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Full Paper

Computation of transient response of a linear time-invariant system using a hybrid of efficient techniques

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Abstract: In this paper a hybrid of efficient techniques for obtaining the transient response vectors of a linear time-invariant system is presented. The proposed method is called the restrictive Padé approximation and is used for approximating the resulting exponential matrix which arises when solving the linear time-invariant system in the Hessenberg form. The suggested method does not require the knowledge of eigen-values of the matrix. Some numerical examples are given to validate the accuracy of the proposed method.

Keywords: transient response vectors, restrictive Padé approximation, exponential matrix, linear time-invariant system, Hessenberg matrix

INTRODUCTION

A linear time-invariant system is described in the following system of ordinary differential equations :

$$\dot{x}(t) = Ax(t) + u(t), \qquad x(0) = x_0, \tag{1}$$

where x and u are *n*-vectors and A is $n \times n$, a constant matrix [1]. The systems with free external input vector u and the transient response vector x(t) at $t = k\tau$ take the following form:

$$x(k\tau) = e^{k\tau A} x_0, \qquad k = 0, 1, \ldots \quad . \tag{2}$$

For a chosen interval time τ, the problem of computing the transient response vectors $x(k\tau)$ depends heavily on evaluating the exponential matrix function, $e^{k\tau A}$ [1]. Several authors have considered this problem; for example, Varga [1] introduced and investigated the use of particular

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Padé approximations, namely $[n,n]$ and $[n-1,n]$, to solve this problem numerically. Liou [2] and Bronson [3] studied the use of Taylor series and Cayley–Hamilton theorem respectively to provide techniques for computing $x(k\tau)$. Recently, Hassan and Ramadan [4] applied the Padé approximations $[1,1]$ and $[2,2]$ to evaluate these response vectors for a special type of matrix A, namely the companion matrix which arises in the case of n^{th}-order differential equations.

The main goal of this paper is devoted to using the restrictive Padé approximation for computing the response vectors $x(k\tau)$ of the linear time-invariant system (1).

PRELIMINARIES AND BASIC CONCEPTS

In this section we present a new technique of Padé approximation, namely the restrictive Padé approximation (RPA) to approximate the function $f(x)$ with parameters determined by Ganapathy and Rao [5]. If these parameters take zero, we get the classical Padé approximation. The main advantage of this technique is that the exact value at a certain value of x is given [6].

Definition 1 [6]. The RPA of a function f(x) is defined as

$$\text{RPA}\ [M+\alpha/N]_{f(x)}(x)=\frac{\sum_{i=0}^{M} a_i x^i+\sum_{i=1}^{\alpha}\varepsilon_i x^{i+M}}{1+\sum_{i=1}^{N} b_i x^i},\qquad \alpha=0,1,\ldots,N. \tag{3}$$

The formula (3) implies that

$$f(x)-\text{RPA}[M+\alpha/N]_{f(x)}(x)=O(x^{M+N+1}). \tag{4}$$

Assuming that $f(x)$ has the following Maclaurin series:

$$f(x)=\sum_{i=0}^{\infty} c_i x^i, \tag{5}$$

then from equations (3)-(5) we can get

$$\left(\sum_{i=0}^{\infty} c_i x^i\right)\left(1+\sum_{i=1}^{N} b_i x^i\right)-\sum_{i=0}^{M} a_i x^i-\sum_{i=1}^{\alpha}\varepsilon_i x^{i+M}=O(x^{M+N+1}).$$

This equation presents a system of (M+N+1) equations with (M+N+1) unknowns due to the vanishing of the first (M+N+1) powers of x. Also, if we assume that

$$\text{RPA}\ [M+\alpha/N]_{f(x)}(x_j)=f(x_j),\qquad j=1,2,\ldots,\alpha,$$

we get a system of α equations. This means that this approximation is exact at $(\alpha+1)$ of points $\{x_0\equiv 0,\ x_1,\ldots,x_\alpha\}$. Hence the $(M+N+\alpha+1)$ coefficients b_i, $i=1,2,\ldots,N$; a_i, $i=0,1,\ldots,M$; and ε_i, $i=1,2,\ldots,\alpha$ can be evaluated.

The local truncation error for the RPA can be summarised in the following theorem.

