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Full Paper

Approximate solution of variational problems by an iterative decomposition method

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Abstract: A numerical method is presented for solving variational problems. The solution of an ordinary differential equation which arises from a variational problem is solved using the method. The solution is presented in the form of a fast convergent infinite series, the components of which are easily evaluated. Numerical examples are presented and results compared with exact solutions to show efficiency and accuracy.

Key words: variational problems, iterative decomposition, error

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Introduction

In several problems arising in mathematics, mechanics, geometry, mathematical physics, other branches of science and even economics, it is necessary to minimise or maximise a certain functional. Because of the important role of this class of problems, considerable attention has been given to them. These problems are called variational problems [1, 2, 3].

The simplest form of a variational problem is given as:

$$V[y(x)] = \int_{x_0}^{x_1} F\big(x, y(x), y'(x)\big)dx \qquad (1)$$

where V is the functional for which we need an extremum. To find the extremum of V, the boundary points of the admissible curve are of the following forms:

$$y(x_0) = \alpha, \quad y(x_1) = \beta \tag{2}$$

Several popular methods have been applied to solve variational problems. One of the most popular methods is the direct method, in which the variational problem is regarded as a limiting case of a finite number of variables. The direct method of Ritz and Galerkin has been applied for this class of problems [4, 1]. A piecewise constant solution was obtained using the Walsh series method [1]. Some orthogonal polynomials have been used to obtain continuous solutions of variational problems. The work of Hwang and Shih [2] is an example of this method. The Fourier series are applied by Razzaghi and Razzaghi [3] to obtain continuous solutions of variational problems. Taylor series are used for the same purpose [5, 3]. The necessary condition for the solution of equation (1) is to satisfy the Euler-Lagrange equation:

$$F_y - \frac{d}{dx} F_{y'} = 0 \tag{3}$$

with the boundary conditions (2). If the solution of the Euler-Lagrange equation satisfies the boundary bonditions, it is unique, and this unique extremal will be the solution of the given variational problem [4]. Therefore, another approach for solving the problem (1) is to find the solution of the ordinary differential equation (3) which satisfies the boundary conditions (2). The general form of the variational problem given by equation (1) is

$$V[y_1, y_2, \dots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n')dx \tag{4}$$

with the given boundary conditions for all functions:

$$\begin{aligned} &y_1(x_0) = \alpha_1 \quad y_2(x_0) = \alpha_2, \dots, y_n(x_0) = \alpha_n \\ &y_1(x_1) = \beta_1, \quad y_2(x_1) = \beta_2, \dots, \quad y_n(x_1) = \beta_n \end{aligned} \tag{5}$$

The Euler-Lagrange equation (3) then takes the form of a system of second order differential equations:

$$F_{y_i} - \frac{d}{dx} F_{y_i} = 0, \qquad i = 1, 2, \dots, n \tag{6}$$

with boundary conditions given by equation (5).

When the functional are dependent on higher-order derivatives, the variational problem is defined as:

$$V[y(x)] = \int_{x_0}^{x_1} F\left(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)\right) dx \tag{7}$$

with the boundary conditions given as:

$$\begin{aligned} &y(x_0) = \alpha_0, y'(x_0) = \alpha_1, y''(x_0) = \alpha_2, \dots, y^{(n-1)}(x_0) = \alpha_{n-1} \\ &y(x_1) = \beta_0, \ y'(x_1) = \beta_1, \ y''(x_1) = \beta_2, \dots, y^{(n-1)}(x_1) = \beta_{n-1} \end{aligned} \tag{8}$$

The function y(x) which extremises the functional given by equation (7) must then satisfy the Euler-Poisson equation:

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + \dots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0 \tag{9}$$

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Equation (9) with boundary conditions (8) forms a 2n-point boundary value problem, and its solution extremises (7).

In the present work, we find the solution of variational problems by applying an iterative decomposition method. The method being presented is useful for solving problems which can be written in the form:

$$y = P(y) + f \qquad (10)$$

where y is unknown, P is a non-linear operator and f is a given function. Equations of the form (10) occur in a wide variety of problems in the applied sciences. The proposed method searches for a solution in the form of a series, by decomposing the non-linear operator into a series, the terms of which are calculated recursively.

