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Full Paper

On weak* Rad-⊕-supplemented modules

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Abstract: We introduce and give various properties of weak* Rad-⊕-supplemented modules, which is the generalisation of ⊕-supplemented and Rad-⊕-supplemented modules for which we provide counter examples. We also prove that for a finitely generated semi-simple R-module M, these three concepts or modules are equivalent. In general, factor modules and direct summands of weak* Rad-⊕-supplemented modules are not weak* Rad-⊕-supplemented modules. Thus, we provide several sufficient conditions for the class of weak* Rad-⊕-supplemented modules to be closed under factor modules and direct summands.

Keywords: Rad-supplemented module, Rad-⊕-supplemented module, weak* Rad-⊕-supplemented module, completely weak* Rad-⊕-supplemented module

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INTRODUCTION

In this paper R represents an associative ring with identity and all modules are unitary left R-modules unless otherwise specified. Let M be an R-module. The notation $N \subseteq M$ means that N is a submodule of M and $RadM$ indicates the Jacobson radical of M. A submodule S of a module M is called small in M, if for any submodule T of M, $M = S + T$ implies $M = T$. A module M is called hollow if every proper submodule of M is small in M, and M is called local if the sum of all proper submodules of M is also a proper submodule of M; equivalently $RadM$ is a maximal submodule of M and $RadM$ is small in M. It is well known that every local module is hollow but the converse need not be true. As an example, the Z-module Z_{p^∞} is hollow but not local, for any prime p. A module M has property (p*), if for any submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K \subseteq Rad(M/K)$, or equivalently, for every submodule $N \subseteq M$ there exists a decomposition $M = K \oplus K'$ with $K \subseteq N$ such that $N \cap K' \subseteq RadK'$ [1].

If N and L are submodules of M, then N is called a supplement (weak supplement) of L if $N + L = M$ and $N \cap L$ is small in N ($N \cap L$ is small in M). A module M is called supplemented (weak supplemented) if each of its submodules has a supplement (weak supplement) in M. Obviously every direct summand of M is a supplement submodule and every supplement submodule is a weak supplement. A module M is called ⊕-supplemented (completely ⊕-supplemented) if every submodule (direct summand) of M has a supplement that is a direct summand of M [2, 3]. A module M is called amply supplemented if for every submodules N and L of M with $M = N + L$, L contains a supplement of N in M.

Let $f: P \to M$ be an epimorphism. We call f a (generalised) cover if ($Kerf \subseteq RadP$) $Kerf$ is small in P and call a (generalised) cover f a (generalised) projective cover if P is a projective module [4]. Xue [4] introduced Rad-supplement module which is also studied by Wang and Ding [5] under the name generalised-supplemented module. A submodule N of a module M has a Rad-supplement K in M if $N + K = M$ and $N \cap K \subseteq RadK$. A module M is called Rad-supplemented if every submodule of M has a Rad-supplement. M is called Rad-⊕-supplemented if every submodule of M has a Rad-supplement that is a direct summand of M. As in Example 3.11 [6], M is Rad-⊕-supplemented but not supplemented, and in Example 2.15(2) [7] the Z-module Q is Rad-⊕-supplemented but not ⊕-supplemented, where Z and Q denote the ring of integers and rational numbers respectively. Every module with (p*) is Rad-⊕-supplemented. An example of M not being a Rad-⊕-supplemented module is given in Example 2.15(1) [7]. A module M is called completely Rad-⊕-supplemented if every direct summand of M is Rad-⊕-supplemented [8].

Motivated by the above notions, we introduce a new concept known as weak* Rad-⊕-supplemented module and completely weak* Rad-⊕-supplemented module, which are the generalisations of ⊕-supplemented and Rad-⊕-supplemented modules. The class of weak* Rad-⊕-supplemented module lies between that of Rad-⊕-supplemented and Rad-supplemented modules. Thus, we have the following implication:

⊕-supplemented ⇒ Rad-⊕-supplemented ⇒ weak* Rad-⊕-supplemented ⇒ Rad-supplemented.

However, the converse need not be true. In this paper under the section of weak* Rad-⊕-supplemented modules we characterise the semi-local modules in terms of weak* Rad-⊕-supplemented modules. Apart from this we show that for a projective module, ⊕-supplemented modules are equivalent to weak* Rad-⊕-supplemented modules. In the last section we obtain the properties of completely weak* Rad-⊕-supplemented modules.

Let M be an R-module. We consider the following conditions.