Theorem 1 [5]. If the function $f(x)$ has $(N+1)$ of derivatives, then for every argument $\bar{x}$ there exists a number λ in the smallest interval I containing the set of points $\{x_0,x_1,\ldots,x_\alpha,\bar{x}\}$ such that

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$$R_{M,N,\alpha}(\overline{x}) = f(\overline{x}) - RPA\,[(M+\alpha)/N]_{f(x)}(\overline{x}) = \frac{\Pi_{\alpha+1}(\overline{x})}{(\alpha+1)!}(R_{M,N,\alpha}(\lambda))^{(\alpha+1)},$$

where $\prod_{\alpha+1}(x) = (x-x_0)(x-x_1)(x-x_2)...(x-x_\alpha)$, and $R_{M,N,\alpha}$ is the local truncation error for RPA.

Test Example

For the function $f(x) = e^{-0.1x}$, we can find the following. The Padé approximation (PA) of $f(x)$ is given by Hassan and El-Barbary [7] :

$$PA[2/2]_{f(x)}(x) = \frac{1-0.05x+0.00083333x^2}{1+0.05x+0.00083333x^2},$$

and the RPA is given as

$$RPA\,[2/2]_{f(x)}(x) = \frac{1-0.04529241561\,x+0.000597954114\,x^2}{1+0.0547075844\,x+0.00106871255\,x^2},$$

where $\alpha = 1$, $x_1 = 30$. In Figure 1 we compare $PA[2/2]_{f(x)}(x)$ and $RPA[2/2]_{f(x)}(x)$.

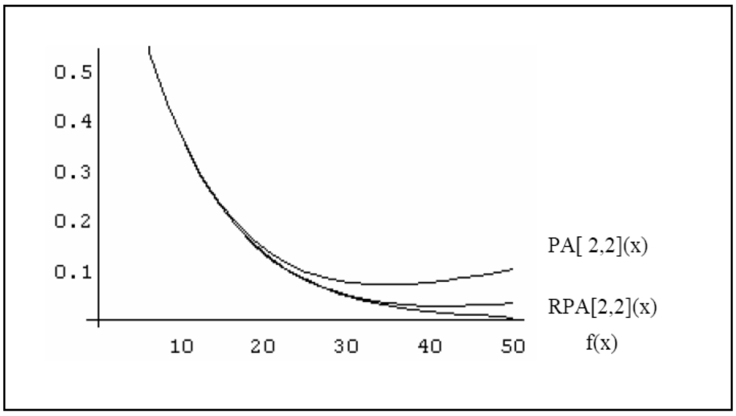

Figure 1. Plots of $PA\,[2/2]_{f(x)}(x)$ and $RPA\,[2/2]_{f(x)}(x)$ with f(x)

PA AND RPA FOR COMPUTING EXPONENTIAL MATRIX

In this section we present the definitions of the diagonal PAs [1,1] and [2,2] of the exponential matrix function e^{tA} and their accuracy [4].

Definition 2 [4]. The diagonal PA[1,1] of the exponential matrix function e^{tA} is given as

$$PA[1,1]_{e^{tA}}(t) = (I - \frac{t}{2}A)^{-1}(I + \frac{t}{2}A), \tag{6}$$

where I is the identity matrix. It is clear that for a companion matrix A, the matrices $I - \frac{t}{2}A$ and $I + \frac{t}{2}A$ are of the Hessenberg form. Also, the formula (6) has an accuracy of order three (truncation error being of $O(t^3)$).

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Definition 3 [4]. The diagonal PA[2,2] of the exponential matrix function e^{tA} is given as

$$\mathrm{PA}[2,2]_{e^{tA}}(t) = (I - \frac{t}{2}A + \frac{t^2}{12}A^2)^{-1}(I + \frac{t}{2}A + \frac{t^2}{12}A^2), \quad (7)$$

this formula (7) having an accuracy of order five (truncation error is of $O(t^5)$).

Now we derive a formula of the RPA[1,1] of the exponential matrix e^{tA} as follows.
Define $\mathrm{RPA}[1,1]_{e^{tA}}(t) = (I + b_1 t)^{-1}(a_0 + a_1 t + \varepsilon_1 t^2)$. (a)

From this formula we find that $f(t) = e^{tA} = R\mathrm{PA}[1,1] + O(t^3)$, (b)

where f(t) is the Maclaurin series of e^{tA}. From (a) and (b) we can write

$$(I + b_1 t)^{-1}(a_0 + a_1 t + \varepsilon_1 t^2) + O(t^3) = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots .$$

Multiplying both sides by $I + b_1 t$, we obtain

$$a_0 + a_1 t + \varepsilon_1 t^2 + O(t^3) = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots + b_1 t + b_1 A t^2 + \frac{1}{2!} b_1 A^2 t^3 + \dots . \quad (c)$$

