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Communication

Modelling of natural convection along isothermal plates and in channels using diffusion velocity method

Ademola A. Dare* and Moses O. Petinrin

Department of Mechanical Engineering,University of Ibadan, Nigeria

* Corresponding author e-mail: ademola.dare@mail.ui.edu.ng

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Abstract: The diffusion velocity method (DVM), a version of vortex element method (VEM), was used to model the steady state, laminar natural convection flows along isothermal vertical plates and in isothermal vertical channels. For each case, numerical models were developed using DVM from the vorticity transport equation and the energy equation. This study shows that the diffusion velocity method is a viable numerical tool at modelling not only fluid flow problems but also the heat transfer problems.

Keywords: natural convection, isothermal plates, isothermal channels, diffusion velocity method

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Introduction

Many engineering problems are represented by non-linear, partial differential equations. In order to simplify these problems which may be difficult to solve analytically, different numerical methods are used to analyse the problems. Some of these numerical methods are: the finite difference method, the finite element method and the Monte Carlo method.

Another numerical method that has been used successfully to study and analyse very complex, unsteady fluid flows and thermal engineering problems is the vortex element method (VEM). This method is simple to implement as it requires simple mathematics compared to other numerical methods that may require many mathematical operations such as variational calculus and matrix inversions [1].

An engineering problem involves flows of gases or liquids over solid bodies. Examples include: air flows over cars and aeroplanes; wind blowing over bridges and buildings; and sea waves

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slashing against the supporting columns of an off-shore oil rig. Often these flows do not follow the contour of the solid surface completely, but may be formed separate from it, e.g. a wake behind a ship. Such separated flows are difficult to handle by conventional numerical schemes. The main reason for basing the numerical method on the vorticity is that typically only a small portion of the flow contains vorticity. This can lead to significant savings in storage and computational effort [2]. A number of numerical schemes model diffusion by vortex methods without using a mesh, which is based on the Lagrangian approach. However, the VEM is grid-free in implementation [2-3].

Recent applications of the vortex methods based on the Biot-Savart law have been extended to numerical prediction of unsteady and complex characteristics of various flows related to difficult engineering problems concerning flow-induced vibration, off-design operation of fluid machinery, automobile aerodynamics, biological fluid dynamics, etc. Kamemoto [4] described how VEM, with its simple algorithms based on physics of flow, has been used to find the virtual operation of fluid machinery (pumps and water turbines), and the calculated flows around bluff bodies, an oscillating airfoil and a swimming fish.

Cheng et al. [5] developed a hybrid vortex method to simulate two-dimensional viscous incompressible flows over a bluff body. It was based on a combination of the diffusion-vortex method and the vortex-in-cell method whereby the flow field is divided into two regions. In the region near the body surface the diffusion-vortex method is used to solve the Navier-Stokes equations while the vortex-in-cell method is used in the exterior domain. They compared the results obtained with those from the finite difference method and other vortex methods and experiments. Which showed that the method is well adapted to calculate two-dimensional external flows at high Reynolds number.

Ghoniem and Oppenheim [6] applied random-walk vortex method to an assortment of problems of diffusion of momentum and energy in one dimension as well as heat conduction in two dimensions in order to assess its validity and accuracy. The numerical solutions obtained were found to be in good agreement with the exact solution except for a statistical error introduced by using a finite number of elements. They claimed further that the error could be reduced by increasing the number of elements or by using ensemble averaging over a number of solutions.

The concept of the diffusion velocity method (DVM), a version of the VEM, was extensively discussed by Ogami [7]. The velocity is defined in order that the vorticity is conserved in the transfer of diffusion process as it is so in the convection process. Unlike the other vortex element methods, this technique handles vorticity equation in a deterministic manner by calculating the diffusion velocity to account for diffusion in the flow [8]. Ogami [8] used the DVM to simulate the diffusion of vorticity and temperature from one-dimensional vorticity transport equation and compared the results with the analytical solutions. The method was successfully used to treat the diffusion equation (Re = 0), the boundary layer and two-dimensional flows around a circular cylinder ($Re = 0.1 \sim 10^7$), aerofoil, the Burger equation, and the equations of incompressible fluid.

This study employs the diffusion velocity technique to model the natural convection along isothermal plates and in isothermal channels. The channels, consisting of two parallel vertical plates, are asymmetrically heated at uniform wall temperature. The Nusselt numbers are obtained with the velocity and the temperature distributions.