(D_1) For every submodule N of M, there exists a decomposition of $M = M_1 \oplus M_2$, such that $M_1 \subseteq N$ and $M_2 \cap N$ is small in M_2.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Lemma 1 [9]. Let $M = \underset{i \in I}{\oplus} M_i$. Then $RadM = \underset{i \in I}{\oplus} RadM_i$.

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For a submodule N of M, in general $N \cap RadM \neq RadN$ [9]. But we have the equality if N is a supplement or Rad-supplement submodule of M.

Lemma 2 [9]**.** Let M be an R -module and N be a supplement (or Rad-supplement) submodule of M. Then $N \cap RadM = RadN$.

Lemma 3 [10]**.** Let $M = S \oplus T = N + T$, where S is $T-$projective. Then $M = S' \oplus T$, where $S' \subseteq N$.

WEAK* RAD-⊕-SUPPLEMENTED MODULES

Definition 1. An R-module M is called a weak* Rad-⊕-supplemented module if every semi-simple submodule of M has a Rad-supplement that is a direct summand of M. An R-module M is called completely weak* Rad-⊕-supplemented module if every direct summand of M has a Rad-supplement which is a direct summand of M; i.e. every direct summand of M is a weak* Rad-⊕-supplemented module.

For example, hollow modules and modules with (p*) are weak* Rad-⊕-supplemented modules. Also, hollow modules are completely weak* Rad-⊕-supplemented modules. Clearly, every Rad-⊕-supplemented module is a weak* Rad-⊕-supplemented module, but the converse may not be true. Thus, we have the following implications, but in general the converse is not necessarily true:

Lifting ⇒ ⊕-supplemented ⇒ Rad-⊕-supplemented ⇒ weak* Rad-⊕-supplemented

Example 1. (i) Let R be a discrete valuation ring with field of fractions K. Let P be the unique maximal ideal of R such that $P = Ra$ for some element $a \in P$. Let $M = (K/R) \oplus (R/P)$; then M is a weak* Rad-⊕-supplemented module but not lifting by Proposition A.7 [10].
(ii) The left Z- module Q is weak* Rad-⊕-supplemented but not ⊕-supplemented.
(iii) Let M be an R-module. By Proposition 4.8 [10], M has (D_1) if and only if M is amply supplemented and every supplement submodule of M is a direct summand. Therefore every (D_1)-module is a weak* Rad-⊕-supplemented module.

Now we have the following simple fact that plays a key role in our study.

Lemma 4. Let $M = M_1 \oplus M_2$ be a module. If N_1 is a Rad-supplement of N_1' in M_1 and N_2 is a Rad-supplement of N_2' in M_2, then $N_1 \oplus N_2$ is a Rad-supplement of $N_1' \oplus N_2'$ in M.

Proof. Assume that N_1 is a Rad-supplement of N_1' in M_1. Then $M_1 = N_1 + N_1'$, $N_1 \cap N_1' \subseteq RadN_1$ and if N_2 is a Rad-supplement of N_2' in M_2, then $M_2 = N_2 + N_2'$, $N_2 \cap N_2' \subseteq RadN_2$. Since $M = M_1 \oplus M_2$, so $M = (N_1 + N_1') \oplus (N_2 + N_2') = (N_1 \oplus N_2) + (N_1' \oplus N_2')$ and now $(N_1 \oplus N_2) \cap (N_1' \oplus N_2') \subseteq (N_1 \cap N_1') \oplus (N_2 \cap N_2') \subseteq RadN_1 \oplus RadN_2 = Rad(N_1 \oplus N_2)$ by Lemma 1. Which shows that $N_1 \oplus N_2$ is a Rad-supplement of $N_1' \oplus N_2'$ in M.

Corollary 1. Let $M = M_1 \oplus M_2$ be a module. If N_1 is a weak* Rad-⊕-supplement of N_1' in M_1 and N_2 is a weak* Rad-⊕-supplement of N_2' in M_2, then $N_1 \oplus N_2$ is a weak* Rad-⊕-supplement of $N_1' \oplus N_2'$ in M.

Proof. The pattern of the proof is similar to that of Lemma 4.

Lemma 5. Let $M = M_1 \oplus M_2$ be a module. If N/M_1 is a Rad-supplement of N'/M_1 in M/M_1, then $N \cap M_2$ is a Rad-supplement of $N' \cap M_2$ in M_2.

