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Full Paper

Common solutions to a system of generalised unrelated variational inequalities

Haider Abbas Rizvi [1], Rais Ahmad [1] and Vishnu Narayan Mishra [2,*]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
[2] Department of Applied Mathematics and Humanities, Sardar Vallabhbhai, National Institute of Technology, Surat 395 007, Gujarat, India
* Corresponding author, e-mail: vishnunarayanmishra@gmail.com

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Abstract: A system of generalised unrelated variational inequalities in a real Hilbert space is considered. We construct an iterative algorithm for finding common solutions to the system and establish strong convergence of the algorithm.

Keywords: generalised unrelated variational inequalities, monotonicity, variational inequality, Hilbert space

INTRODUCTION

In the early sixties Hartman and Stampacchia [1] introduced and studied variational inequalities as a natural and significant extension of variational principles. The variational inequalities have numerous applications in diverse areas of basic sciences and prove to be productive and innovative. Pang [2] showed that the equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities.

Using the projection technique, one may usually establish the equivalence between a system of variational inequalities and fixed point problems. Most of the iterative methods for solving the system of variational inequalities have been considered in a convex setting. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general when the sets are non-convex.

Ceng and Yao [3] introduced and studied an implicit process with perturbed mappings for finding a common fixed point of infinitely many non-expansive mappings, and Yao et al. [4] introduced and considered an iterative scheme for finding a common element of the set of solutions to an equilibrium problem and the set of common fixed points of infinitely many non-expansive mappings in Hilbert spaces. Zegeye and Shahzad [5] introduced an iterative process with strong convergence to a common solution to the variational inequality problem for two monotone mappings in Banach spaces. Censor et al. [6] studied a new variational inequality problem involving the common solutions to variational inequalities. Other items of related work can be found in Kumam et al. [7] and references therein.

In this paper we generalise the problem of Censor et al. [6] and obtain the common solutions to a system of generalised unrelated variational inequalities in a real Hilbert space. We present an iterative procedure to solve the system and establish the procedure's strong convergence.

PRELIMINARIES

Throughout the paper we assume that H is a real Hilbert space with its norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. When $\{x_n\}_{n\in N}$ converges weakly to x, we denote it by $w\text{-}\lim_{n\to\infty} x^n = x$, and when $\{x_n\}_{n\in N}$ converges strongly to x, we denote it by $\lim_{n\to\infty} x^n = x$. The following concepts and results are essential to prove the main result of this paper.

Condition 1 (Opial Condition**)** [8]**.** A Hilbert space H is said to satisfy the Opial condition if, for every $x \in H$ and each sequence $\{x_n\}_{n\in N} \in H$ that converges weakly to x,

$$\lim_{n\to+\infty}\inf \|x^n - x\| < \lim_{n\to+\infty}\inf \|x^n - y\| \tag{1}$$

holds for $y \neq x$.

Condition 2 (Kadec-Klee property**)** [9]**.** A Hilbert space H is said to satisfy the Kadec-Klee property if $\|x^n - x\| \to 0$ whenever $\|x^n\| \to \|x\|$ and $x_n \to x$ weakly.

Definition 1. A function $g: H \to (-\infty, +\infty]$ is called weakly lower semicontinuous if

$$g(x) \leq \lim_{n\to+\infty}\inf g(x^n) \tag{2}$$

for any sequence $\{x_n\}_{n\in N}$ which satisfies $w\text{-}\lim_{n\to+\infty} x^n = x$.

Definition 2. Let $f: H \to H$ be a single-valued mapping; then

(i) f is monotone if, for all $x, y \in H$,

$$\langle f(x) - f(y), x - y\rangle \geq 0;$$

(ii) f is Lipschitz continuous if there exists a constant $\alpha > 0$ such that for all $x, y \in H$,

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|.$$

Definition 3. A set-valued mapping $A: H \to 2^H$ is said to be

(i) monotone if

$$\langle u - v, x - y \rangle \geq 0 \text{ for all } x, y \in H,\ u \in A(x) \text{ and } v \in A(y); \tag{3}$$

(ii) maximal monotone if it is monotone and the graph of A is not properly contained in the graph of any other monotone mapping.