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The Governing Equations

The continuity equation and the Navier-Stokes equations for Newtonian, two-dimensional, incompressible flow are presented as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+f_x+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+f_y+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}) \tag{3}$$

Also, the energy equation, assuming no viscous dissipation or thermal generation, is

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}) \tag{4}$$

Variables u and v are the velocity components in x and y directions respectively, f_x and f_y are the components of the gravitational acceleration in the x and y directions respectively, T is the fluid temperature, P is the fluid pressure, t is the time, α is the fluid thermal diffusivity, ν is the kinematic viscosity, and ρ is the fluid density [9-10].

Following the Lagrangian scheme, the alternative expression of the governing equations of viscous and incompressible flow gives the vorticity transport equation as

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+g\beta\frac{\partial T}{\partial y} \tag{5}$$

where vorticity, $w=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ (6)

Considering the diffusive term only in the vorticity equation, then

$$\frac{\partial w}{\partial t}=\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}) \tag{7}$$

The solution to equation (7) was given by Batchelor [11] as

$$w(r,t)=\frac{\Gamma}{4\pi\nu t}e^{(-r^2/4\nu t)} \tag{8}$$

where $r=\sqrt{(x^2+y^2)}$ and Γ is the vortex strength or circulation [12]

Numerical Formulations

The initial velocity, γ, is induced by a buoyancy effect created by the temperature difference between the plate and the fluid in contact with its surface, and is given as

$$\gamma=-g\beta\frac{\partial T}{\partial y}\Delta t\Delta s \tag{9}$$

where Δs is the elemental length, Δt is the time step, β is the volumetric thermal expansion coefficient, and g is acceleration due to gravity.

Velocity and temperature vortices with strengths Γ and Θ respectively, which are created on one elemental surface of m elements into which a plate is divided are given by

$$\Gamma_q=\gamma\Delta s \tag{10}$$

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$$\Theta_q = Tu_{Tq}\Delta t = -\alpha\Delta t \left.\frac{\partial T}{\partial y}\right|_{y=\delta} \quad (11)$$

where $m = \frac{L}{\Delta s}$; $\delta = \sqrt{\alpha \Delta t}$; q = 1, 2, 3,...m; and L is the plate length.

The elemental length has a great influence on the strength of the vortices. The vortices are initially distributed and separated at distance Δs before diffusing. The vorticity of each velocity vortex and the temperature of each temperature particle are respectively given by

$$w(r,\Delta t) = \frac{\Gamma_i}{4\pi\nu\Delta t}\left[e^{(-r_1^2/4\nu\Delta t)} - e^{(-r_2^2/4\nu\Delta t)}\right] \quad (12)$$

$$T(r,\Delta t) = \frac{\Theta_i}{4\pi\alpha\Delta t}\left[e^{(-r_1^2/4\alpha\Delta t)} - e^{(-r_2^2/4\alpha\Delta t)}\right] \quad (13)$$

For flow on a flat plate, each velocity vortex or temperature particle has a corresponding negative image. The distances between a vortex and the inducing vortex and its corresponding image can be deduced respectively by

$$r_1(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \text{ and } r_2(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j + y_i)^2}$$

Here $i = j$ = 1, 2, 3,…N, and N is the number of vortices on the plate surface and in space. However, i is not equal to j most of the times. Figure 1 shows an inducing vortex with its image and how r_1 and r_2 are determined from a vortex of interest.

The divergences of equations (12) and (13) are respectively $\nabla.w$ and $\nabla.T$, which can be written in the forms:

$$\nabla.w = \frac{\partial w}{\partial xj} + \frac{\partial w}{\partial y_j} \quad (14)$$

$$\text{and } \nabla.T = \frac{\partial T}{\partial xj} + \frac{\partial T}{\partial y_j} \quad (15)$$

The diffusion velocity of each velocity vortex is induced by all the velocity vortices and their corresponding images. Also, the diffusion velocity of temperature particle is induced by all the temperature particles and their corresponding images. As discussed by Ogami [8], the diffusion velocities of each velocity vortex and temperature particle are now respectively given as

$$u_{wj} = \sum_{i=1}^{N}[-\frac{\nu}{w}\nabla.w] \quad (16)$$

$$u_{Tj} = \sum_{i=1}^{N}[-\frac{\alpha}{T}\nabla.T] \quad (17)$$

The diffusion distances of each velocity vortex and temperature particle can be expressed respectively as

$$d_{vj} = u_{wj} \times \Delta t \quad (18)$$

$$\text{and } d_{Tj} = u_{Tj} \times \Delta t \quad (19)$$

These distances are added to the original positions to move the vortex and particle to other positions. For convection, each vortex is repositioned to a new location using the induced velocities (u, v) by neighbouring velocity vortices.