Now without loss of generality we can choose $\varepsilon_1 = EA$, where E is a restrictive matrix. By comparing the coefficients of t^i in equation (c), where $i = 0,1,2$, we get the following system of equations:

$$a_0 = I, \quad a_1 = A + b_1, \quad EA = \frac{1}{2}A^2 + b_1 A.$$

The solution of this system is given by

$$a_0 = I, \quad a_1 = E + \frac{1}{2}A, \quad \text{and} \quad b_1 = E - \frac{1}{2}A.$$

By substituting in equation (a) we can write the RPA[1,1] in the following form :

$$\mathrm{RPA}[1,1]_{e^{tA}}(t) = (I + (E - \frac{1}{2}A)t)^{-1}(I + (E + \frac{1}{2}A)t), \quad (8)$$

which has an accuracy of order two (truncation error is of $O(t^2)$).

In the same way, we can derive a formula of the RPA[2,2] of the exponential matrix e^{tA} as follows. Define

$$\mathrm{RPA}[2,2]_{e^{tA}}(t) = (I + b_1 t + b_2 t^2)^{-1}(a_0 + a_1 t + a_2 t^2 + \varepsilon_1 t^3 + \varepsilon_2 t^4). \quad (d)$$

From this formula we find that

$$f(t) = \mathrm{RPA}[2,2]_{e^{tA}}(t) + O(t^5), \quad (e)$$

where f(t) is the Malaren series of e^{tA}. From (d) and (e) we can write

$$(I + b_1 t + b_2 t^2)^{-1}(a_0 + a_1 t + a_2 t^2 + \varepsilon_1 t^3 + \varepsilon_2 t^4) + O(t^5) = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \frac{1}{4!}(At)^4 + \dots .$$

Multiplying both sides by $I + b_1 t + b_2 t^2$, we obtain

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$$a_0 + a_1 t + a_2 t^2 + \varepsilon_1 t^3 + \varepsilon_2 t^4 + O(t^5) = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$$
$$+ b_1 t + b_1 A t^2 + \frac{1}{2!} b_1 A^2 t^3 + \frac{1}{6} b_1 A^3 t^4 + \ldots + b_2 t^2 + b_2 A t^3 + \frac{1}{2} b_2 A^2 t^4 + \ldots \quad \text{(f)}$$

Also, without loss of generality we can choose $\varepsilon_1 = EA$ and $\varepsilon_2 = \frac{1}{2}EA^2$, where E is a restrictive matrix. By comparing the coefficients of t^i in equation (f), where $i = 0,1,2,3,4$, we get the following system of equations :

$$a_0 = I, \quad a_1 = A + b_1, \quad a_2 = \frac{1}{2}A^2 + b_1 A + b_2,$$
$$EA = \frac{1}{6}A^3 + \frac{1}{2}b_1 A^2 + b_2 A, \quad \frac{1}{2}EA^2 = \frac{1}{24}A^4 + \frac{1}{6}b_1 A^3 + \frac{1}{2}b_2 A^2.$$

The solution of this system is given by

$$a_0 = I, \quad a_1 = \frac{1}{2}A, \quad a_2 = E + \frac{1}{12}A^2, \quad b_1 = -\frac{1}{2}A, \quad b_2 = E + \frac{1}{12}A^2.$$

So by substituting in equation (d) we obtain the RPA[2,2] as

$$RPA[2,2]_{e^{tA}}(t) = [I - \frac{t}{2}A + (E + \frac{1}{12}A^2)t^2]^{-1}[I + \frac{t}{2}A + (E + \frac{1}{12}A^2)t^2]. \quad (9)$$

This formula has an accuracy of order three (truncation error is of $O(t^3)$) [1].

Remark: The restrictive matrix E can be obtained by using the condition $RPA[M+\alpha, N]_{f(t)}(t) = f(t)$ at a certain value of t. If $E = 0$ in (8) and (9), we obtain the classical PAs (6) and (7) respectively.

RPA[1,1] FOR COMPUTING TRANSIENT RESPONSE VECTORS

In this section we rewrite the solution (2) in a stepwise fashion and apply the RPA[1,1]. The main idea is to keep the resulting matrices after applying this approximation in the Hessenberg form as done by Antsaklis and Michel [8]. At the same time we evaluate the square restrictive matrix which is obtained from the use of RPA efficiently, taking into account the advantage of the special structure of the matrix A. Thus, the stable LU-factorisation can be used effectively and the total efficiency is maintained [9, 10].