Let K be a non-empty, closed and convex subset of H. For each point $x \in H$, there exists a unique nearest point in K, denoted by $P_K(x)$, i.e.

$$\|x - P_K(x)\| \leq \|x - y\|, \tag{4}$$

for all $y \in K$. The operator $P_K: H \to K$ is called the metric projection of H onto K. It is well known that P_K is the non-expansive operator from H onto K. The projection P_K is characterised [10] by the following two properties:

$$P_K(x) \in K \quad (5)$$

and

$$\langle x - P_K(x), y - P_K(x) \rangle \leq 0, \quad (6)$$

for all $x \in H$, $y \in K$. If K is a hyperplane, then (6) becomes an equality. It is easy to check that (6) is equivalent to

$$\|x - P_K(x)\|^2 + \|y - P_K(x)\|^2 \leq \|x - y\|^2, \quad (7)$$

for all $x \in H, y \in K$. We denote by $N_K(v)$ the normal cone of K at $v \in K$, i.e.

$$N_K(v) = \{ z \in H: \langle z, y - v \rangle \leq 0, \ \forall\, y \in K \}. \quad (8)$$

We recall that in a real Hilbert space H the following property holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad (9)$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 1 [6]**.** Consider the half-space

$$H(x,y) = \{ z \in H : \langle x - y, z - y \rangle \leq 0 \}. \quad (10)$$

Given two points x and y in H and setting $y_\lambda = \lambda x + (1-\lambda)y$ for any $\lambda \in [0,1]$, then $H = H(x,y) \subseteq H(x, y_\lambda)$.

FORMULATION OF THE PROBLEM AND ITERATIVE ALGORITHM

In this section first we formulate the problem of finding common solutions to a system of generalised unrelated variational inequalities in a real Hilbert space.

For $i = 1,2,\cdots,n$, let $A_i, B_i: H \to 2^H$ be the set-valued mappings and $K_i \subseteq H$ with $\cap_{i=1}^n K_i \neq \emptyset$ be the non-empty, closed and convex subsets. Let $f, g: H \to H$ be single-valued mappings. We consider the following problem of obtaining common solutions to a system of generalised unrelated variational inequalities:

Find a point $x^* \in \cap_{i=1}^n K_i$ such that for each $i = 1,2,\cdots,n$, there exist $u_i^* \in A_i(x^*)$, $v_i^* \in B_i(x^*)$ and

$$\langle f(u_i^*) - g(v_i^*), x - x^* \rangle \geq 0. \quad (11)$$

We denote the solution set of problem (11) by $SLSVI(A_i, B_i, K_i)$.

When $f, g = I$, the identity mappings and $B_i \equiv 0$, then problem (11) coincides with problem (1.1) of Censor et al. [6], and consequently one can obtain the convex feasibility problem and the common fixed-point problem from (11) for $n > 1$ by simple observations. For more details, see Konnov [11], Ansari and Yao [12] and Kassay and Kolumban [13]. It is clear that for suitable choices of operators involved in the formulation of problem (11), one can obtain many related problems studied previously.

We now construct an iterative algorithm for finding the common solution to a system of generalised unrelated variational inequalities (11).

Algorithm 1. For $i = 1,2,\cdots,n$, let $A_i, B_i: H \to CB(H)$ be the set-valued mappings and all other mappings and conditions be the same as stated in problem (11).

