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Full Paper

Free vibration analysis of symmetrically laminated composite rectangular plates using extended Kantorovich method

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Abstract: Free vibration of symmetrically laminated composite rectangular plates with various boundary conditions is analysed by an extended Kantorovich method, in which a separable function to the dynamic-system energy equation is applied in order to reduce the partial differential equations to ordinary differential equations in the direction of x, y coordinates with a constant coefficient. The beam function is used as an initial trial function in the iterative calculation, which is employed to evaluate the natural frequency and force the final solution needed to satisfy the boundary conditions. To verify the accuracy of the present method, the frequency parameters are evaluated in comparison with previous work on the subject. A good agreement proves that the method can be used to evaluate the natural frequencies of unidirectional 0°, unidirectional 90° and cross-ply symmetrically laminated composite rectangular plates.

Keywords: Kantorovich method, free vibration analysis, composite rectangular plates

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INTRODUCTION

Composite materials are increasingly being used for rectangular plate structures constructed on mechanical, civil, aerospace and marine engineering projects. Due to their high strength, low weight, good fatigue resistance and good corrosion resistance, their properties meet the requirements of most specific designs. Regarding the bending, buckling or vibrating problem found in rectangular plates, the difficulty involved is in solving the related partial differential equations. The exact method used in doing this is possible when at least a pair of opposite edges is simply supported. Otherwise, an approximate method such as the Galerkin method, the Rayleigh-Ritz method, the extended Kantorovich method and the finite element method (FEM) is usually employed.

The extended Kantorovich method is used to reduce the partial differential equations to ordinary differential equations in the direction of x, y coordinates. The iterative calculation is used to evaluate the deflection, buckling load or natural frequency and to force the final solution required to satisfy the boundary conditions. The extended Kantorovich method has been reviewed by several researchers. For example, Dalaei and Kerr derived a closed-form approximate solution for a uniformly lateral distributed load [1] and the natural frequency [2] of an orthotropic rectangular clamped plate. An initial trial function which satisfies the boundary conditions along the y coordinate direction was used in the iterative calculation. It was found that the final solution can be obtained from the fourth iteration and that this is independent of an initial trial function. Sakata et al. [3] evaluated the natural frequency of an orthotropic rectangular plate with various boundary conditions. An initial trial function which satisfies the boundary conditions along one direction was used in the iterative calculation. The results showed that the convergence of the final solution is rapid and the particular natural frequency can be obtained separately with a good accuracy while the Rayleigh-Ritz method uses a large number of shape functions if a higher natural frequency is required. Rajalingham et al. [4] improved the convergence of the natural frequency of an isotropic rectangular clamped plate. The shape functions obtained from the extended Kantorovich method were used in the Rayleigh-Ritz method. It was discovered that these shape functions enhance the effectiveness of the Rayleigh-Ritz method. Bercin [5] evaluated the low natural frequency of an orthotropic rectangular clamped plate. An initial trial function such as that used by Dalaei and Kerr [2] was used in the iterative calculation. It was found that the convergence of the solution is very rapid.

Lee et al. [6] derived the free vibration of symmetrically laminated composite rectangular plates with all edges elastically restrained against rotation based on first-order anisotropic shear deformation plate theory. The Timoshenko beam function was used in the iterative calculation as an initial trial function. The results indicated that the extended Kantorovich method can be more effectively and accurately applied to the free vibration of a symmetrically laminated composite with a cross-ply rectangular plate than the Rayleigh-Ritz method, but cannot be applied to the free vibration of a symmetrically laminated composite with an angle-ply rectangular plate. Rajalingham et al. [7] derived a closed-form approximate solution for the natural frequency of an isotropic and clamped plate. As the plate characteristic function was used in the iterative calculation as an initial trial function, the modal parameters were found to be suitable for evaluating a higher natural frequency while the Rayleigh-Ritz method involved a large order matrix eigenvalue problem, plus the finite element method could not provide accurate values for higher natural frequencies. Ungbhakorn and Singhatanadgid [9] evaluated the critical buckling load of symmetrically laminated composite with unidirectional 0° and cross-ply rectangular plates with various boundary conditions. Although an arbitrary function was used in the iterative calculation as an initial trial function, the final solution was automatically forced to satisfy the boundary conditions and the critical buckling load was obtained from the fourth iteration.

The purpose of this study is to evaluate the natural frequencies of symmetrically laminated composite rectangular plates with various boundary conditions using the extended Kantorovich method.

METHODS

Derivation of the Iterative Differential Equations

Hamiton's principle is a generalisation of the principle of virtual displacement within the dynamics of a system. The principle assumes that the system under consideration is characterised by two energy functions, namely the kinetic energy and the potential energy [8]:

$$\delta \int_{t_1}^{t_2} [K-(V+U)]dt = 0 \tag{1}$$

where K is the kinetic energy and $V+U$ is the potential energy.

The potential energy and the kinetic energy of a symmetrically laminated composite rectangular plate, as shown in Figure 1, can be written as follows.

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2}\int_0^b\int_0^a \left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 4D_{16}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right) + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right.\right.$$
$$\left.\left. + 4D_{26}\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right) + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dx\, dy - \frac{1}{2}\int_0^b\int_0^a m(\omega w)^2 dx\, dy \right\} dt = 0 \tag{2}$$

where D_{ij} is the bending stiffness of the composite plate, w is the lateral deflection, m is mass per unit area of plate and ω is the natural circular frequency.