Iterative Decomposition Method

Consider the Euler-Lagrange equation (3). For the completion of the iterative decomposition, we can write equation (3) as:

$$L(y) - N(y) = f \qquad (11)$$

for $x_0 \leq x \leq x_1$, where $L = \frac{d^2}{dx^2}$ is the second-order derivative operator, N is a nonlinear operator which contains differential operators of order less than two, and f is a given function. Suppose the inverse operator L^{-1} exists and can be taken as the two-fold definite integral of the following form:

$$L^{-1}(\cdot) = \int_{x_0}^{x} \int_{x_0}^{t_2} (\cdot)\, dt_1\, dt_2 \qquad (12)$$

Applying the inverse operator L^{-1} to both sides of (11) , we have

$$L^{-1}L(y) = L^{-1}N(y) + L^{-1}f \qquad (13)$$

Thus, we have

$$y(x) - y(x_0) - xy'(x_0) = L^{-1}N(y) + L^{-1}f \qquad (14)$$

Thus, by the decomposition procedure, we construct the unknown function y(x) as the sum of the components of a decomposition series. The equation (14) is of the form:

$$y(x) = K(y) + c \qquad (15)$$

where c is a constant and K denotes the nonlinear term on the RHS of (14).

It is convenient to find the solution of (15) in series form as:

$$y(x) = \sum_{i=0}^{\infty} y_i \qquad (16)$$

We decompose the nonlinear operator K as:

$$K(y) = K(y_0) + \sum_{i=0}^{\infty} \{ K(\sum_{j=0}^{i} y_j) - K(\sum_{j=0}^{i-1} y_j) \} \qquad (17)$$

From equations (16) and (17), equation (15) is equivalent to

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$$\sum_{i=0}^{\infty} y_i = c + K(y_0) + \sum_{i=0}^{\infty}\left\{K(\sum_{j=0}^{i} y_j) - K(\sum_{j=0}^{i-1} y_j)\right\} \qquad (18)$$

We then define the following iterative scheme:

$y_0(x) = c$

$y_1(x) = K(y_0)$

$y_2 = K(y_0 + y_1) - K(y_0)$

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$$y_{n+1} = K(y_0 + y_1 + y_2 + \cdots + y_n) - K(y_0 + y_1 + y_2 + \cdots + y_{n-1}) \qquad (19)$$

Substituting the components y_i of (19) in (16) gives the solution of the equation.

Numerical Experiments

We now apply the decomposition method discussed in the previous section to solving some variational problems and then compare the solutions with the known exact solution of each problem.

Example 1

We consider the variational problem:

$$\min V = \int_0^{\pi/2} ((y'')^2 - y^2 + x^2)dx \qquad (20)$$

which satisfies the conditions:

$$y(0) = 1, \quad y'(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0, \quad y'\left(\frac{\pi}{2}\right) = -1 \qquad (21)$$

The exact solution of this problem is $\cos x$.

The corresponding Euler-Poisson equation is:

$$y^{(iv)} - y = 0 \qquad (22)$$

We can put equation (22) in operator form as:

$$Ly = y \qquad (23)$$

where $L = \frac{d^4}{dx^4}$

Applying the iterative decomposition method, we have

$$y(x) = 1 + \frac{Ax^2}{2!} + \frac{Bx^3}{3!} + L^{-1}(y) \qquad (24)$$

where A = y (0) and B = y'(0).