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For a given initial point $x^1 \in H$, compute the iterative procedure as:

$$\begin{cases} y_i^n = P_{K_i}\big(x^n - \lambda_i^n f(u_i^n) - \beta_i^n g(v_i^n)\big), \;\; u_i^n \in A_i(x^n), v_i^n \in B_i(x^n). \\ find\; w_i^n \in A_i(y_i^n)\; and\; s_i^n \in B_i(y_i^n)\; such\; that \\ z_i^n = P_{K_i}\big(x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n)\big), \\ \tilde{C}_i^n = \{z \in H: \|z_i^n - z\| \leq \|x^n - z\|\}, \\ C^n = \cap_{i=1}^n \tilde{C}_i^n, \\ W^n = \{z \in H: \langle x^1 - x^n, z - x^n \rangle \leq 0\}, \\ x^{n+1} = P_{\tilde{C}_i^n \cap W^n}(x^1). \end{cases} \tag{12}$$

The elements w_i^n , s_i^n, together with u_i^n, v_i^n respectively, satisfy the following conditions:

$$\begin{cases} \|w_i^n - u_i^n\| \leq \hat{H}\big(A_i(y_i^n), A_i(x^n)\big) \leq k_1 \|y_i^n - x^n\|, \\ and \\ \|s_i^n - v_i^n\| \leq \hat{H}\big(B_i(y_i^n), B_i(x^n)\big) \leq k_2 \|y_i^n - x^n\|, \end{cases} \tag{13}$$

where $\hat{H}(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(H)$.

We remark that from our Algorithm 1, one can easily get Algorithm 3.1 of Censor et al. [6] by simple computation, and thus our algorithm is more general.

STRONG CONVERGENCE THEOREM

In this section we discuss the convergence criteria of the proposed iterative algorithm. We need the following lemma for the proof of our main result.

Lemma 2. Let $A: H \to 2^H$ be the maximal monotone operator. Then

(i) graph of A is closed;

(ii) the solution set $SLSVI(A_i, B_i, K_i)$ is closed and convex for all closed and convex subsets $K_i \subseteq H$.

Proof. The proof is similar to that for Lemma 2.4(ii) of Cruz and Iusem [14].

We now prove our main result, which is a generalisation of Theorem 3.6 of Censor et al. [6].

Theorem 1. *Let $A_i, B_i: H \to 2^H$ be maximal monotone and Lipschitz continuous mappings, with Lipschitz constants k_1 and k_2 respectively. Let $f, g: H \to H$ be single-valued monotone and Lipschitz continuous mappings, with Lipschitz constants α_1 and α_2 respectively. Let the common solution set $F = \bigcap_{i=1}^n SLSVI(A_i, B_i, K_i)$ be non-empty. Let the sequences $\{\lambda_i^n\}, \{\beta_i^n\} \subset [0,1]$ be such that*

$(i)\; \lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2 \leq 1;$

$(ii)\; \frac{\beta_i^n}{\lambda_i^n} \leq 1;$

$(iii)\; |\lambda_i^n||\beta_i^n| \leq ab\;\; where\; 0 < a < b < \frac{1}{k_i}, i = 1,2.$

Then the sequences $\{x^n\}_{n\in N}$, $\{y_i^n\}_{n\in N}$ and $\{z_i^n\}_{n\in N}$, generated by Algorithm 1, converge strongly to $P_F(x^1)$.

Proof. For the convenience, we divide the proof into the following four steps.

Step 1. The projection $P_F(x^1)$ and the sequence $\{x^n\}_{n\in N}$ are well defined.

Clearly by Lemma 2, $SLSVI(A_i, B_i, K_i)$ is a closed and convex subset of H. Hence F is non-empty, closed and convex so that $P_F(x^1)$ is well defined. Also, it is clear that both C_i^n and W^n are

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closed half-spaces for all $n \geq 1$. Therefore, C^n and $C^n \cap W^n$ are closed and convex for all $n \geq 1$. Further, we are to show that $C^n \cap W^n \neq \emptyset$ for all n.

