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ for (MD) thus follows from weak duality Theorem 2.

Theorem 4 (Strict converse duality). *Let* x^* *and* $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ *be the optimal of* (P) *and* (MD) *respectively. Suppose that* $\left[\sum_{i=1}^{s^*} t_i^*\left(f(\cdot,\bar{y}_i^*)-\lambda^* h(\cdot,\bar{y}_i^*)\right),\sum_{j=1}^{m}\mu_j^* g_j(\cdot)\right]$ *is strictly second order Type I at* z^**, and* $\nabla g_j(x^*)$, $j\in J(x^*)$ *are linearly independent. Then* $z^*=x^*$*, i.e.* z^* *is an optimal solution of* (P).

Proof. Suppose it is contrary to the result that $z^*\neq x^*$. Since x^* and $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*)$ are the optimal of (P) and (MD) respectively, and $\nabla g_j(x^*)$, $j\in J(x^*)$ are linearly independent, from the Strong Duality Theorem 3, therefore, we reach: $\sup_{y^*\in Y}\frac{f(x^*,y^*)}{h(x^*,y^*)}=\lambda^*$

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Thus, we have $f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*) \le 0$ for all $\bar{y}_i^* \in Y(x^*)$, $i = 1,2,...,s^*$.

Now proceeding as in Theorem 2, we get:

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) - \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) - \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$+ \frac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^* + \frac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* \le 0. \tag{5}$$

The strictly second-order Type-I assumption on $\left[\sum_{i=1}^{s^*} t_i^* \left(f(\cdot, \bar{y}_i^*) - \lambda^* h(\cdot, \bar{y}_i^*)\right), \sum_{j=1}^{m} \mu_j^* g_j(\cdot)\right]$ at z gives:

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) - \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right)$$
$$+ \frac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*$$
$$> \eta^T(x^*, z^*) \left[\nabla \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*\right],$$

$$-\sum_{j=1}^{m} \mu_j^* g_j(z^*) + \frac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* > \eta^T(x^*, z^*) \left[\nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*\right].$$

Combining the above two inequalities, we get:

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) - \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) - \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$+ \frac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^* + \frac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*$$
$$> \eta^T(x^*, z^*) \left[\nabla \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) + \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^*\right.$$
$$\left. + \nabla \sum_{j=1}^{m} \mu_j^* g_j(z^*) + \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^*\right],$$

which along with (1), implies:

$$\sum_{i=1}^{s^*} t_i^* \left(f(x^*, \bar{y}_i^*) - \lambda^* h(x^*, \bar{y}_i^*)\right) - \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) - \sum_{j=1}^{m} \mu_j^* g_j(z^*)$$
$$+ \frac{1}{2} p^{*T} \nabla^2 \sum_{i=1}^{s^*} t_i^* \left(f(z^*, \bar{y}_i^*) - \lambda^* h(z^*, \bar{y}_i^*)\right) p^* + \frac{1}{2} p^{*T} \nabla^2 \sum_{j=1}^{m} \mu_j^* g_j(z^*) p^* > 0$$

which contradicts (5). Hence $z^* = x^*$.

SECOND DUALITY MODEL

Now, we consider the following dual for (P) and establish weak, strong and strict converse duality theorems:

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(GMD) $$\max_{(s,t,\bar{y})\in K(z)} \sup_{(z,\mu,\lambda,p)\in H_2(s,t,\bar{y})} \lambda,$$

where $H_2(s,t,\bar{y})$ denotes the set of all $(z,\mu,\lambda,p)\in R^n \times R_+^m \times R_+ \times R^n$ satisfying:

$$\nabla\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p + \nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p = 0, \tag{6}$$

$$\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z) -\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z)\right]p \geq 0, \tag{7}$$

$$\sum_{j\in J_\alpha}\mu_j g_j(z)-\frac{1}{2}p^T\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p \geq 0, \quad \alpha = 1,2,\ldots,r, \tag{8}$$

where $J_\alpha \subseteq M$, $\alpha = 0,1,2,\ldots,r$, with $\bigcup_{\alpha=0}^{r} J_\alpha = M$ and $J_\alpha \cap J_\beta = \phi$ if $\alpha \neq \beta$. If, for a triplet $(s,t,\bar{y})\in K(z)$, the set $H_2(s,t,\bar{y}) = \phi$, then we define the supremum over it to be -∞.

Theorem 5 (Weak duality). *Let x and $(z,\mu,\lambda,s,t,\bar{y},p)$ be the feasible solutions of* (P) *and* (GMD) *respectively. Assume that* $\left[\sum_{i=1}^{s} t_i\left(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(\cdot),\ \sum_{j\in J_\alpha}\mu_j g_j(\cdot),\ \alpha = 1,2,\ldots,r\right]$ *is second order pseudoquasi Type I at z, with $\eta(x,z)>0$. Then*

$$\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} \geq \lambda .$$

Proof. Suppose it is contrary to the result that $\sup_{y\in Y}\frac{f(x,y)}{h(x,y)} < \lambda$.