Assuming the solution is

$$w(x,y) = X(x)\,Y(y) \quad , \tag{3}$$

substitute equation (3) into equation (2):

$$\delta \int_{t_1}^{t_2} \left\{ \frac{1}{2}\int_0^b\int_0^a \left[D_{11}\left(\frac{\partial^2 X}{\partial x^2}Y\right)^2 + 2D_{12}\left(\frac{\partial^2 X}{\partial x^2}Y\right)\left(X\frac{\partial^2 Y}{\partial y^2}\right) + D_{22}\left(X\frac{\partial^2 Y}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right)^2 \right.\right.$$
$$\left.\left. + 4\left(D_{16}\left(\frac{\partial^2 X}{\partial x^2}Y\right) + D_{26}\left(X\frac{\partial^2 Y}{\partial y^2}\right)\right)\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right) \right] dxdy - \frac{1}{2}\int_0^b\int_0^a \left[mX^2Y^2\omega^2\right] dx\, dy \right\} dt = 0 \tag{4}$$

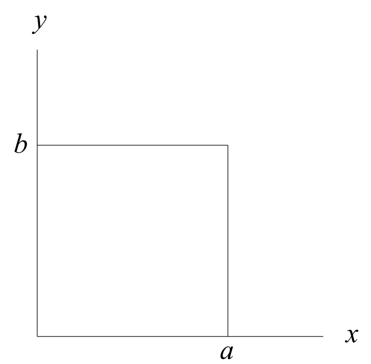


Figure 1. The rectangular plate

If $X(x)$ is defined a priori, equation (4) can be rewritten as

$$\delta\int_{t_1}^{t_2}\left\{\frac{1}{2}\int_0^b\left[S_{1x}D_{11}Y^2+2S_{2x}D_{12}Y\left(\frac{\partial^2 Y}{\partial y^2}\right)+S_{3x}D_{22}\left(\frac{\partial^2 Y}{\partial y^2}\right)^2+4S_{4x}D_{66}\left(\frac{\partial Y}{\partial y}\right)^2+4S_{5x}D_{16}Y\left(\frac{\partial Y}{\partial y}\right)\right.\right.$$
$$\left.\left.+4S_{6x}D_{26}\left(\frac{\partial Y}{\partial y}\right)\left(\frac{\partial^2 Y}{\partial y^2}\right)\right]dy-\frac{1}{2}\int_0^b S_{3x}mY^2\omega^2 dy\right\}dt=0 \qquad (5)$$

where

$$S_{1x}=\int_0^a\left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx, \qquad S_{2x}=\int_0^a\left(X\frac{\partial^2 X}{\partial x^2}\right)dx$$

$$S_{3x}=\int_0^a X^2 dx, \qquad S_{4x}=\int_0^a\left(\frac{\partial X}{\partial x}\right)^2 dx$$

$$S_{5x}=\int_0^a\left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial^2 X}{\partial x^2}\right)dx, \qquad S_{6x}=\int_0^a X\left(\frac{\partial X}{\partial x}\right)dx$$

The variational method and integration by parts of equation (5) yield a fourth-order ordinary differential equation as in equation (6) and the boundary conditions along $y=0$ and $y=b$ as shown in equations (7) and (8) respectively.

$$S_{3x}D_{22}\frac{d^4Y}{dy^4}+(2S_{2x}D_{12}-4S_{4x}D_{66})\frac{d^2Y}{dy^2}+(S_{1x}D_{11}-S_{3x}m\omega^2)Y=0 \qquad (6)$$

$$V_y=S_{3x}D_{22}\frac{d^3Y}{dy^3}+(S_{2x}D_{12}-4S_{4x}D_{66})\frac{dY}{dy}-2S_{5x}D_{16}Y \qquad (7)$$

$$M_y=S_{3x}D_{22}\frac{d^2Y}{dy^2}+2S_{6x}D_{26}\frac{dY}{dy}+S_{2x}D_{12}Y \qquad (8)$$

Similarly, when $Y(y)$ is defined a priori, a fourth-order ordinary differential equation can be written as equation (9) and the boundary conditions along $x=0$ and $x=a$ as equations (10) and (11) respectively.

$$S_{3y}D_{11}\frac{d^4X}{dx^4}+(2S_{2y}D_{12}-4S_{4y}D_{66})\frac{d^2X}{dx^2}+(S_{1y}D_{22}-S_{3y}m\omega^2)X=0 \qquad (9)$$

$$V_x=S_{3y}D_{11}\frac{d^3X}{dx^3}+(S_{2y}D_{12}-4S_{4y}D_{66})\frac{dX}{dx}-2S_{5y}D_{26}X \qquad (10)$$

$$M_x=S_{3y}D_{11}\frac{d^2X}{dx^2}+2S_{6y}D_{16}\frac{dX}{dx}+S_{2y}D_{12}X \qquad (11)$$

where

$$S_{1y}=\int_0^b\left(\frac{\partial^2 Y}{\partial y^2}\right)^2 dy, \qquad S_{2y}=\int_0^b\left(Y\frac{\partial^2 Y}{\partial y^2}\right)dy$$

$$S_{3y}=\int_0^b Y^2 dy, \qquad S_{4y}=\int_0^b\left(\frac{\partial Y}{\partial y}\right)^2 dy$$

$$S_{5y} = \int_0^b \left(\frac{\partial Y}{\partial y}\right)\left(\frac{\partial^2 Y}{\partial y^2}\right) dy, \qquad S_{6y} = \int_0^b Y\left(\frac{\partial Y}{\partial y}\right) dy$$

Solution of the Iterative Differential Equations

The fourth-order ordinary differential equation (6) can be rewritten in a simple form as

$$\frac{d^4 Y}{dy^4} + \left(\frac{2S_{2x}D_{12} - 4S_{4x}D_{66}}{S_{3x}D_{22}}\right)\frac{d^2 Y}{dy^2} + \left(\frac{S_{1x}D_{11} - S_{3x}m\omega^2}{S_{3x}D_{22}}\right)Y = 0$$

where the form of the general solution is composed of the following four forms:

$$Y(y) = C_{1y}\sin(q_1 y) + C_{2y}\cos(q_1 y) + C_{3y}\sin(q_2 y) + C_{4y}\cos(q_2 y)$$
$$Y(y) = C_{1y}\sin(q_1 y) + C_{2y}\cos(q_1 y) + C_{3y}\sinh(q_2 y) + C_{4y}\cosh(q_2 y)$$
$$Y(y) = [C_{1y}\sin(q_2 y) + C_{2y}\cos(q_2 y)]\cosh(q_1 y) + [C_{3y}\sin(q_2 y) + C_{4y}\cos(q_2 y)]\sinh(q_1 y)$$
$$Y(y) = C_{1y}\sinh(q_1 y) + C_{2y}\cosh(q_1 y) + C_{3y}\sinh(q_2 y) + C_{4y}\cosh(q_2 y)$$

In this study, considering a case $S_{1x}D_{11} < S_{3x}\, m\omega^2$, the solution can be written as follows.