Proof. Let us assume that N/M_1 is a Rad-supplement of N'/M_1 in M/M_1. Then $M/M_1 = N/M_1 + N'/M_1$ and $(N/M_1) \cap (N'/M_1) \subseteq Rad(N/M_1)$, which implies that $M = N + N'$ and $N \cap N' \subseteq RadN$; i.e. $N \cap N'$ is small in N. Now we have $M = M_1 \oplus (N \cap M_2) + M_1 \oplus (N' \cap M_2) = M_1 \oplus \left[(N \cap M_2) + (N' \cap M_2)\right]$, which implies that $M_2 = (N \cap M_2) + (N' \cap M_2)$. If K is a submodule of $N \cap M_2$ such that $N \cap M_2 = (N \cap M_2 \cap N') + K$, then $(N \cap M_2) + M_1 = (N \cap M_2 \cap N') + K + M_1$, which implies that $(N \cap M_2 \cap N') + K + M_1 = N$. So $(N \cap N') + K + M_1 = N$ implies that $K + M_1 = N$ because $N \cap N'$ is small in N. But $K \subseteq (N \cap M_2)$ implies that $N = K \oplus M_1$ and $K = N \cap M_2$. We conclude that $(N \cap M_2 \cap N')$ is small in $N \cap M_2$, i.e. $(N \cap M_2 \cap N')$, which shows that $N \cap M_2$ is a Rad-supplement of $N' \cap M_2$ in M_2.

Lemma 6. Let $M = M_1 \oplus M_2$ be a module. If N_1/M_1 is a Rad-supplement of N_1'/M_1 in M/M_1 and N_2/M_2 is a Rad-supplement of N_2'/M_2 in M/M_2, then $N_1 \cap N_2$ is a Rad-supplement of $\left[(N_1' \cap M_2) \oplus (N_2' \cap M_1)\right]$ in M.

Proof. Let $M = M_1 \oplus M_2$. Then clearly $N_1 = M_1 \oplus (N_1 \cap M_2)$ and $N_2 = (N_2 \cap M_1) \oplus M_2$, so that $N_1 \cap N_2 = (N_1 \cap M_2) \oplus (N_2 \cap M_1)$. By Lemma 5, $(N_1 \cap M_2)$ is a Rad-supplement of $(N_1' \cap M_2)$ in M_2 and $(N_2 \cap M_1)$ is a Rad-supplement of $(N_2' \cap M_1)$ in M_1. Therefore by Lemma 4 we conclude that $(N_1 \cap M_2) \oplus (N_2 \cap M_1)$ is a Rad-supplement of $(N_1' \cap M_2) \oplus (N_2' \cap M_1)$ in M; i.e. $N_1 \cap N_2$ is a Rad-supplement of $(N_1' \cap M_2) \oplus (N_2' \cap M_1)$ in M.

Corollary 2. Let $M = M_1 \oplus M_2$ be a module. If N_1/M_1 is a weak* Rad-⊕-supplement of N_1'/M_1 in M/M_1 and N_2/M_2 is a weak* Rad-⊕-supplement of N_2'/M_2 in M/M_2, then $N_1 \cap N_2$ is a weak* Rad-⊕-supplement of $\left[(N_1' \cap M_2) \oplus (N_2' \cap M_1)\right]$ in M.

Proof. The proof is immediate from Lemma 6.

Lemma 7. Let N, L be semi-simple submodules of a module M such that $N + L$ has a weak* Rad-⊕-supplement H in M and $N \cap (H + L)$ has a weak* Rad-⊕-supplement G in N. Then $H + G$ is a weak* Rad-⊕-supplement of L in M.

Proof. Let H be a weak* Rad-⊕-supplement of $N + L$ in M. Then $M = (N + L) + H$, $(N + L) \cap H \subseteq RadH$ and H is a direct summand of M. Now since $N \cap (H + L)$ has a weak* Rad-⊕-supplement G in N, we have $N = [N \cap (H + L)] + G$, $(H + L) \cap G \subseteq RadG$ and G is a direct summand of N. Then $M = N + H + L = [N \cap (H + L) + G] + H + L = L + (H + G)$ and $L \cap (H + G) \subseteq [H \cap (L + G)] + [G \cap (L + H)] \subseteq H \cap (L + N) + G \cap (L + H) \subseteq RadH + RadG \subseteq Rad(H + G)$. Consequently, we have $M = (H + G) + L$, $(H + G) \cap L \subseteq Rad(H + G)$ and obviously $H + G$ is a direct summand of M. Hence $H + G$ is a weak* Rad-⊕-supplement of L in M.

Definition 2. A module M is called semi-local if $M / RadM$ is semi-simple.