Thus, we have $f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i) < 0$ for all $\bar{y}_i \in Y(x)$, $i = 1,2,\ldots,s$.

It follows from $t_i \geq 0$, $i = 1,2,\ldots,s$, that $t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right) \leq 0$,

with at least one strict inequality since $t = (t_1,t_2,\ldots,t_s) \neq 0$. Taking summation over i and using $\sum_{i=1}^{s} t_i = 1$, we have: $\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right) < 0$.

The above inequality, together with the feasibility of x for (P), $\mu \geq 0$ and (7), implies:

$$\sum_{i=1}^{s} t_i\left(f(x,\bar{y}_i)-\lambda h(x,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(x) < 0 \leq \sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z) -\frac{1}{2}p^T\nabla^2\left[\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(z)\right]p . \tag{9}$$

Also from (8), we have:

$$-\sum_{j\in J_0}\mu_j g_j(z)+\frac{1}{2}p^T\nabla^2\sum_{j\in J_0}\mu_j g_j(z)p \leq 0, \quad \alpha = 1,2,\ldots,r . \tag{10}$$

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The inequalities (9), (10) and the second order pseudoquasi Type I assumption on $\left[\sum_{i=1}^{s} t_i\left(f(\cdot,\bar{y}_i)-\lambda h(\cdot,\bar{y}_i)\right)+\sum_{j\in J_0}\mu_j g_j(\cdot),\ \sum_{j\in J_\alpha}\mu_j g_j(\cdot),\ \alpha=1,2,...,r\right]$ at z implies:

$$\eta^T(x,z)\left[\nabla\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p\right.$$
$$\left.+\nabla\sum_{j\in J_0}\mu_j g_j(z)+\nabla^2\sum_{j\in J_0}\mu_j g_j(z)p\right]<0,$$

$$\eta^T(x,z)\left[\nabla\sum_{j\in J_\alpha}\mu_j g_j(z)+\nabla^2\sum_{j\in J_\alpha}\mu_j g_j(z)p\right]\le 0,\quad \alpha=1,2,...,r.$$

Combining these inequalities with $\eta(x,z)>0$, we get:

$$\nabla\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)+\nabla^2\sum_{i=1}^{s} t_i\left(f(z,\bar{y}_i)-\lambda h(z,\bar{y}_i)\right)p$$
$$+\nabla\sum_{j=1}^{m}\mu_j g_j(z)+\nabla^2\sum_{j=1}^{m}\mu_j g_j(z)p<0,$$

which contradicts (6). This completes the proof.

Theorem 6 (Strong duality). *Assume that* x^* *is an optimal solution of* (P) *and* $\nabla g_j(x^*)$, $j\in J(x^*)$ *are linearly independent. Then there exist* $(s^*,t^*,\bar{y}^*)\in K(x^*)$ *and* $(x^*,\mu^*,\lambda^*,p^*=0)\in H_2(s^*,t^*,\bar{y}^*)$ *such that* $(x^*,\ \mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ *is a feasible solution of* (GMD) *and the two objectives have the same values. Further, if the hypothesis of Theorem 5 (weak duality) holds for all feasible solutions* $(z,\mu,\lambda,s,t,\bar{y},p)$ *of* (GMD)*, then* $(x^*,\ \mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*=0)$ *is an optimal solution of* (GMD).

Proof: The proof of the above theorem is similar to that of Theorem 3 and hence omitted.

Theorem 7 (Strict converse duality). *Let* x^* *and* $(z^*,\mu^*,\lambda^*,s^*,t^*,\bar{y}^*,p^*)$ *be the optimal of* (P) *and* (GMD) *respectively. Suppose that* $\left[\sum_{i=1}^{s^*} t_i^*\left(f(\cdot,\bar{y}_i^*)-\lambda^* h(\cdot,\bar{y}_i^*)\right)+\sum_{j\in J_0}\mu_j^* g_j(\cdot),\ \sum_{j\in J_\alpha}\mu_j^* g_j(\cdot),\quad \alpha=1,2,...,r\right]$ *is second order strictly pseudoquasi Type I at* z^**, and* $\nabla g_j(x^*)$, $j\in J(x^*)$ *are linearly independent. Then* $z^*=x^*$*, i.e.* z^* *is an optimal solution of* (P).

Proof: It can be proved by a contradiction, applying Theorem 6.

CONCLUSIONS

We have established weak, strong and strict converse duality theorems for a class of generalised fractional minimax programming problems possessing some second-order Type-I invexity property. This paper extends earlier work in which duality results were obtained for a generalised fractional optimisation problem by applying a convexity assumption.

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