$$Y(y) = C_{1y}\sin(q_1 y) + C_{2y}\cos(q_1 y) + C_{3y}\sinh(q_2 y) + C_{4y}\cosh(q_2 y) \tag{12}$$

where q_1 and q_2 are modal parameters in y coordinate direction, and

$$q_1^2 - q_2^2 = \frac{2S_{2x}D_{12} - 4S_{4x}D_{66}}{S_{3x}D_{22}} \tag{13}$$

$$q_1^2\, q_2^2 = \frac{S_{3x}m\omega^2 - S_{1x}D_{11}}{S_{3x}D_{22}} \tag{14}$$

Similarly, the fourth-order ordinary differential equation (9) can be rewritten in a simple form as

$$\frac{d^4 X}{dx^4} + \left(\frac{2S_{2y}D_{12} - 4S_{4y}D_{66}}{S_{3y}D_{11}}\right)\frac{d^2 X}{dx^2} + \left(\frac{S_{1y}D_{22} - S_{3y}m\omega^2}{S_{3y}D_{11}}\right)X = 0$$

where the form of the general solution is composed of the following four forms:

$$X(x) = C_{1x}\sin(p_1 x) + C_{2x}\cos(p_1 x) + C_{3x}\sin(p_2 x) + C_{4x}\cos(p_2 x)$$
$$X(x) = C_{1x}\sin(p_1 x) + C_{2x}\cos(p_1 x) + C_{3x}\sinh(p_2 x) + C_{4x}\cosh(p_2 x)$$
$$X(x) = [C_{1x}\sin(p_2 x) + C_{2x}\cos(p_2 x)]\cosh(p_1 x) + [C_{3x}\sinh(p_2 x) + C_{4x}\cosh(p_2 x)]\sinh(p_1 x)$$
$$X(x) = C_{1x}\sinh(p_1 x) + C_{2x}\cosh(p_1 x) + C_{3x}\sinh(p_2 x) + C_{4x}\cosh(p_2 x)$$

In this study, considering a case $S_{1y}D_{22} < S_{3y}\, m\omega^2$, the solution can be written as follows.

$$X(x) = C_{1x}\sin(p_1 x) + C_{2x}\cos(p_1 x) + C_{3x}\sinh(p_2 x) + C_{4x}\cosh(p_2 x) \tag{15}$$

where p_1 and p_2 are modal parameters in the x coordinate direction, and

$$p_1^2 - p_2^2 = \frac{2S_{2y}D_{12} - 4S_{4y}D_{66}}{S_{3y}D_{11}} \tag{16}$$

$$p_1^2\, p_2^2 = \frac{S_{3y}m\omega^2 - S_{1y}D_{22}}{S_{3y}D_{11}} \tag{17}$$

Iterative Calculation Procedure

The iterative calculation is used to evaluate the natural frequency and to develop a final solution to satisfy the boundary conditions by the following steps.

(1) The iterative calculation begins by choosing an initial trial function in the x or y coordinate direction using the procedure shown in Figure 2 and choosing the $X_0(x)$ as an initial trial function. S_{1x} through S_{6x} are calculated from $X_0(x)$.

(2) In the first iteration, substitute the solution equation (12) in the boundary conditions and use q_2 as a function of q_1, or q_1 as a function of q_2, from the relationship equation (13). Then find the eigenvalue q_1 or q_2 and the eigenvector $Y_1(y)$.

(3) In the second iteration, substitute the solution equation (15) in the boundary conditions and use p_2 as a function of p_1, or p_1 as a function of p_2, from the relationship equation (16). S_{1y} through S_{6y} are calculated from the eigenvector $Y_1(y)$ obtained from equation (2). Then find the eigenvalue p_1 or p_2 and the eigenvector $X_1(x)$.

(4) In the third iteration, substitute the solution equation (12) in the boundary conditions and use q_2 as a function of q_1, or q_1 as a function of q_2, from the relationship equation (13). S_{1x} through S_{6x} are calculated from $X_1(x)$. Then find the eigenvalue q_1 or q_2 and the eigenvector $Y_2(y)$.

(5) Compare q_1 and q_2 from equations (4) and (2). If the difference satisfies the specified tolerance level, the last q_1 and q_2 can be taken as the final solution. Otherwise, continue the iterative calculation by repeating steps (2) to (4).

(6) The natural frequency is calculated from equation (14) or (17).

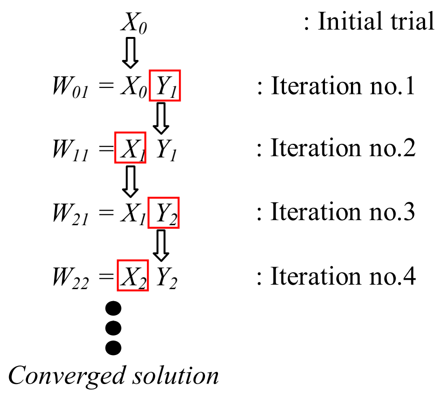


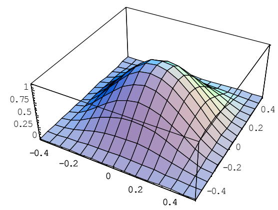
Figure 2. Iteration procedure

Numerical Verification and Accuracy

For numerical calculation, the present method is applied to a rectangular plate with various aspect ratios b/a, natural frequencies and boundary conditions. The method considers individual plate modes as the product of a separable function in the x, y coordinate directions. The notation for the plate mode (i, j) is a plate mode which is the product of the ith mode for the x coordinate direction and the jth mode for the y coordinate direction. The notation for boundary condition, for example CFCS, is as follows. The first and third letters mean the boundary condition along $x=0$ and $x=a$ respectively, and the second and fourth letters mean the boundary condition along $y=0$ and $y=b$ respectively. The letters C, S and F mean the clamped, simply supported and free boundary conditions respectively.

The iteration example of the (1, 1) plate mode of an isotropic square plate with CCCC boundary condition is illustrated in Table 1. The (1, 1) plate mode is the product of the first mode of the x and y coordinate directions. The first iteration chooses the first mode of the free vibration of the clamped beam in the x coordinate direction, A, as an initial trial function. S_{1x} through S_{6x} are calculated for the substitution of q_2 as a function of q_1. The first eigenvalues q_1 and q_2 and the eigenvector in the y coordinate direction are *4.304*, *6.567* and B respectively. In the second iteration, the eigenvector in the y coordinate direction from the first iteration is used to determine S_{1y} through S_{6y} for the substitution of p_2 as a function of p_1. The first eigenvalues p_1 and p_2 and the eigenvector in the x coordinate direction are *4.312*, *6.525* and C respectively. The iterative calculation is repeated until the difference of the modal parameter in the x or y coordinate direction satisfies the specified tolerance. The fourth iteration is the end iteration due to the modal parameter in the x coordinate direction of the second and fourth iteration being identified, where $p_1 = 4.312$ and $p_2 = 6.525$. The plate mode shape is the product of the eigenvector in the x and y coordinate directions, $w(x,y) = C\,D$, as shown in the last column.