The following proposition (Prosition 1) characterises the semi-local module in terms of weak* Rad-⊕-supplemented modules:

Proposition 1. For a left R-module M, the following statements are equivalent:

(i) M is semi-local;

(ii) M is weak* Rad-⊕-supplemented;

(iii) There is a decomposition $M = M_1 \oplus M_2$ such that M_1 is semi-simple and M_2 is semi-local with $RadM$ essential in M_2.

Proof. The proof is immediate by Result 17.2 [11].

Proposition 2. Let N be a submodule of a finitely generated Rad-supplemented module M. Then N is a Rad-supplement submodule if and only if it is a co-closed submodule of M.

Proof. Let N be a Rad-supplement of K in M. Then $M = N + K$ and $N \cap K \subseteq RadN$. By Result 3.2 (1) [11], for any submodule L of N such that $L \subset N$ is co-small in M, we have $N + K = M$, which implies $L + K = M$. Applying the modular law, we have $N = N \cap M = L + (N \cap K)$, which implies that $N = L$ because $N \cap K$ is small in N. Hence N is a co-closed submodule of M. Conversely, assume that N is a co-closed submodule of M and K is a Rad-supplement of N. Then $N + K = M$ and $N \cap K \subseteq RadK$. By Lemma 2, $N \cap K \subseteq K \cap RadM$ implies that $N \cap K \subseteq RadM$; i.e. $N \cap K$ is small in M as M is finitely generated. By Result 3.7(3) [11], $N \cap K$ is small in N, that is $N \cap K \subseteq RadN$. Thus, N is a Rad-supplement of K in M and hence N is a Rad-supplement submodule.

Recall that a module M is called radical if M has no maximal submodule, i.e. $RadM = M$ [11]. For example, for ${}_ZQ$, we have $Rad\, {}_ZQ = {}_ZQ$ since ${}_ZQ$ has no maximal submodule. Every divisible Z-module is a radical module. For a module M, $P(M)$ will indicate the sum of all radical submodules of M. Note that $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, then M is called reduced. Also, $M / P(M)$ is reduced for every module M.

Lemma 8. Every radical module M is weak* Rad-⊕-supplemented.

Proof. Assume that M is a radical module, i.e. $RadM = M$. So M has a trivial weak* Rad-⊕-supplement 0 in M. Consequently, M is weak* Rad-⊕-supplemented.

For example, ${}_ZQ$ is weak* Rad-⊕-supplemented but not ⊕-supplemented because it is not torsion. An element $x \in M$ is a torsion element if $r.x = 0$ for some nonzero $r \in R$ and module M is called torsion if every element of M is torsion.

Corollary 3. Let M be any module. Then $P(M)$ is weak* Rad-⊕-supplemented.

Proposition 3. Every weak* Rad-⊕-supplemented module has a radical direct summand.

Proof. Let M be a weak* Rad-⊕-supplemented module. Then for any semi-simple submodule N of M, there exists a direct summand K of M such that $M = RadM + N$, $RadM \cap N \subseteq RadN$ and $M = N \oplus K$. By Lemma 1, $RadM = RadN \oplus RadK$. Then $M = RadM + N = RadN \oplus RadK + N = RadK + N$. Using the modular law, we get $K = K \cap M = RadK$. Thus, the direct summand K is radical.

Definition 3. A submodule N of M is called a fully invariant submodule if for every $f \in S$, we have $f(N) \subseteq N$, where $S = End_R(M)$. An R-module M is called a duo module if every submodule of M is fully invariant.

In the following results, we discuss about the direct sum of weak* Rad-⊕-supplemented module and prove that the finite direct sum of weak* Rad-⊕-supplemented module is weak* Rad-

⊕-supplemented. However, this is not true for the infinite direct sum. Thus, we have the result if M is a duo module.

Proposition 4. Let M be a countably infinite direct sum of weak* Rad-⊕-supplemented module M_i, $i \in I$, where I is a countably infinite set. If M is a duo module, then it is weak* Rad-⊕-supplemented.

Proof. Let K be any semi-simple submodule of M. Since M is a duo module, then K can be written as $K = \oplus_{i \in I}(M_i \cap K)$. Since each M_i, $i \in I$ is weak* Rad-⊕-supplemented, there exists a direct summand L_i of M_i such that $M_i = (M_i \cap K) + L_i$ and $(M_i \cap K) \cap L_i = K \cap L_i \subseteq RadL_i$ for every $i \in I$. Let us form a sum $L = \oplus_{i \in I} L_i$; then L is a direct summand of M. Thus, $M = K + L$ and $K \cap L$ $= [\oplus_{i \in I}(M_i \cap K)] \cap [\oplus_{i \in I} L_i] \subseteq \oplus_{i \in I}(RadL_i) = RadL$ by Lemma 1, which shows that L is a weak* Rad-⊕-supplement of K. Hence M is weak* Rad-⊕-supplemented.