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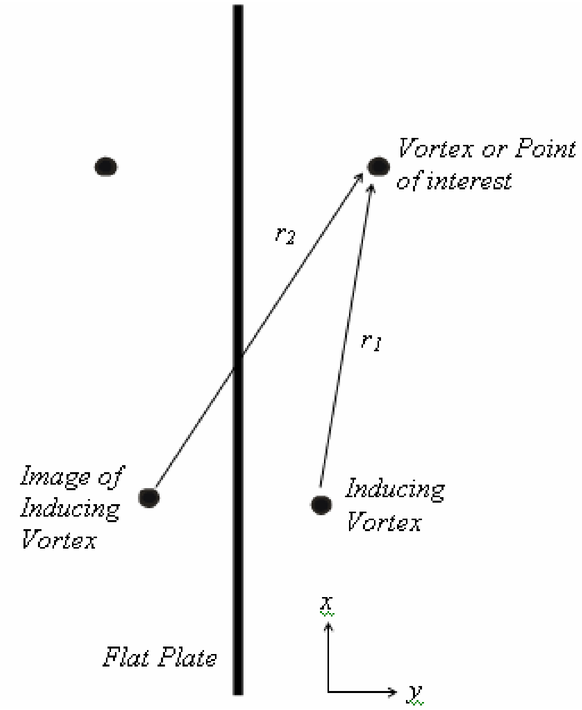

Figure 1. Interaction of vortices on a vertical flat plate

The slip flow condition is met when the velocity on each elemental surface is approximately zero and the numerically calculated temperature is approximately equal to the initial temperature of the plate. To check for the slip flow condition, the velocity on the elemental surface can be determined as

$$v_e = \sum_{i=1}^{N} \left[\frac{\Gamma_i}{2\pi r_1} - \frac{\Gamma_i}{2\pi r_2} \right] \tag{20}$$

The corresponding temperature on the surface is determined as

$$T_e = \sum_{i=1}^{N} \frac{\Theta_i}{4\pi\alpha\Delta t} \left(e^{\left(-r_1^2 / 4\alpha\Delta t\right)} - e^{\left(-r_2^2 / 4\alpha\Delta t\right)} \right) + T_B \tag{21}$$

The practical substitution for the given boundary condition is

$$T_B = T_w erfc\left(\frac{y_j}{2\sqrt{\alpha\Delta t}}\right) \tag{22}$$

where T_w is the wall temperature. At each time step, the strength of a vortex is increased by adding $-g\beta \frac{\partial T}{\partial y} \Delta t \Delta s$ to equation (9) and T_e is substituted for T in equation (11) until the slip flow condition is met.

When a vortex is very close to the wall or an inducing vortex, i.e. when $r < \frac{\Delta s}{\pi}$ (r can be r_1 or r_2), then $\frac{\Gamma_i r}{2\pi r_0^2}$ replaces $\frac{\Gamma_i}{2\pi r}$ where it is appropriate. The minimal distance, r_0, is equal to $\frac{\Delta s}{\pi}$. Likewise for temperature particles, $\frac{\Theta_i r}{2\pi r_0^2}$ replaces $\frac{\Theta_i}{2\pi r}$.

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The velocity and temperature distributions at specific locations in the hydrodynamic and thermal boundary layers are determined when the slip flow condition is meant for the velocities and temperatures on the plate surfaces. The two components of the velocity distribution are obtained as

$$v_{mnx}(r) = \sum_{i=1}^{N}\left[\frac{\Gamma_i}{2\pi r_1}\left(\frac{y_j - y_i}{r_1}\right) - \frac{\Gamma_i}{2\pi r_2}\left(\frac{y_j + y_i}{r_2}\right)\right] \tag{23}$$

$$v_{mny}(r) = \sum_{i=1}^{N}\left[\frac{\Gamma_i}{2\pi r_2}\left(\frac{x_j - x_i}{r_2}\right) - \frac{\Gamma_i}{2\pi r_1}\left(\frac{x_j - x_i}{r_1}\right)\right] \tag{24}$$

The temperature at any point can be obtained from

$$T_{mn}(r) = \sum_{i=1}^{N}\left[\frac{\Theta_i}{2\pi r_1} - \frac{\Theta_i}{2\pi r_2}\right] + T_B \tag{25}$$

where m and n are the positions of the velocity or temperature along and normal to the plate respectively.