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Then

$$y_0 = 1 + \frac{Ax^2}{2!} + \frac{Bx^3}{3!}$$

$$y_1 = L^{-1} = \frac{x^4}{4!} + \frac{Ax^6}{6!} + \frac{Bx^7}{7!}$$

$$y_2 = L^{-1}(y_0 + y_1) - L^{-1}(y_0) = \frac{x^8}{8!} + \frac{Ax^{10}}{10!} + \frac{Bx^{11}}{11!}$$

$$y_3 = \frac{x^{12}}{12!} + \frac{Ax^{14}}{14!} + \frac{Bx^{15}}{15!} \tag{25}$$

$$y_4 = \frac{x^{16}}{16!} + \frac{Ax^{18}}{18!} + \frac{Bx^{19}}{19!}$$

$$y_5 = \frac{x^{20}}{20!} + \frac{Ax^{22}}{22!} + \frac{Bx^{23}}{23!}$$

Summing the terms of equations (25) gives

$$y(x) = 1 + \frac{Ax^2}{2!} + \frac{Bx^3}{3!} + \frac{x^4}{4!} + \frac{Ax^6}{6!} + \frac{Bx^7}{7!} + \frac{x^8}{8!} + \frac{Ax^{10}}{10!} + \frac{Bx^{11}}{11!} + \frac{x^{12}}{12!} + \frac{Ax^{14}}{14!} + \frac{Bx^{15}}{15!} + \frac{x^{16}}{16!} + \frac{Ax^{18}}{18!} + \frac{Bx^{19}}{19!} + \frac{x^{20}}{20!} + \frac{Ax^{22}}{22!} + \frac{Bx^{23}}{23!} \tag{26}$$

Applying the boundary conditions (21) to equation 26, we obtain the values of the constants:

$A = -1.000000001 \; and \; B = 2.000000004E-09$

Then y(x) is approximated as:

$$\begin{aligned} y(x) = 1 &- 0.5x^2 + (3.33333334\,E-10)x^3 + 0.041666666x^4 \\ &- (1.38888889E-03)x^6 + (3.968253976E-13)x^7 \\ &+ (2.48015873E-05)x^8 - (2.755731925E-07)x^{10} \\ &+ (5.010421687E-17)x^{11} + (2.087675699E-09)x^{12} \\ &- (1.147074561E-11)x^{14} + (1.529432749E-21)x^{15} \\ &+ (4.779477332E-14)x^{16} - (1.561920698E-16)x^{18} \\ &+ (1.644127053E-26)x^{19} + (4.110317623E-19)x^{20} \\ &- (8.896791401E-22)x^{22} + (7.736340357E-32)x^{23} \end{aligned} \tag{27}$$

Table 1 gives the approximate values of y(x) for some points in the interval $0 \leq x \leq \frac{\pi}{2}$ and compares the approximate solution with the exact solution at those points. The very small errors generated by our decomposition approximation can be observed. Errors of less than 10E-10 are written as zero. In fact better accuracy is possible if more terms of the approximation series are taken. Recall that for the current problem, only five terms of the series have been taken to generate our approximate solution.

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Table 1. Approximate values by iterative decomposition method (IDM) vs. exact values of y(x) in Example 1

x \ y	Exact	IDM Approximate	Error
0	1.000000000	1.000000000	0.000000000
$\frac{\pi}{20}$	0.98768834	0.98768834	5.95E-10
$\frac{\pi}{10}$	0.951056516	0.951056515	1.295E-09
$\frac{3\pi}{20}$	0.891006524	0.891006524	1.88E-10
$\frac{\pi}{5}$	0.809016994	0.809016994	3.75E-10
$\frac{\pi}{4}$	0.707106781	0.707106781	1.86E-10
$\frac{3\pi}{10}$	0.587785252	0.587785252	2.93E-10
$\frac{7\pi}{20}$	0.453990499	0.453990499	7.37E-10
$\frac{2\pi}{5}$	0.309016994	0.309016994	3.71E-10
$\frac{9\pi}{20}$	0.156434465	0.156434465	3.60E-11
$\frac{\pi}{2}$	0.000000000	-4.73732750E-09	4.74E-09

Example 2

Consider the variational problem [6]:

$$\min v = \int_0^1 \left(y(x) + y'(x) - 4e^{3x}\right)^2 dx \qquad (28)$$

with given boundary conditions:

$$y(0) = 1, \quad y(1) = e^3 \qquad (29)$$

The corresponding Euler-Lagrange equation is found to be:

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$$y''(x) - y(x) - 8e^{3x} = 0 \tag{30}$$

with the same boundary conditions as (29).