Corollary 4. The finite direct sum of a weak* Rad-⊕-supplemented module is weak* Rad-⊕-supplemented.

Proof. Let M_1 and M_2 be weak* Rad-⊕-supplemented. We claim that $M = M_1 \oplus M_2$ is weak* Rad-⊕-supplemented. Let L be a semi-simple submodule of M; then $M = M_1 + M_2 + L$ and so $M = M_1 + M_2 + L$ has a Rad-⊕-supplement 0 in M. Let H be a Rad-⊕-supplement of $M_2 \cap (M_1 + L)$ in M_2 such that H is a direct summand of M_2. By Lemma 7, H is a Rad-⊕-supplement of $M_1 + L$ in M. Let $M_1 \cap (H + K = H \oplus K)$ and K be a Rad-⊕-supplement of $M_1 \cap (L + H)$ in M_1 such that K is a direct summand of M_1. Again by Lemma 7, we have that $H + K$ is a Rad-⊕-supplement of L in M. Since H is a direct summand of M_2 and K is a direct summand of M_1, it follows that $H + K = H \oplus K$ is a direct summand of M. Thus, $M = M_1 \oplus M_2$ is a weak* Rad-⊕-supplemented module.

Corollary 5. Any finite direct sum of modules with (p∗) is weak* Rad-⊕-supplemented.

Corollary 6. Any finite direct sum of hollow (or local) modules is weak* Rad-⊕-supplemented.

Lemma 9. Let a module M be weak* Rad-⊕-supplemented. Then $RadM$ is small in M if M is projective.

Proof. Let $M = RadM + N$ for some semi-simple submodule N of M. We claim that $N = M$. Since M is weak* Rad-⊕-supplemented, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \subseteq RadK$. Since K is a direct summand of a projective module M, clearly K is projective. We now define a map $f: K \to M/N$ by $f(K) = k + N \quad \forall\ k \in K$, which is an epimorphism; thus $Kerf = N \cap K$. Therefore f is a generalised projective cover because $Kerf = (N \cap K) \subseteq RadK$. From $M = RadM + N$, we have $Rad(M/N) = M/N$. Since M/N has a generalised projective cover, it is easy to see that $M/N = 0$, which implies that $M = N$. Hence we obtain the result that $RadM$ is small in M.

It is clear that every ⊕-supplemented module is weak* Rad-⊕-supplemented. However, the converse is not always true. Now we provide the necessary and sufficient conditions for these two structures to be equivalent.

Proposition 5. For a projective module M, the following statements are equivalent:

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(i) M is ⊕-supplemented;

(ii) M is weak* Rad-⊕-supplemented.

Proof. Straightforward.

Definition 4. A ring *R* is called left perfect if every left R-module has a projective cover.

It is well known that *R* is left perfect if and only if every projective left R-module is ⊕-supplemented. Using this fact along with Proposition 5, we obtain Corollary 7.

Corollary 7. A ring *R* is left perfect if and only if every projective left R-module is weak* Rad-⊕-supplemented.

Corollary 8. For a finitely generated semi-simple module M, the following statements are equivalent:

(i) M is ⊕-supplemented;

(ii) M is Rad-⊕-supplemented;

(iii) M is weak* Rad-⊕-supplemented.

Proof. It is obvious that $(i) \Rightarrow (ii) \Rightarrow (iii)$. For $(iii) \Rightarrow (i)$, assume that M is weak* Rad-⊕-supplemented and N is any submodule of M. Since M is semi-simple, N is a direct summand and hence a semi-simple submodule of M. By hypothesis, there exists a direct summand K of M such that $M = N + K$ and $(N \cap K) \subseteq RadK$. It is easily seen that $RadK$ is small in K since M is finitely generated; i.e. $RadM$ is small in M. So we get that $N \cap K$ is small in K. Hence K is a supplement of N in M. Therefore M is ⊕-supplemented.

Proposition 6. Let M be an indecomposable module. Then M is weak* Rad-⊕-supplemented if and only if $M = RadM$ or M is a local module.