Natural Convection in Channels

The type of channels described here are two parallel plates placed vertically. The number of images of a vortex or a particle is infinite, but for easy computation the number of the images is reduced to eight; therefore, we have one positive vortex with four positive images and four negative images [1].

As discussed by Petinrin [13], the vorticity of each velocity vortex and the temperature of each temperature particle can then be respectively given as

$$w(r,\Delta t) = \frac{\Gamma_i}{4\pi\nu\Delta t}\Big[e^{\left(-r_1^2/4\nu\Delta t\right)} - e^{\left(-r_2^2/4\nu\Delta t\right)} + e^{\left(-r_3^2/4\nu\Delta t\right)} + e^{\left(-r_4^2/4\nu\Delta t\right)} + e^{\left(-r_5^2/4\nu\Delta t\right)} + e^{\left(-r_6^2/4\nu\Delta t\right)} - e^{\left(-r_7^2/4\nu\Delta t\right)} - e^{\left(-r_8^2/4\nu\Delta t\right)} - e^{\left(-r_9^2/4\nu\Delta t\right)}\Big] \tag{26}$$

$$T(r,\Delta t) = \frac{\Theta_i}{4\pi\alpha\Delta t}\Big[e^{\left(-r_1^2/4\alpha\Delta t\right)} - e^{\left(-r_2^2/4\alpha\Delta t\right)} + e^{\left(-r_3^2/4\alpha\Delta t\right)} + e^{\left(-r_4^2/4\alpha\Delta t\right)} + e^{\left(-r_5^2/4\alpha\Delta t\right)} + e^{\left(-r_6^2/4\alpha\Delta t\right)} - e^{\left(-r_7^2/4\alpha\Delta t\right)} - e^{\left(-r_8^2/4\alpha\Delta t\right)} - e^{\left(-r_9^2/4\alpha\Delta t\right)}\Big] \tag{27}$$

Also, the distances of each vortex from the inducing vortex and its images can be determined as r_1 to r_9, where h is the distance of the plates apart, i.e.

$$r_1(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}\ ,$$

$$r_2(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j + y_i)^2}\ ,$$

$$r_3(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - (y_i + 2h))^2}\ ,$$

$$r_4(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - (y_i + 4h))^2}\ ,$$

$$r_5(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - y_i + 2h)^2}\ ,$$

$$r_6(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - y_i + 4h)^2}\ ,$$

$$r_7(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (2h - y_j - y_i)^2}\ ,$$

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$$r_8(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (4h - y_j - y_i)^2} \ ,$$
$$r_9(x_j, y_j) = \sqrt{(x_j - x_i)^2 + (2h + y_j + y_i)^2} \qquad (28)$$

Results and Discussion

The results of the numerical analysis, which was solved with Visual Basic programming language, are automatically displayed on Microsoft Excel Workbook. The velocity and temperature distributions are displayed on the Sheet 1 and Sheet 2 of the workbook respectively.

The input parameters used to simulate natural convection on a single plate lying vertically are listed in Table 1. The fluid (air) thermophysical properties of the fliud are taken at film temperature.

Table 1. The input parameters for natural convection over a single plate

Length of the plates	0.5 m
Fluid (air) temperature	10^0C
Plate wall temperature	30^0C
Coefficient of thermal expansion	0.00341 (K^{-1})
Kinematic viscosity of air at 20^0C	0.0000157 m^2/s
Thermal diffusivity of air at 20^0C	0.000022 m^2/s

The logarithmic plot of Nusselt number against Rayleigh number is presented in Figure 2 by varying the plate wall temperature, *Tw*, from 40^0C to 120^0C while keeping the fluid (air) temperature constant at 10^0C. The fluid properties are taken at film temperature. It can be deduced that the Nusselt number increases with the Rayleigh number for a fixed plate length as in Table 2. The slope of the plot is 3.47 while the intercept is -28.22 on the log scale and 6.08E-29 on the normal scale.