From Algorithm 1, we have

$$\tilde{C}_i^n = \{ z \in H : \|z_i^n - z\| \leq \|x^n - z\| \}.$$

It follows from Lemma 1 that

$$\tilde{C}_i^n \subseteq \{ z \in H : \langle x - y, z - y \rangle \leq 0 \}.$$

Let $\tilde{C}^n = \bigcap_{i=1}^n \tilde{C}_i^n$. Now we show that $F \subseteq \tilde{C}^n \cap W^n$ for all $n \in N$. First, we prove $F \subseteq \tilde{C}^n$. For this, let $t \in F$, $b_i^n \in A_i(t)$ and $d_i^n \in B_i(t)$. Using the monotonicity of f and g, equation (7) and the fact that $t \in SLSVI(A_i, B_i, K_i)$, it follows that

$$\begin{aligned}
\|z_i^n - t\|^2 &= \left\|P_{K_i}\left(x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n)\right) - t\right\|^2 \\
&\leq \|x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n)) - t\|^2 - \|x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n)) - z_i^n\|^2 \\
&= \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\langle t - z_i^n, \lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n)\rangle \\
&= \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n\langle f(w_i^n), t - z_i^n\rangle + 2\beta_i^n\langle g(s_i^n), t - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n\langle f(w_i^n), t - y_i^n\rangle + 2\lambda_i^n\langle f(w_i^n), y_i^n - z_i^n\rangle \\
&\quad + 2\beta_i^n\langle g(s_i^n), t - y_i^n\rangle + 2\beta_i^n\langle g(s_i^n), y_i^n - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n\langle f(w_i^n) - f(b_i^n) + f(b_i^n), t - y_i^n\rangle \\
&\quad + 2\lambda_i^n\langle f(w_i^n), y_i^n - z_i^n\rangle + 2\beta_i^n\langle g(s_i^n) - g(d_i^n) + g(d_i^n), t - y_i^n\rangle \\
&\quad + 2\beta_i^n\langle g(s_i^n), y_i^n - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n\langle f(w_i^n) - f(b_i^n), t - y_i^n\rangle + 2\lambda_i^n\langle f(b_i^n), t - y_i^n\rangle \\
&\quad + 2\beta_i^n\langle g(s_i^n) - g(d_i^n), t - y_i^n\rangle + 2\beta_i^n\langle g(d_i^n), t - y_i^n\rangle + 2\beta_i^n\langle g(d_i^n), t - y_i^n\rangle \\
&\quad + 2\langle \lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n), y_i^n - z_i^n\rangle \\
&\leq \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n\langle f(b_i^n), t - y_i^n\rangle + 2\beta_i^n\langle g(d_i^n), t - y_i^n\rangle \\
&\quad + 2\langle \lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n), y_i^n - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - z_i^n\|^2 + 2\langle\lambda_i^n f(b_i^n) + \beta_i^n g(d_i^n), t - y_i^n\rangle \\
&\quad + 2\langle\lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n), y_i^n - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - y_i^n + y_i^n - z_i^n\|^2 + 2\langle\lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n), y_i^n - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - z_i^n\|^2 - 2\langle x^n - y_i^n, y_i^n - z_i^n\rangle \\
&\quad + 2\langle\lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n), y_i^n - z_i^n\rangle \\
&= \|x^n - t\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - z_i^n\|^2 \\
&\quad + 2\langle\lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n) + y_i^n - x^n, y_i^n - z_i^n\rangle. \qquad (14)
\end{aligned}$$

Using equation (6), we have

$$\begin{aligned}
\langle\lambda_i^n f(w_i^n) + \beta_i^n g(s_i^n) + y_i^n - x^n, y_i^n - z_i^n\rangle &= \langle x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n) - y_i^n, z_i^n - y_i^n\rangle \\
&= \langle x^n - \lambda_i^n f(u_i^n) - \beta_i^n g(v_i^n) - y_i^n, z_i^n - y_i^n\rangle \\
&\quad + \langle \lambda_i^n f(u_i^n) + \beta_i^n g(v_i^n) - \lambda_i^n f(w_i^n) \\
&\quad - \beta_i^n g(s_i^n), z_i^n - y_i^n\rangle \\
&\leq \lambda_i^n\langle f(u_i^n) - f(w_i^n), z_i^n - y_i^n\rangle \\
&\quad + \beta_i^n\langle g(v_i^n) - g(s_i^n), z_i^n - y_i^n\rangle. \qquad (15)
\end{aligned}$$