Proof. Assume that M is weak* Rad-⊕-supplemented and $M \neq RadM$. Let N be any proper semi-simple submodule of M. Then there exists a direct summand K of M such that $M = N + K$ and $N \cap K \subseteq RadK$. Since M is indecomposable and $N \neq M$, we have $K = M$, which implies that N is small in M. Consequently, M is a local module and the converse is clear by Lemma 8.

Proposition 7. The following statements are equivalent for a finitely generated indecomposable module M:

(i) M is ⊕-supplemented;

(ii) M is Rad-⊕-supplemented;

(iii) M is hollow.

Proof. It is clear that $(i) \Rightarrow (ii)$. For $(ii) \Rightarrow (iii)$, let N be a proper submodule of M. Assume that M is Rad-⊕-supplemented; then there exists a direct summand K of M such that $M = N + K$ and $N \cap K \subseteq RadK$. Since M is indecomposable, $M = N + K$ implies that $M = K$. Then $N = N \cap K \subseteq RadK$ is small in $K \subset^{\oplus} M$; that is, N is small in M, which shows that M is a hollow module. For $(iii) \Rightarrow (i)$, this is obvious.

Lemma 10. Let M be a module and N be a submodule of M. If U is a Rad-supplement of N in M, then $(U + L)/L$ is a Rad-supplement of N/L in M/L for every submodule L of N.

Proof. Let M be a module and N, U be submodules of M. Since U is a Rad-supplement of N in M, then $M = N + U$ and $U \cap N \subseteq RadU$. Therefore we have $M/L = N/L + (U+L)/L$ for every submodule L of N and $U \cap N \subseteq Rad(U+L)$. Hence we get $(N/L) \cap (U+L)/L = (U \cap N + L)/L$. Now let us consider the canonical epimorphism $\phi : (U+L) \to (U+L)/L$. By Result 2.8(1) [11], we have $\phi(U+N) \subseteq \phi(Rad(U+L)) \subseteq Rad((U+L)/L)$. Now $N/L \cap (U+L)/L = [L+(U \cap N)]/L = \phi(U \cap N) \subseteq Rad((U+L)/L)$, i.e. $N/L \cap (U+L)/L \subseteq Rad((U+L)/L)$. Hence $(U+L)/L$ is a Rad-supplement of N/L in M/L.

In the following propositions we investigate the sufficient condition for weak* Rad-⊕-supplemented modules to be closed under direct summands and factor modules.

Proposition 8. Let M be a weak* Rad-⊕-supplemented module with (D_3). Then every direct summand of M is a weak* Rad-⊕-supplemented module.

Proof. Let M be a weak* Rad-⊕-supplemented module. Let N be a direct summand of M and A be a semi-simple submodule of N. Since M is a weak* Rad-⊕-supplemented module, there exists a direct summand B of M such that $M = A + B$ and $A \cap B \subseteq RadB$. Now we have $N = A + (N \cap B)$. Since M has (D_3), $N \cap B$ is a direct summand of M and so it is also a direct summand of N. Now $A \cap (N \cap B) = A \cap B$, $A \cap B \subseteq RadM$ and $A \cap B \subseteq N \cap B$. By Lemma 2, $A \cap B \subseteq (N \cap B) \cap RadM = Rad(N \cap B)$, i.e. $A \cap (N \cap B) \subseteq Rad(N \cap B)$. Hence N is a weak* Rad-⊕-supplemented module.

Proposition 9. Let M be a nonzero weak* Rad-⊕-supplemented R-module and let A be a fully invariant semi-simple submodule of M. Then

(i) The factor module M/A is a weak* Rad-⊕-supplemented module.

(ii) Moreover, if A is a direct summand of M, then A is also a weak* Rad-⊕-supplemented module.

Proof. (i) Let M be an R-module. For $A \subseteq L \subseteq M$, where L is a semi-simple submodule of M, let L/A be a semi-simple submodule of M/A. Since M is a weak* Rad-⊕-supplemented module, there exist submodules K and K' of M such that $M = L + K$, $L \cap K \subseteq RadK$ and $M = K \oplus K'$. By Lemma 10, $(K+A)/A$ is a Rad-supplement of L/A in M/A. Since $f(A) \subseteq A$ for each $f \in End_R(M)$, we have $A = (A \cap K) \oplus (A \cap K')$. Hence $(K+A) \cap (K'+A) \subseteq A$ and so we have $(K+A)/A \cap (K'+A)/A = 0$; i.e. $(K+A)/A$ is a direct summand of M/A. Thus, M/A is a weak* Rad-⊕-supplemented module.