Therefore, the relationship between Nusselt number and Rayleigh number is $Nu = (6.08E\text{-}29)Ra^{3.47}$. Comparing this relationship with $Nu = 0.59Ra^{0.25}$ as reported by Incropera and Dewitt [10], there is much deviation in the two correlations which may be due to convergence difficulties at the plate surface. The correlation coefficient of the plot is 0.9065.

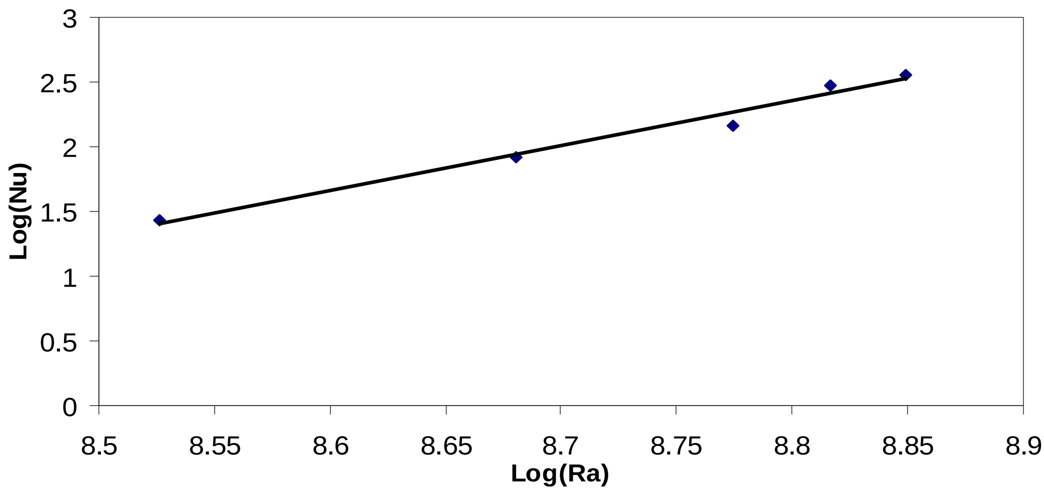

Figure 2. Logarithmic plot of Nusselt number (Nu) against Rayleigh number (Ra) for a vertical plate

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Table 2. Distribution of Rayleigh number with Nusselt number for natural convection over a single plate

Tw (^0C)	*Ra*	*Nu*	*Log(Ra)*	*Log(Nu)*
40	336123348	27.0125	8.5265	1.4316
60	479235019	84.0570	8.6805	1.9246
80	594247932	144.6929	8.7740	2.1604
100	655562931	297.4379	8.8166	2.4734
120	706467195	362.2841	8.8491	2.5590

The Nusselt number increases with the wall temperature as presented in Figure 3. However, the relationship between them is not linear owing to the Nusselt number being also dependent on some of the fluid properties, e.g. the thermal conductivity, which keeps changing as the wall temperature changes.

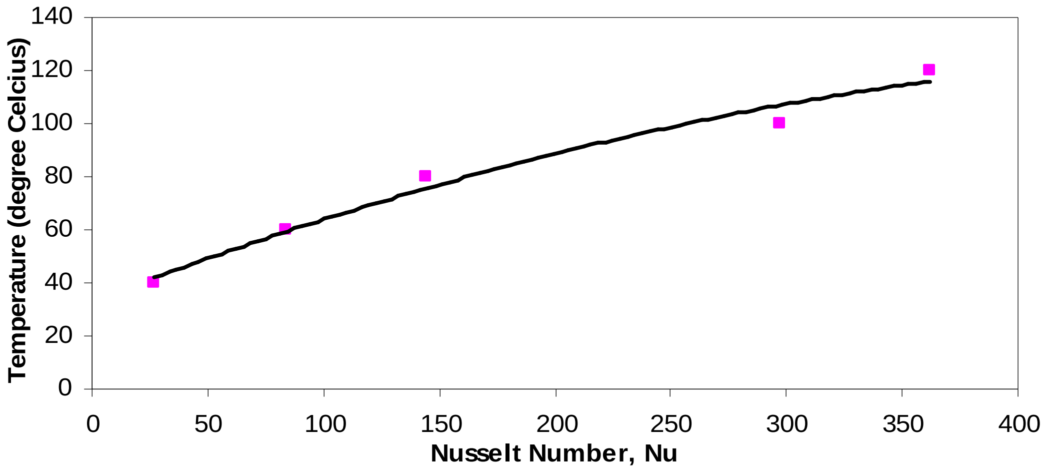

Figure 3. Effect of wall temperature on Nusselt number

The input parameters for the symmetrically heated, isothermal plates listed in Table 3 were used to simulate natural convection in channels. The channels are two parallel plates placed vertically.