Table 1. (1, 1) Plate mode of an isotropic square plate with CCCC boundary condition

Iteration	Assumed solution	Solution		
No.	*X(x)* or *Y(y)*	Eigenvalue	Eigenvector *X(x)* or *Y(y)*	Mode shape
1	*A*	$q_1 = 4.304$ $q_2 = 6.567$	*B*	
2	*B*	$p_1 = 4.312$ $p_2 = 6.525$	*C*	
3	*C*	$q_1 = 4.312$ $q_2 = 6.526$	*D*	
4	*D*	$p_1 = 4.312$ $p_2 = 6.525$	*C*	

Note: A = cos(4.730x) + 0.132cosh(4.730x), B = cos(4.304y) + 0.041cosh(6.567y)
C = cos(4.312x) + 0.042cosh(6.525x), D = cos(4.312y) + 0.042cosh(6.526y)

For the plate mode shape, the contour representing the lateral deflection is zero and is called the nodal line. The nodal line is defined by the dash line. The nodal lines (i, j) of the (1, 1) through (3, 3) square plate mode with CCCC boundary conditions are illustrated in Figure 3. The nodal lines of the (i, j) plate mode are the i-1 and j-1 lines in the x and y coordinate directions respectively. For example, the nodal lines for the (3, 2) plate mode are 2 and 1 lines in the x and y coordinate directions respectively. The frequency parameter $\omega ab\sqrt{m/D}$ is evaluated by substituting q_1 and q_2 obtained from the third iteration into equation (14), or by substituting p_1 and p_2 obtained from the fourth iteration into equation (17), producing *35.998*, as shown in Table 2.

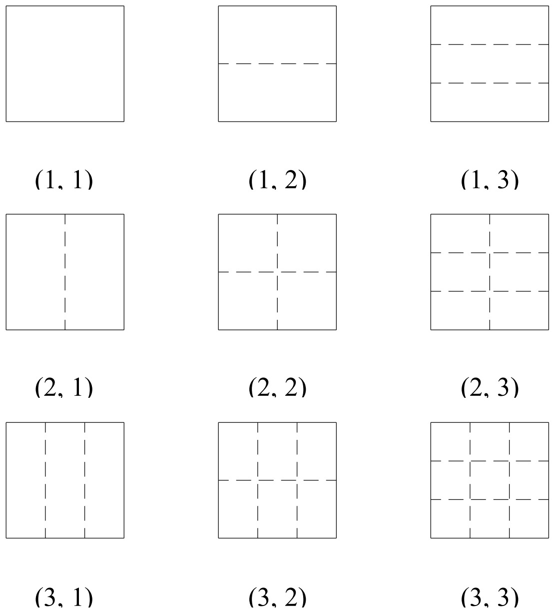


Figure 3. Nodal lines of the mode of square plate with CCCC boundary condition

Table 2. (1, 1) Frequency parameters $\omega ab\sqrt{m/D}$ of an isotropic square plate

Boundary condition	Method				
	Ref [2]	Ref [3]	Ref [4]	Ref [7]	Present
CCCC	35.999	35.999	35.998	35.998	35.998
SSSS	19.739				19.739

The (1, 1) frequency parameters of an isotropic square plate, an orthotropic rectangular plate and the $[0/90]_s$ laminated composite rectangular plate with SSSS boundary condition are evaluated to verify the accuracy of the present method as illustrated in Tables 2-4. In the case of the isotropic square plate, one can consider the bending stiffness of the composite plate as $D_{11} = D_{22} = (D_{12}+D_{66}) = D$, and in the case of the orthotropic rectangular plate, as $D_{11} = D_{22}$, $(D_{12}+D_{66}) = 0.5D_{11}$. The mechanical properties of the $[0/90]_s$ laminated composite rectangular plate are $G_{12} = 0.5E_2$ and $\nu_{12} = 0.25$.

The natural frequencies of the symmetrically laminated composite rectangular plate are evaluated by the present method and the FEM (ANSYS). In the present method, the natural frequency is obtained by dividing the natural circular frequency from equation (14) or equation (17) with 2π. In the FEM, a mesh size 64 x 64 of an 8-node, 3-D shell element with six degrees of freedom at each node is employed. The convergence of the mesh size of the (1, 1) natural frequencies of the $[0/90]_s$ laminated composite square plate is illustrated in Table 5. The natural frequencies of the unidirectional 0° and cross-ply symmetrically laminated composite rectangular plates are illustrated in Tables 6 and 7 respectively. The mechanical properties of the Kevlar 49 and the plate dimensions are E_1 = 138 GPa, E_2 = 8.96 GPa, G_{12} = 7.1 GPa, G_{23} = 2.82 GPa, ν_{12} = 0.3, ν_{23} = 0.59, mass per unit volume = 1600 kg/m^3, a = 1 m, aspect ratio b/a = 0.5, 1.0 and 2.0, and thickness = 2.5 mm.