The exact solution of this problem is $y = e^{3x}$. Using the operator form of (28), we have

$$Ly = y + 8e^{3x} \tag{31}$$

Thus,

$$y(x) = 1 + Ax + L^{-1}(8e^{3x}) + L^{-1}(y(x)) \tag{32}$$

Using the previous decomposition procedure,

$$\sum_{n=0}^{\infty} y_n(x) = 1 + Ax + L^{-1}(8e^{3x}) + L^{-1}(\sum_{k=0}^{\infty} y_{k-1}(x)) \tag{33}$$

Then,

$$y_0(x) = \frac{1}{9} + Ax - \frac{8}{3}x + \frac{8}{9}e^{3x}$$

$$y_1(x) = -\frac{8}{81} - \frac{8}{27}x + \frac{x^2}{18} + \frac{Ax^3}{6} - \frac{4x^3}{9} + \frac{8}{81}e^{3x}$$

$$y_2(x) = -\frac{8}{729} - \frac{8x}{243} - \frac{4x^2}{81} - \frac{4x^3}{81} + \frac{x^4}{216} + \frac{Ax^5}{120} - \frac{x^5}{45} + \frac{8}{729}e^{3x}$$

$$y_3(x) = -\frac{8}{6561} - \frac{8x}{2187} - \frac{8x^2}{1458} - \frac{4x^3}{729} - \frac{x^4}{243} - \frac{x^5}{405} + \frac{x^6}{6480} + \frac{Ax^7}{5040} - \frac{x^7}{1890} + \frac{8}{6561}e^{3x}$$

and so on. Then the four-term approximation of y(x) is given as:

$$y(x) = \frac{1}{6561} + Ax - \frac{6560}{2187}x + \frac{x^2}{1458} + \frac{Ax^3}{6} - \frac{364x^3}{729} + \frac{x^4}{1941} + \frac{Ax^5}{120} - \frac{2x^5}{81} + \frac{x^6}{6480} + \frac{Ax^7}{5040} - \frac{x^7}{1890} + \frac{6560}{6561}e^{3x} \tag{34}$$

To evaluate the constant A, we impose the boundary condition at x=1 on the approximate solution (34). Then we obtain A= 3.00003101. The approximation is then given as:

$$\begin{aligned} y(x) = {} & (1.524157903E-04) + (4.88257E-04)x + (6.858710562E-04)x^2 \\ & + (6.9104E-04)x^3 + (5.144032922E-04)x^4 + (3.11222E-04)x^5 \\ & + (1.543209877E-04)x^6 + (6.61437189E-04)x^7 \\ & + (0.999847584)e^{3x} \end{aligned} \tag{35}$$

Table 2 gives a comparison of the exact solution with approximation of the solution at points $x \in [0,1]$. From this example it is obvious that the decomposition algorithm can be considered an efficient method. A better approximation to the solution of the problem may be achieved by taking more components of y(x), as shown in example 1.

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Table 2. Approximate values by iterative decomposition method (IDM) vs. exact values of y(x) in Example 2

x	Exact	IDM Approximate	Error
0.0	**1.000000000**	**0.999999999**	**1.000000E-09**
0.1	**1.349858808**	**1.349861914**	**3.106000E-06**
0.2	**1.8221188**	**1.822120252**	**1.451610E-06**
0.3	**2.459603111**	**2.459612688**	**9.576590E-06**
0.4	**3.320116923**	**3.32013064**	**1.371726E-05**
0.5	**4.48168907**	**4.481709836**	**2.076566E-05**
0.6	**6.049647464**	**4.049683533**	**3.606826E-05**
0.7	**8.166169913**	**8.166241**	**7.108743E-05**
0.8	**11.02317638**	**11.02332391**	**1.475294E-04**
0.9	**14.87973172**	**14.88003464**	**3.029109E-04**
1.0	**20.08553692**	**20.08613453**	**5.976101E-04**

Conclusions

The iterative decomposition algorithm is used for the solution of the ordinary differential equations which arise from variational problems. It is noteworthy that this method does not require the discretisation of the variables .The implementation of the method is straightforward, requiring no restrictive assumptions or linearisation techniques.

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