Now the transient response vectors (2) can be written in an iterative form as

$$x(k\tau) = e^{\tau A} x((k-1)\tau), \qquad k = 1,2,\ldots, \quad (10)$$

where τ is the step time. By using the formula (8) of RPA[1,1] for $e^{\tau A}$, the transient response vectors (10) take the following form:

$$x(k\tau) = [I + (E - \frac{1}{2}A)\tau]^{-1}[I + (E + \frac{1}{2}A)\tau]\, x((k-1)\tau). \quad (11)$$

Without loss of generality we can choose the matrix E of the diagonal form, i.e. $E = diag[e_1, e_2, e_3, \ldots, e_n]$; the elements e_i, which depend on τ, will be determined after specifying the value of τ and can be generally written $e_i(\tau)$. Now by using the condition that the truncation error of RPA[1,1] is equal to zero at a certain value of $k\tau$, we can compute the restrictive matrix E. Assuming that at $k = 1$ the truncation error is zero, we rewrite (11) in the form

$$[I + (E - \frac{1}{2}A)\tau]\,\mathbf{x}(\tau) = [I + (E + \frac{1}{2}A)\tau]\,x(0), \qquad (12)$$

where $x(0)$ is the given initial condition and $\mathbf{x}(\tau)$ is the exact solution computed at τ.

Remark: In general, if the exact solution at the level k is unknown, we can use another highly accurate numerical method to evaluate the matrix E and then continue as before.

We can calculate the elements e_i by using the following lemma.

Lemma 1. The restrictive matrix E is determined with the cost of O(n) operations.

Proof. Equation (12) can be written as

$$(I + \tau E)(x(\tau) - x(0)) = \frac{1}{2}\tau A(x(\tau) + x(0)).$$

By setting $d = \mathbf{x}(\tau) - \mathbf{x}(0)$ and $b = \mathbf{x}(\tau) + \mathbf{x}(0)$, we can get

$$(I + \tau E)\,d = 0.5\,\tau\,Ab. \qquad (13)$$

Since the matrix E takes the form $E = \mathrm{diag}[e_1, e_2, e_3, \ldots, e_n]$, we can write d and b as $d = [d_1, d_2, d_3, ..., d_n]^T$ and $b = [b_1, b_2, b_3, ..., b_n]^T$ respectively, and the n^{th} row c of the matrix A as $c = [c_1, c_2, c_3, ..., c_n]$.

Now if we take $\mu = 0.5\tau$ and $\lambda = \frac{1}{\tau}$ in (13), we can obtain the following formula to find the elements e_i as

$$e_n = \lambda(\frac{\mu\, cb}{d_n} - 1), \qquad e_i = \lambda(\frac{\mu\, b_{i+1}}{d_i} - 1), \quad i = 1, 2, \ldots, n-1. \qquad (14)$$

For a certain value of τ we can calculate the exact solution $x(\tau)$ (i.e. E depending on the exact solution). Thus, we get the entries of the matrix E using scalar vectors multiplication, where such computation requires O(n) of operations. This proves the lemma.

PROPOSED ALGORITHM

The approximate transient response vectors (11) can be written as

$$Hx(k\,\tau) = Nx((k-1)\tau), \qquad k = 1,2,..., \qquad (15)$$

where the matrices $H = [I + (E - \frac{1}{2}A)\tau]$ and $N = [I + (E + \frac{1}{2}A)\tau]$ are of the Hessenberg forms. The approximation $x(k\tau)$ is computed using the following steps:

1. Compute the vector $y_{k-1} = Nx((k-1)\tau)$ where it takes $o(n^2)$ flops for each k.

2. Solve the Hessenberg linear system $Hx(k\tau) = y_{k-1}$ using the stable Gaussian elimination method to factorise H into LU form, where in this case L will be of bi-diagonal form and U will be of upper triangular form. It is clear that LU-factorisation costs $\frac{n^2}{2}$ of operations and will only be carried out once it does not depend on k as the case in the direct method.

3. Solve the resulting system $LUx(k\tau) = y_{k-1}$ for the vector $x(k\tau)$ through two steps:

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[i] Solve the bi-diagonal system $Lz_k = y_{k-1}$;

[ii] Solve the upper triangular system $Ux(k\tau) = z_k$.

It is easy to find that the first step needs $o(n)$ operations while the second step takes $\frac{n^2}{2}$ operations. Therefore, the total cost of computing $x(k\tau)$ for each k still takes $o(n^2)$ flops and thus the efficiency of the proposed algorithm is the same in the case of using regular PA[1,1].