Table 3. (1, 1) Frequency parameters $\omega ab\sqrt{m/D_{22}}$ of an orthotropic rectangular plate

Boundary condition	b = 0.5a		b = a		b = 2a	
	Ref [3]	Present	Ref [3]	Present	Ref [3]	Present
CCCC	95.391	95.391	33.917	33.917	23.848	23.848
CCCS	69.687	69.687	29.625	29.625	23.447	23.447
CCSS	67.497	67.497	24.610	24.610	16.874	16.874
CSCS	50.349	50.349	26.809	26.809	23.172	23.172
CSSS	47.325	47.325	21.163	21.163	16.497	16.497
SSSS	45.228	45.228	17.095	17.095	11.307	11.307

Table 4. (1, 1) Frequency parameters $(\omega ab/\pi^2)\sqrt{m/D_{22}}$ of the $[0/90]_s$ laminated composite rectangular plate with SSSS boundary condition

a/b	$E_1 = 10E_2$		$E_1 = 20E_2$		$E_1 = 40E_2$	
	Ref [8]	Present	Ref [8]	Present	Ref [8]	Present
0.5	8.515	8.515	9.355	9.355	9.917	9.917
1.0	2.519	2.519	2.638	2.638	2.721	2.721
1.5	1.531	1.531	1.536	1.536	1.539	1.539
2.0	1.246	1.246	1.229	1.229	1.216	1.216
2.5	1.138	1.138	1.119	1.119	1.105	1.105
3.0	1.087	1.087	1.071	1.071	1.059	1.059

Table 5. Convergence of mesh size of the (1, 1) natural frequencies of the $[0/90]_s$ laminated composite square plate

Boundary condition	Mesh size					
	2x2	4x4	8x8	16x16	32x32	64x64
CCCC	89.087	100.588	101.086	101.504	101.558	101.558
CCCS	86.286	95.144	96.431	96.872	96.927	96.927
CCSS	67.547	71.164	71.726	71.924	71.949	71.949
CFCC	78.816	87.636	89.978	90.526	90.605	90.605
CFCF	78.110	87.030	88.901	89.349	89.405	89.405
CFCS	78.664	87.383	89.556	90.062	90.132	90.132
CFSC	57.985	62.094	63.124	63.374	63.404	63.404
CFSF	57.651	60.838	61.570	61.748	61.771	61.771
CFSS	57.754	61.583	62.498	62.718	62.742	62.742
CSCS	83.499	91.839	93.552	94.004	94.059	94.059
CSSS	63.207	67.047	67.846	68.048	68.072	68.072
FSCS	24.904	25.671	25.971	26.047	26.063	26.063
FSFS	17.935	17.965	17.981	17.986	17.986	17.986
SSFS	19.866	20.534	20.702	20.744	20.752	20.752
SSSS	45.264	47.607	48.044	48.133	48.144	48.144

Table 6. Natural frequencies of the $[0]_4$ laminated composite rectangular plate

(1) Boundary condition CCCC

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	146.130	145.397	102.081	101.592	96.696	96.295
(1, 2)	302.299	299.966	127.192	126.600	100.315	99.831
(1, 3)	555.059	548.308	178.585	177.714	107.671	107.158
(2, 1)	298.399	295.269	269.499	266.664	265.060	262.274
(2, 2)	423.333	418.502	288.738	285.706	268.635	265.806
(2, 3)	653.828	644.244	327.230	323.824	275.099	272.198
(3, 1)	545.963	536.114	522.745	513.407	518.569	509.310
(3, 2)	647.953	636.133	540.051	530.388	522.185	512.838
(3, 3)	847.768	830.895	572.655	562.375	528.440	518.980

(2) Boundary condition CCCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	127.192	126.462	100.315	99.810	96.608	96.140
(1, 2)	256.654	254.955	120.275	119.661	99.656	99.166
(1, 3)	485.547	480.822	164.861	164.035	106.085	105.561
(2, 1)	288.738	285.501	268.635	265.783	264.972	262.184
(2, 2)	389.848	385.232	285.121	282.037	268.276	265.436
(2, 3)	593.597	585.588	318.92	315.462	274.271	271.335
(3, 1)	540.051	530.183	522.18	512.825	518.516	509.247
(3, 2)	624.615	612.754	537.713	527.981	521.918	512.579
(3, 3)	799.550	783.608	567.276	551.803	527.872	518.375

(3) Boundary condition CCSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	105.833	105.243	72.184	71.920	67.1591	66.965
(1, 2)	245.820	244.214	97.462	97.007	71.280	71.035
(1, 3)	479.172	474.492	148.400	107.666	79.732	79.414
(2, 1)	242.543	240.403	219.274	217.680	215.031	213.535
(2, 2)	354.715	350.935	238.289	236.371	218.841	217.268
(2, 3)	569.048	561.692	276.635	274.196	225.751	224.050
(3, 1)	471.233	464.645	451.405	445.473	447.343	441.525
(3, 2)	563.812	554.870	468.578	462.184	451.129	445.196
(3, 3)	750.246	736.857	501.090	493.857	457.681	451.580

Table 6. (continued)

(4) Boundary condition CFCC

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	100.644	100.090	96.647	96.161	95.930	95.458
(1, 2)	156.852	155.620	105.935	105.342	97.687	97.197
(1, 3)	311.322	308.453	132.549	131.718	101.911	101.381
(2, 1)	268.865	265.950	264.940	262.115	264.137	261.333
(2, 2)	317.926	313.990	274.476	271.486	266.241	263.402
(2, 3)	447.216	440.793	297.021	293.648	270.708	267.792
(3, 1)	522.206	512.763	518.409	509.081	517.594	508.283
(3, 2)	567.928	557.136	528.024	518.431	519.820	510.460
(3, 3)	680.763	666.671	549.166	539.005	524.408	514.915

(5) Boundary condition CFCF

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	95.749	95.211	95.743	95.242	95.744	95.261
(1, 2)	106.432	105.693	98.460	97.930	96.407	95.921
(1, 3)	167.132	165.526	109.758	109.042	98.757	98.237
(2, 1)	263.922	260.964	263.921	261.044	263.921	261.085
(2, 2)	278.501	275.256	267.520	264.605	264.787	261.945
(2, 3)	336.718	331.867	280.142	276.937	267.729	264.826
(3, 1)	517.401	507.858	517.385	507.975	517.388	508.023
(3, 2)	533.275	523.320	521.267	511.767	518.302	508.900
(3, 3)	591.097	579.101	534.415	524.491	521.484	511.979

(6) Boundary condition CFCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	98.460	97.919	96.407	95.922	95.902	95.432
(1, 2)	137.590	136.492	103.550	102.962	97.419	96.929
(1, 3)	266.094	263.893	125.300	124.488	101.101	100.568
(2, 1)	267.520	264.602	264.787	261.963	264.119	261.316
(2, 2)	305.380	301.505	272.993	269.991	266.067	263.228
(2, 3)	412.832	406.862	292.648	289.260	270.211	267.288
(3, 1)	521.267	511.810	518.295	508.974	517.591	508.272
(3, 2)	558.869	548.060	526.952	517.356	519.709	510.334
(3, 3)	654.862	641.040	546.097	535.900	524.052	514.554