(ii) Let M be a module over an associative ring R. Let A be a direct summand of M and B be a semi-simple submodule of A. Since M is weak* Rad-⊕-supplemented, then there exist submodules C and C' of M such that $M = B + C$, $B \cap C \subseteq RadC$ and $M = C \oplus C'$. Now we have $A = B + (A \cap C)$, and also $A = (A \cap C) \oplus (A \cap C')$ since $M = C \oplus C'$. Thus, $(A \cap C)$ is a direct summand of A. Now we know that $B \cap (A \cap C) = B \cap C \subseteq RadM$ and $B \cap C \subseteq A \cap C$. By Lemma 2, $B \cap C \subseteq (A \cap C) \cap RadM = Rad(A \cap C)$, i.e. $B \cap (A \cap C) \subseteq Rad(A \cap C)$. Hence A is a weak* Rad-⊕-supplemented module.

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Theorem 1. Let M be a weak* Rad-⊕-supplemented module and N be a semi-simple submodule of M. If for every direct summand L of M, $(N+L)/N$ is a direct summand of M/N, then M/N is a weak* Rad-⊕-supplemented module.

Proof. Let M be an R-module and N be a semi-simple submodule of M. Let X be a semi-simple submodule of M containing N. Since M is a weak* Rad-⊕-supplemented module, there exists a direct summand L of M such that $M = X + L = L \oplus L'$ and $X \cap L \subseteq RadL$ for some submodule L' of M. Therefore we have $M/N = X/N + (L+N)/N$, where X/N is a semi-simple submodule of M/N. By hypothesis, $(N+L)/N$ is a direct summand of M/N. Now we have $(X/N) \cap (L+N)/N = [X \cap (L+N)]/N = [N + (L \cap X)]/N$. Since $X \cap L \subseteq RadL$, we have $[(L \cap X)+N]/N \subseteq Rad((L+N)/N)$. Hence $(N+L)/N$ is a Rad-supplement submodule of X/N in M/N. Thus, M/N is a weak* Rad-⊕-supplemented module.

Proposition 10. Let N be a submodule of a weak* Rad-⊕-supplemented module M with $N \cap RadM = 0$. Then N is semi-simple. In particular, a weak* Rad-⊕-supplemented module M with $RadM = 0$ is semi-simple.

Proof. Since M is a weak* Rad-⊕-supplemented module, for any semi-simple submodule K of N, there exists submodule $L \subseteq M$ such that $K + L = M$ and $K \cap L \subseteq RadL$. Thus, $N = N \cap M = N \cap (K+L) = K + (N \cap L)$ by the modular law. Since $K \cap L \subseteq RadL$ considers $K \cap N \cap L = K \cap L \subseteq N \cap RadL \subseteq N \cap RadM = 0$, so $N = K \oplus (N \cap L)$, which proves that N is semi-simple.

Proposition 11. Let M be a weak* Rad-⊕-supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is a semi-simple module and M_2 is a module with essential radical.

Proof. For $RadM$, there exists submodule $M_1 \subseteq M$ such that $M_1 \cap RadM = 0$ and $M_1 \oplus RadM \subseteq^e M$. By Proposition 10, the existing submodule is semi-simple. Since M is a weak* Rad-⊕-supplemented module, there exists submodule $M_2 \subseteq M$ such that $M_1 + M_2 = M$ and $M_1 \cap M_2 \subseteq Rad(M_2)$ with $M_2 \subseteq^{\oplus} M$. Since $M_1 \cap M_2 = M_1 \cap (M_1 \cap M_2) \subseteq M_1 \cap \mathrm{Rad}M_2 \subseteq M_1 \cap \mathrm{Rad}M = 0$, so $M = M_1 \oplus M_2$. Now $RadM = RadM_1 \oplus RadM_2 = RadM_2$ because M_1 is semi-simple. Since $M_1 \oplus RadM \subseteq^e M = M_1 \oplus M_2$, i.e. $M_1 \oplus RadM_2 \subseteq^e M = M_1 \oplus M_2$, this gives $RadM_2 \subseteq^e M_2$, which is the required result.

Definition 5. The modules M_i $(1 \le i \le n)$ are called relatively projective if M_i is M_j-projective for all $1 \le i \le n$, where n is a positive integer.

Theorem 2. Let M_i $(1 \le i \le n)$ be any finite collection of relatively projective modules. Then the module $M = M_1 \oplus M_2 \ldots\ldots\ldots\ldots\ldots\ldots \oplus M_n$ is a weak* Rad-⊕-supplemented module if and only if M_i is a weak* Rad-⊕-supplemented module for each $1 \le i \le n$.