NUMERICAL RESULTS

In this subsection we consider the following first-order system of three equations:

$$\dot{x} = Ax,$$

with $x(0) = [2 \quad -2.5 \quad 3.75]^T$ and matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.75 & -2.75 & -3 \end{pmatrix}$.

To test the accuracy of the proposed algorithm we compute the approximate response vectors $x(k\tau)$ at $k\tau = 0.1, 0.2, 0.3, 0.4, 0.5$ by using two different time steps τ.

Case I

We compute the exact solution $x(\tau)$ at $\tau = 0.1$ and evaluate the matrix E by using (14) :

$E = \text{diag}[\,0.019510595109799 \quad 0.020426896827462 \quad 0.020191207544100\,2\,]$.

Then for a fixed time step $\tau = 0.1$ and $k = 1, 2, 3, 4, 5$ we can compute the approximate transient response vectors $x(k\tau)$ by using the iterative process (11) with the help of RPA[1,1] and the regular (E=0) PA[1,1]. The numerical results for this case are summarised in Table 1, where x_e, x_{rp} and x_p denote the exact solution and the approximate solution by using RPA[1,1] and PA[1,1] respectively. Also, in this table the infinite norm errors ($\|x_e - x_{rp}\|_\infty$, $\|x_e - x_p\|_\infty$) are introduced.

Table 1. Comparison of infinite norm errors for RPA[1,1] with exact solution ($\|x_e - x_{rp}\|_\infty$) and PA[1,1] with exact solution ($\|x_e - x_p\|_\infty$) for $\tau = 0.1$

$k\tau$	$\|x_e - x_{rp}\|_\infty$	$\|x_e - x_p\|_\infty$
0.1	0	1.01983545929096e-03
0.2	3.03785716249116e-06	1.74908521276010e-03
0.3	8.82407835600670e-06	2.24943727514315e-03
0.4	1.70675743839823e-05	2.57099390525894e-03
0.5	2.73425751020628e-05	2.75430489329698e-03

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Case II

We test the proposed algorithm on the same test example as that for a different smaller step $\tau = 0.01$. For this case we compute the exact solution $x(\tau)$ at $\tau = 0.01$ and the corresponding restrictive matrix E is given by

$$E = diag[\,0.001957698052313 \quad 0.002041851921386 \quad 0.00201714687400045\,].$$

Then for the fixed time step $\tau = 0.01$, the approximate transient response vectors $x(k\tau)$ for $k = 1, 2, 3, 4, 5$ are computed by using the RPA[1,1] and the PA[1,1] as in Case I. The obtained numerical results are summarised in Table 2. In this case the number of iterations is different for each $k\tau$; for example, at $k\tau = 0.1$ we use ten iterations. All numerical results are obtained by using MATLAB version 7.1 in single precision.

Table 2. Comparison of infinite norm errors for RPA[1,1] with exact solution ($\|x_e - x_{rp}\|_\infty$) and PA[1,1] with exact solution ($\|x_e - x_p\|_\infty$) for $\tau = 0.01$

$k\tau$	$\|x_e - x_{rp}\|_\infty$	$\|x_e - x_p\|_\infty$
0.1	1.43655642936833e-08	1.01644171861537e-05
0.2	5.65894933135524e-07	1.74350979085247e-05
0.3	1.25602971579042e-07	2.42583089362430e-05
0.4	2.17672743385577e-07	2.56352253273917e-05
0.5	3.29511339103306e-07	2.74669075999915e-05

HIGHLY ACCURATE METHOD FOR COMPUTATION OF TRANSIENT RESPONSE VECTORS

In this section we use the formula (9) of RPA[2,2] to improve the accuracy with a little more cost of computing the transient response vectors $x(k\tau)$. To achieve this aim, we rewrite the transient response vectors $x(k\tau)$ (Eq.(2)) in the following iterative form :

$$x(k\tau) = (I - \frac{\tau}{2}A + (E + \frac{1}{12}A^2)\tau^2)^{-1}(I + \frac{\tau}{2}A + (E + \frac{1}{12}A^2)\tau^2)\, x((k-1)\tau). \tag{16}$$

Also, by using the condition that the truncation error of RPA[2,2] is equal to zero at a certain restrictive $k\tau$, we can compute the restrictive matrix E. Assuming that at $k = 1$ the truncation error is zero, we can rewrite (16) as

$$(I - \frac{\tau}{2}A + (E + \frac{1}{12}A^2)\tau^2)x(\tau) = (I + \frac{\tau}{2}A + (E + \frac{1}{12}A^2)\tau^2)\, x(0), \tag{17}$$

where $x(0)$ and $x(\tau)$ are the initial vector and the exact solution which is computed for a certain value of τ respectively. Also, E is assumed to have the diagonal form $E = diag[e_1, e_2, e_3, \ldots, e_n]$.