Table 6. (continued)

(7) Boundary condition CFSC

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	72.610	72.293	67.215	66.999	66.234	66.049
(1, 2)	139.205	138.057	79.481	79.082	68.618	68.396
(1, 3)	301.769	299.031	111.804	111.093	74.255	73.955
(2, 1)	219.557	217.910	214.998	213.462	214.068	212.567
(2, 2)	275.258	272.268	225.998	224.230	216.493	214.941
(2, 3)	415.360	409.687	251.767	249.520	221.648	219.985
(3, 1)	451.463	445.440	447.247	441.358	446.343	440.490
(3, 2)	501.785	494.174	457.858	451.664	448.810	442.883
(3, 3)	623.742	612.383	481.156	474.313	453.865	447.795

(8) Boundary condition CFSF

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	65.980	65.727	65.980	65.760	65.980	65.781
(1, 2)	79.834	79.245	69.625	69.352	66.880	66.676
(1, 3)	149.981	148.437	84.179	83.628	70.035	69.771
(2, 1)	213.819	212.153	213.818	212.238	213.819	212.281
(2, 2)	230.45	228.388	217.953	216.323	214.815	213.277
(2, 3)	295.472	291.526	232.396	230.388	218.197	216.576
(3, 1)	446.125	440.018	446.113	440.148	446.116	440.202
(3, 2)	463.589	457.000	450.394	444.340	447.121	441.187
(3, 3)	526.653	517.535	464.876	458.312	448.080	444.572

(9) Boundary condition CFSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	69.625	69.330	66.880	66.666	66.197	66.012
(1, 2)	117.344	116.325	76.345	75.957	68.248	68.025
(1, 3)	255.010	252.805	103.208	102.520	73.162	72.862
(2, 1)	217.523	216.307	214.815	213.281	214.048	212.546
(2, 2)	260.969	258.200	224.245	222.470	216.287	214.734
(2, 3)	378.438	373.259	246.702	244.436	221.051	219.388
(3, 1)	450.394	444.360	447.124	441.237	446.324	440.477
(3, 2)	491.695	484.121	456.672	450.458	448.665	442.740
(3, 3)	595.709	585.020	477.710	470.837	453.468	447.385

Table 6. (continued)

(10) Boundary condition CSCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	114.779	114.051	99.113	98.604	96.485	96.017
(1, 2)	217.513	216.163	114.779	114.149	99.113	98.619
(1, 3)	421.958	418.724	152.854	152.046	104.728	104.194
(2, 1)	282.135	278.869	267.948	265.087	264.225	262.104
(2, 2)	362.621	358.081	282.135	279.016	267.948	265.106
(2, 3)	540.018	533.026	311.768	308.252	273.490	270.560
(3, 1)	535.686	525.759	521.686	512.332	518.448	509.188
(3, 2)	605.709	593.804	535.686	525.915	521.686	512.341
(3, 3)	757.555	742.228	562.335	551.913	527.322	517.827

(11) Boundary condition CSSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	90.654	90.069	70.534	70.266	66.987	66.793
(1, 2)	204.740	203.455	90.655	90.183	70.534	70.285
(1, 3)	414.685	411.499	135.004	134.292	77.942	77.616
(2, 1)	234.777	232.659	218.455	216.855	214.937	213.436
(2, 2)	324.789	321.127	234.777	232.823	218.455	216.876
(2, 3)	513.130	506.817	268.453	265.971	224.838	223.134
(3, 1)	466.287	459.682	450.866	444.915	447.276	441.459
(3, 2)	542.997	534.098	466.287	459.859	450.866	444.928
(3, 3)	705.590	692.887	495.624	488.405	457.077	450.964

(12) Boundary condition FSCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	51.696	51.479	22.408	22.321	16.573	16.543
(1, 2)	178.770	178.132	51.696	51.545	22.408	22.341
(1, 3)	393.258	390.834	104.156	103.887	34.054	33.960
(2, 1)	125.740	124.626	101.457	100.940	96.027	95.612
(2, 2)	240.372	238.154	125.740	124.833	101.457	100.935
(2, 3)	447.414	443.104	171.454	170.131	111.146	110.474
(3, 1)	290.303	287.215	270.198	267.831	265.542	263.160
(3, 2)	386.963	381.572	290.304	287.360	270.198	267.682
(3, 3)	575.454	566.813	328.028	324.248	278.302	275.571

Table 6. (continued)

(13) Boundary condition FSFS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	42.969	42.940	10.729	10.733	2.682	2.683
(1, 2)	172.013	171.569	42.969	42.940	10.735	10.733
(1, 3)	387.159	384.950	96.727	96.586	24.162	24.153
(2, 1)	60.005	59.462	23.613	23.393	10.865	10.775
(2, 2)	190.950	189.852	60.005	59.635	23.613	23.446
(2, 3)	406.362	403.381	115.158	114.61	39.690	39.459
(3, 1)	136.407	135.033	105.802	105.32	98.249	97.913
(3, 2)	261.636	258.604	136.407	135.311	105.802	105.292
(3, 3)	473.670	468.161	188.229	186.475	118.489	117.734

(14) Boundary condition SSFS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	47.737	47.552	15.001	14.916	5.903	5.864
(1, 2)	176.906	176.283	47.737	47.604	15.001	14.938
(1, 3)	392.079	389.664	101.590	101.328	28.789	28.707
(2, 1)	102.365	101.423	74.511	74.203	68.061	67.895
(2, 2)	224.713	222.705	102.365	101.649	74.511	74.221
(2, 3)	436.568	432.438	152.234	151.105	85.799	85.342
(3, 1)	243.464	241.356	220.832	219.563	215.544	214.333
(3, 2)	348.803	344.303	243.464	241.546	220.832	219.466
(3, 3)	546.216	538.45	285.155	282.340	229.993	228.382

(15) Boundary condition SSSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	73.313	72.782	48.313	48.143	43.585	43.516
(1, 2)	195.924	194.707	73.313	72.905	48.313	48.166
(1, 3)	409.306	406.184	122.783	122.131	57.859	57.613
(2, 1)	193.253	191.815	174.343	173.533	170.246	169.562
(2, 2)	293.255	290.111	193.253	192.000	174.343	173.558
(2, 3)	491.133	485.278	231.440	229.562	181.765	180.816
(3, 1)	402.477	398.250	385.382	381.925	381.416	378.103
(3, 2)	486.526	479.726	402.477	398.449	385.382	381.941
(3, 3)	659.824	648.987	434.818	429.885	392.272	388.619