Proof. The sufficient part of the theorem is proved in Corollary 4. Conversely, we only have to prove that M_1 is a weak* Rad-⊕-supplemented module. Let A be a semi-simple submodule of M_1. Since M is a weak* Rad-⊕-supplemented module, there exists a direct summand B of M such that $M = A + B$ and $A \cap B \subseteq RadB$. Since $M = A + B = M_1 + B$, then by Lemma 3 there exists $B_1 \subseteq B$ such that $M = M_1 \oplus M_2$; then $B = B_1 \oplus (M_1 \cap B)$. Now we have $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of M_1. We know that $A \cap B = A \cap (M_1 \cap B)$, $A \cap B \subseteq RadM$ and $A \cap B \subseteq M_1 \cap B$.

Therefore $A\cap B\subseteq (M_1\cap B)\cap RadM = Rad(M_1\cap B)$ by Lemma 2, i.e. $A\cap(M_1\cap B) \subseteq Rad(M_1\cap B)$. Hence M_1 is a weak* Rad-⊕-supplemented module.

Proposition 12. Let M be a module with (D_3). Then the following statements are equivalent:

(1) There exists a decomposition $M = M_1 \oplus M_2$, where M_1 is a semi-simple module and M_2 is a weak* Rad-⊕-supplemented module with essential radical;

(2) M is weak* Rad-⊕-supplemented;

(3) Every direct summand of M is weak* Rad-⊕-supplemented.

Proof. From Corollary 4 it follows that (1) $\Rightarrow$ (2). From Proposition 8 it is clear that (2) $\Rightarrow$ (3). From Proposition 11 it can be obtained that (3) $\Rightarrow$ (1).

COMPLETELY WEAK* RAD-⊕-SUPPLEMENTED MODULES

In this section we study the properties of completely weak* Rad-⊕-supplemented modules. In the previous section we have observed that in general the direct summand of weak* Rad-⊕-supplemented modules are not weak* Rad-⊕-supplemented, but under some conditions it will be weak* Rad-⊕-supplemented. So we recall that a module M is completely weak* Rad-⊕-supplemented if every direct summand of M is weak* Rad-⊕-supplemented.

Proposition 13. Let M be a weak* Rad-⊕-supplemented module with (D_3). Then M is a completely weak* Rad-⊕-supplemented module.

Proof. The proof follows from Proposition 8.

Definition 6. An R-module M is said to have the summand sum property (summand intersection property) if the sum (intersection) of any pair of direct summands of M is a direct summand of M; i.e. if N and L are direct summands of M, then $N+L$ ($N\cap L$) is also a direct summand of M [11].

Theorem 3. Let M be a weak* Rad-⊕-supplemented module with the summand sum property (SSP). Then M is a completely weak* Rad-⊕-supplemented module.

Proof. Let M be a weak* Rad-⊕-supplemented module and let N be a direct summand of M. Then $M = N\oplus N'$, where N' is supposed to be a semi-simple submodule of M. We have to show that M/N' is a weak* Rad-⊕-supplemented module. Let us suppose that L is a direct summand of M. Since M has SSP, $L+N'$ is a direct summand of M. Let $M=(L+N')\oplus K$ for some semi-simple submodule $K\subseteq M$. Then $M/N' = (L+N')/N' \oplus (K+N')/N'$. Therefore M/N' is a weak* Rad-⊕-supplemented module by Theorem 1.

Corollary 9. Let M be a weak* Rad-⊕-supplemented-duo module. Then M is a completely weak* Rad-⊕-supplemented module.

Lemma 11. Let an R-module M be an indecomposable module. Then M is a hollow module if and only if M is a completely weak* Rad-⊕-supplemented module.

Proof. This is clear from definitions.

Proposition 14. Let $M = A\oplus B$ such that A and B have local endomorphism rings. Then M is a completely weak* Rad-⊕-supplemented module if and only if A and B are hollow modules.

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Proof. For (:⇒), this follows from Lemma 11. For (⇐:), let K be a direct summand of M. If $K = M$, then by Corollary 6, K is a weak* Rad-⊕-supplemented module. Let us consider that $K \neq M$. Then either $K \cong A$ or $K \cong B$ by Krull-Schmidt-Azumaya Theorem [12]. In either case K is a weak* Rad-⊕-supplemented module. Thus, M is a completely weak* Rad-⊕-supplemented module.

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