We can calculate the elements e_i of the matrix E by using the following lemma.

Lemma 2. The restrictive matrix E is determined with the cost of $O(n^2)$ operations.

Proof. Equation (17) can be rewritten as

$$(I + \tau^2 E)(x(\tau) - x(0)) = \frac{\tau}{2}A\,(x(\tau) + x(0)) - \frac{\tau^2}{12}A^2(x(\tau) - x(0)),$$

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or

$$(I + \tau^2 E)\, d = \alpha\, Ab - \beta\, A^2 d, \tag{18}$$

where $d = x(\tau) - x(0)$, $b = x(\tau) + x(0)$, $\alpha = 0.5\tau$, $\beta = \frac{\tau^2}{12}$ and $\gamma = \frac{1}{\tau^2}$.

Since matrix E takes the form $E = \mathrm{diag}[e_1, e_2, e_3, \ldots, e_n]$, d and b can be written as $d = [d_1, d_2, d_3, \ldots, d_n]^T$ and $b = [b_1, b_2, b_3, \ldots, b_n]^T$ respectively, and the n^{th} row c of matrix A is $c = [c_1, c_2, c_3, \ldots, c_n]$. Now from (18) the elements e_i of matrix E can be computed as

$$e_{n-1} = \gamma\left[\frac{\alpha\, b_n - \beta\, cd}{d_{n-1}} - 1\right], \qquad e_n = \gamma\left[\frac{\alpha\, cb - \beta\,[cdc_n + \sum_{i=1}^{n-1} c_i d_{i+1}]}{d_n} - 1\right],$$
$$e_i = \gamma\left[\frac{\alpha\, b_{i+1} - \beta\, d_{i+2}}{d_i} - 1\right], \quad i = 1, 2, \ldots n-2. \tag{19}$$

Then for a certain value of τ, we can calculate the exact solution $x(\tau)$ (i.e. E depending on the exact solution). Thus, we obtain the entries of matrix E by using scalar vectors multiplication, where such computation requires $O(n^2)$ flops. This proves the lemma.

NUMERICAL IMPLEMENTATION

We consider the same previous example $\dot{x} = Ax$ to test the accuracy of the proposed algorithm. To compute the approximate response vectors $x(k\tau)$ at $k\tau = 0.1, 0.2, 0.3, 0.4, 0.5$, we compute the exact solution $x(\tau)$ at $\tau = 0.1$ and evaluate matrix E using equation (19), which is given by

$$E = \mathrm{diag}\,[-0.00007880717300 \quad -0.00008085870755074 \quad -0.0000785856598].$$

Then for the fixed time step $\tau = 0.1$, the approximate transient response vectors $x(k\tau)$ for $k = 1, 2, 3, 4, 5$ are computed by using RPA[2,2] and PA[2,2]. The obtained numerical results are summarised in Table 3.

Table 3. Comparison of infinite norm errors for RPA[2,2] with exact solution ($\|x_e - x_{rp}\|_\infty$) and PA[2,2] with exact solution ($\|x_e - x_p\|_\infty$) for $\tau = 0.1$

$k\tau$	$\|x_e - x_{rp}\|_\infty$	$\|x_e - x_p\|_\infty$
0.1	1.332267629550188e-15	3.965757566353100e-07
0.2	1.754661749231445e-09	6.815807349624947e-07
0.3	4.617700000153491e-09	8.784833589459140e-07
0.4	8.093338443870834e-09	1.006373423262375e-06
0.5	1.180780251353042e-08	1.080726866220516e-06

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CONCLUSIONS

The main goal of this work is to introduce and implement the RPA for computing the transient response vectors of a linear time-invariant system. In the light of the obtained numerical results, we found that the proposed method is more efficient and more accurate than the method based on PA. The infinite norm errors of the given solutions from RPA are less than those from the classical Padé method. It is clear that for a smaller time step we obtain a better accuracy where the RPA[1,1] is used. The drawback for the smaller time step is that the number of iterations needed to compute $x(k\tau)$ is increased. However, for large systems we can see that the number of operations depends essentially on the dimension of the system.

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