The graph of logarithm of Nusselt number against logarithm of Rayleigh number in Figure 4 is obtained by varying the magnitude of the space, *S*, between the channel plates from *S* = 0.1m to 0.3m at a step of 0.05m. Input parameters are fed from Table 3. The slope of the plot is 0.51 while the intercept is -0.75 on the log scale and 0.18 on the normal scale.

Therefore, the relationship between Nusselt number and Rayleigh number is $Nu = 0.18Ra^{0.51}$ (Figure 4). The Figure also shows a deviation in the Nusselt-Rayleigh relationship from other workers. Ofi and Hetherington [14] with finite element method obtained a relationship $Nu = 0.699Ra^{0.26}$; Bodoia and Osterle[15] obtained $Nu = 0.56Ra^{0.25}$; and Ogundare[1] obtained $Nu = 0.43Ra^{0.21}$.

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Table 3. Input parameters for natural convection in vertical channel

Length of plates	0.5 m
Gap between plates	0.1m
Fluid (air) temperature	10^0C
Wall temperature of first plate	60^0C
Wall temperature of second plate	60^0C
Coefficient of thermal expansion	0.00325
Kinematic viscosity of air at 35^0C	0.0000171 m^2/s
Thermal diffusivity of air at 35^0C	0.0000241 m^2/s

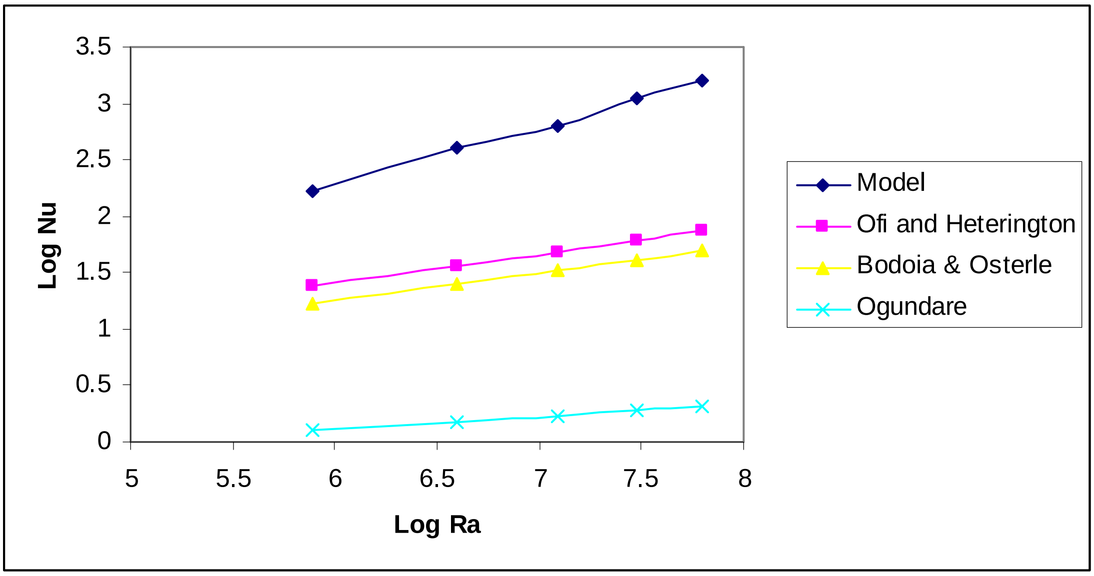

Figure 4. Logarithm plot of Nusselt number (Nu) against Rayleigh number (Ra) for vertical channel

Conclusions

Modelling of natural convection in isothermal vertical plates and channels has been successfully carried out with the diffusion velocity method, a version of the vortex element method. However, a large deviation recorded for correlation of Nusselt number and Rayleigh number for both the plate and the channel with existing correlations may stem from convergence difficulties encountered at the plate surface.

From the results obtained, it is established that as the wall temperature increases while keeping the mainstream fluid temperature constant, the thermal boundary layer thickness increases. The study has also established that the diffusion velocity method is a viable numerical tool capable of modelling fluid and heat transfer problems.

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