Table 7. Natural frequencies of the $[0/90]_s$ laminated composite rectangular plate

(1) Boundary condition CCCC

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	193.481	192.319	102.125	101.558	91.450	90.899
(1, 2)	469.651	464.362	152.510	151.765	97.518	96.953
(1, 3)	898.655	881.793	249.217	247.785	111.734	111.147
(2, 1)	313.322	309.597	256.188	252.850	249.409	246.115
(2, 2)	549.079	541.605	288.967	285.445	254.144	250.807
(2, 3)	959.142	940.036	361.009	356.951	263.753	260.352
(3, 1)	533.285	521.866	493.182	482.191	487.584	476.672
(3, 2)	720.807	706.154	518.596	507.273	491.953	480.954
(3, 3)	1090.160	1064.000	573.029	561.292	500.115	488.978

(2) Boundary condition CCCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	152.510	151.625	97.518	96.927	91.089	90.534
(1, 2)	387.666	384.487	136.904	136.197	95.912	95.336
(1, 3)	778.985	768.074	222.186	221.039	107.889	107.290
(2, 1)	288.967	285.182	254.144	250.773	249.227	245.930
(2, 2)	478.949	473.995	280.300	276.711	253.354	250.015
(2, 3)	846.367	832.686	341.883	337.882	261.669	258.423
(3, 1)	518.596	506.985	491.953	480.927	487.462	476.548
(3, 2)	666.875	652.987	513.249	501.830	491.447	480.437
(3, 3)	990.124	968.900	560.252	548.256	498.895	487.745

(3) Boundary condition CCSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	137.269	136.574	72.242	71.949	63.536	63.310
(1, 2)	381.317	378.238	119.737	119.247	70.075	69.806
(1, 3)	775.394	764.560	211.588	210.591	85.471	85.143
(2, 1)	248.488	246.061	208.084	206.211	202.334	200.563
(2, 2)	453.917	449.041	238.428	236.248	207.152	205.306
(2, 3)	831.045	818.053	307.398	304.631	217.091	215.086
(3, 1)	455.339	447.527	425.601	418.623	419.229	413.743
(3, 2)	616.821	606.346	449.298	441.847	425.031	418.053
(3, 3)	954.976	936.472	501.235	493.040	433.317	426.167

Table 7. (continued)

(4) Boundary condition CFCC

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	97.635	96.982	91.175	90.605	90.202	89.648
(1, 2)	200.611	199.067	106.120	105.409	92.514	91.935
(1, 3)	475.593	470.517	157.102	156.112	99.276	98.646
(2, 1)	254.611	251.173	249.265	245.944	248.278	244.981
(2, 2)	332.550	327.751	261.871	258.325	250.730	247.387
(2, 3)	568.059	559.489	297.811	293.799	256.467	253.030
(3, 1)	492.296	481.197	487.439	476.489	486.647	475.549
(3, 2)	557.483	544.619	499.170	487.891	488.975	477.976
(3, 3)	751.789	734.541	528.782	561.800	494.415	483.262

(5) Boundary condition CFCF

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	89.949	89.392	89.974	89.405	89.975	89.416
(1, 2)	101.379	100.501	92.923	92.303	90.699	90.131
(1, 3)	209.883	208.139	109.979	109.134	93.633	93.017
(2, 1)	248.018	244.635	248.018	244.678	248.019	244.703
(2, 2)	263.801	259.951	251.944	248.524	248.972	245.642
(2, 3)	350.810	345.196	267.746	263.932	252.300	248.888
(3, 1)	486.070	475.161	486.216	475.229	486.223	475.262
(3, 2)	503.507	491.831	490.453	479.334	487.231	476.245
(3, 3)	580.894	566.391	505.892	494.201	490.712	479.597

(6) Boundary condition CFCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	92.923	92.288	90.699	90.132	90.150	89.597
(1, 2)	161.439	160.225	100.989	100.292	91.979	91.400
(1, 3)	393.984	390.870	141.452	140.530	97.796	96.866
(2, 1)	251.944	248.513	248.972	245.651	248.245	244.948
(2, 2)	306.302	301.720	258.907	255.360	250.402	247.057
(2, 3)	498.276	491.216	288.384	284.383	255.468	252.031
(3, 1)	490.453	479.348	487.231	476.283	486.449	475.526
(3, 2)	539.019	526.260	497.116	485.824	488.739	477.741
(3, 3)	696.579	680.394	522.430	510.418	493.734	482.570

Table 7. (continued)

(7) Boundary condition CFSC

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	72.342	71.979	63.653	63.404	62.316	62.100
(1, 2)	188.598	187.222	83.137	82.688	65.468	65.211
(1, 3)	470.997	465.073	142.015	141.225	74.453	74.117
(2, 1)	208.623	206.702	202.383	200.583	201.235	199.472
(2, 2)	296.625	293.078	217.008	214.935	204.075	202.255
(2, 3)	545.876	538.326	258.099	255.479	210.707	208.803
(3, 1)	425.999	418.926	420.594	413.685	419.189	412.651
(3, 2)	497.904	488.802	433.606	426.333	422.293	415.336
(3, 3)	706.154	692.229	466.400	458.186	428.323	421.164

(8) Boundary condition CFSF

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	62.005	61.757	62.005	61.771	62.005	61.782
(1, 2)	76.702	76.058	65.950	65.645	62.986	62.753
(1, 3)	197.907	196.358	87.702	87.110	66.936	66.636
(2, 1)	200.839	199.086	200.934	199.125	200.935	199.152
(2, 2)	218.906	216.506	205.443	203.547	202.032	200.237
(2, 3)	315.813	311.367	223.598	221.231	205.869	203.979
(3, 1)	419.234	412.221	419.228	412.298	419.241	412.332
(3, 2)	438.236	430.514	423.906	416.841	420.354	413.430
(3, 3)	522.780	512.084	440.948	433.274	421.378	417.137

(9) Boundary condition CFSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	65.950	65.618	62.986	62.742	62.243	62.028
(1, 2)	146.474	145.460	76.576	76.148	64.727	64.472
(1, 3)	387.312	384.362	124.569	123.863	72.096	71.762
(2, 1)	205.443	203.523	202.032	200.234	201.195	199.429
(2, 2)	267.255	263.985	213.507	211.425	203.683	201.862
(2, 3)	473.111	466.783	247.305	244.696	209.555	207.618
(3, 1)	423.906	416.837	420.357	413.452	419.208	412.625
(3, 2)	477.432	468.889	431.274	424.002	422.027	415.070
(3, 3)	647.417	634.742	459.289	451.260	427.532	420.418

Table 7. (continued)

(10) Boundary condition CSCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	124.247	123.437	94.660	94.059	90.835	90.279
(1, 2)	315.594	313.694	124.247	123.545	94.660	94.077
(1, 3)	668.684	662.073	197.927	196.964	104.649	104.037
(2, 1)	273.482	269.661	252.712	249.328	249.079	245.779
(2, 2)	420.358	415.174	273.482	269.842	252.712	249.352
(2, 3)	744.034	734.132	325.479	321.481	260.211	256.764
(3, 1)	508.925	497.250	491.006	479.964	487.354	476.440
(3, 2)	623.666	610.057	508.925	497.450	491.006	479.984
(3, 3)	901.507	883.204	549.377	537.294	497.837	486.660

(11) Boundary condition CSSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	105.089	104.501	68.370	68.072	63.178	62.952
(1, 2)	307.847	306.072	105.089	104.619	68.370	68.095
(1, 3)	664.551	658.038	186.008	185.207	81.370	81.033
(2, 1)	230.460	228.042	206.365	204.488	202.156	200.383
(2, 2)	391.801	387.709	230.460	228.243	206.366	204.515
(2, 3)	726.720	717.624	289.151	286.432	215.090	213.121
(3, 1)	444.409	436.686	424.522	417.518	420.486	413.621
(3, 2)	570.055	560.030	444.409	436.909	424.520	417.540
(3, 3)	863.000	847.658	489.148	480.905	432.105	424.936

(12) Boundary condition FSCS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	77.306	77.092	26.144	26.063	16.162	16.131
(1, 2)	292.035	290.730	77.306	77.142	26.144	26.083
(1, 3)	651.767	645.784	166.421	165.958	46.757	46.667
(2, 1)	134.512	133.347	97.272	96.700	90.527	90.061
(2, 2)	331.637	329.028	134.512	133.577	97.272	96.705
(2, 3)	684.864	677.398	212.575	211.179	111.199	110.494
(3, 1)	281.887	278.383	255.083	252.356	249.755	247.033
(3, 2)	441.442	435.532	281.886	278.585	255.083	252.227
(3, 3)	769.934	758.631	341.051	336.930	265.252	262.185

Table 7. (continued)

(13) Boundary condition FSFS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	72.036	71.911	18.009	17.986	4.502	4.497
(1, 2)	288.147	286.870	72.036	71.911	18.009	17.986
(1, 3)	648.331	642.336	162.082	161.629	40.521	40.462
(2, 1)	83.270	82.790	27.674	27.480	11.448	11.368
(2, 2)	299.700	298.050	83.270	82.934	27.674	27.533
(2, 3)	659.780	653.365	173.610	172.935	51.282	51.084
(3, 1)	144.436	143.043	101.638	101.141	92.706	92.361
(3, 2)	347.366	344.075	144.436	143.358	101.638	101.129
(3, 3)	702.318	693.893	226.291	224.535	118.408	117.669

(14) Boundary condition SSFS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	74.925	74.743	20.817	20.752	6.918	6.887
(1, 2)	290.979	289.694	74.925	74.781	20.817	20.770
(1, 3)	651.105	645.140	164.944	164.497	43.403	43.331
(2, 1)	115.310	114.393	72.463	72.145	64.315	64.140
(2, 2)	321.400	319.050	115.310	114.624	72.463	72.173
(2, 3)	678.360	671.119	198.802	197.654	88.909	88.472
(3, 1)	239.294	236.990	208.880	207.455	202.801	201.444
(3, 2)	411.835	407.033	239.294	237.237	208.880	207.376
(3, 3)	750.287	739.984	304.678	301.741	220.470	218.733

(15) Boundary condition SSSS

(i, j)	b = 0.5a		b = a		b = 2a	
	Present	FEM	Present	FEM	Present	FEM
(1, 1)	92.209	91.729	48.313	48.144	41.282	41.207
(1, 2)	302.581	300.897	92.209	91.846	48.313	48.172
(1, 3)	661.475	655.043	178.202	177.492	64.767	64.550
(2, 1)	193.252	191.688	165.130	164.200	160.189	159.386
(2, 2)	368.837	365.419	193.252	191.910	165.130	164.232
(2, 3)	712.813	704.268	259.070	257.126	175.379	174.334
(3, 1)	385.271	380.406	363.096	359.037	358.611	354.703
(3, 2)	522.682	515.204	385.271	380.653	363.096	359.062
(3, 3)	829.881	816.669	434.817	429.336	371.543	367.306

CONCLUSIONS

The extended Kantorovich method has been employed to analyse the free vibration of a symmetrically laminated composite rectangular plate with various boundary conditions. The frequency parameters of an isotropic square plate, an orthotropic rectangular plate and a cross-ply symmetrically laminated composite rectangular plate were evaluated in order to compare with previously published results on the topic. A good agreement with the results verifies the accuracy of the present method. The natural frequencies of symmetrically laminated composite rectangular plates were also evaluated by the present method and the finite element method. A good agreement with the finite element method verifies that the present method can be used to evaluate the natural frequencies of unidirectional 0°, unidirectional 90° and cross-ply symmetrically laminated composite rectangular plates.

The advantages of the present method are as follows.

(1) The rectangular plate vibration problem can be resolved by the use of ordinary differential equations.

(2) Any arbitrary function can be used as an initial trial function in the iterative calculation. An initial trial function which satisfies the boundary conditions will make the convergence of the final solution rapid.

(3) The final solution converges rapidly; therefore, the final solution can be obtained from the fourth iteration.

(4) The particular natural frequency can be obtained separately with good accuracy. In the Rayleigh-Ritz method, a larger number of shape functions must be used if a higher natural frequency is required.

However, due to the fact that the present method considers individual modes as a product of separable functions in the x and y coordinate directions, it cannot provide for modes with curved nodal lines.

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