Using Cauchy-Schwarz inequality and inequality (15), we have

$$\begin{aligned}
\langle x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n) - y_i^n, z_i^n - y_i^n\rangle &\leq \lambda_i^n\|f(u_i^n) - f(w_i^n)\|\,\|z_i^n - y_i^n\| \\
&\quad + \beta_i^n\|g(v_i^n) - g(s_i^n)\|\|z_i^n - y_i^n\|.
\end{aligned}$$

Using Lipschitz continuity of f, g, A_i and B_i, the above inequality becomes

$$\begin{aligned}\langle x^n - \lambda_i^n f(w_i^n) - \beta_i^n g(s_i^n) - y_i^n, z_i^n - y_i^n\rangle &\leq \lambda_i^n \alpha_1 k_1 \|x^n - y_i^n\| \|z_i^n - y_i^n\| \\ &\quad + \beta_i^n \alpha_2 k_2 \|x^n - y_i^n\| \|z_i^n - y_i^n\| \\ &= (\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2) \|x^n - y_i^n\| \\ &\quad \|z_i^n - y_i^n\|. \end{aligned} \tag{16}$$

Using (16), the inequality (14) becomes

$$\begin{aligned}\|z_i^n - t\|^2 \leq \|x^n - t\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - z_i^n\|^2 \\ + 2(\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)\|x^n - y_i^n\| \|z_i^n - y_i^n\|. \end{aligned} \tag{17}$$

Now

$$\begin{aligned} 0 &\leq \left((\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)\|x^n - y_i^n\| - \|z_i^n - y_i^n\|\right)^2 \\ &= (\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)^2 \|x^n - y_i^n\|^2 + \|z_i^n - y_i^n\|^2 \\ &\quad - 2(\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)\|x^n - y_i^n\| \|z_i^n - y_i^n\|, \end{aligned} \tag{18}$$

or

$$\begin{aligned} 2(\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)\|x^n - y_i^n\| \|z_i^n - y_i^n\| \leq (\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)^2 \|x^n - y_i^n\|^2 \\ + \|z_i^n - y_i^n\|^2. \end{aligned} \tag{19}$$

Therefore, inequality (17) becomes

$$\begin{aligned} \|z_i^n - t\|^2 &\leq \|x^n - t\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - z_i^n\|^2 + (\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)^2 \|x^n - y_i^n\|^2 \\ &\quad + \|z_i^n - y_i^n\|^2 \\ &= \|x^n - t\|^2 - \|x^n - y_i^n\|^2 + (\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)^2 \|x^n - y_i^n\|^2 \\ &= \|x^n - t\|^2 - \|x^n - y_i^n\|^2 \left(1 - (\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2)^2\right). \end{aligned} \tag{20}$$

Since $\lambda_i^n k_1 \alpha_1 + \beta_i^n k_2 \alpha_2 \leq 1$, we have $\|z_i^n - t\|^2 \leq \|x^n - t\|^2$. Therefore, $t \in C^n$. Consequently $F \subseteq \tilde{C}_i^n$ for all $n \geq 1$.

Applying mathematical induction, we then show that sequence $\{x^n\}$ is well defined. Clearly, $F \subseteq \tilde{C}^1$ and $F \subseteq W^1 = H$, so it follows that $F \subseteq \tilde{C}^1 \cap W^1$ and therefore $x^2 = P_{\tilde{C}^1 \cap W^1}(x^1)$ is well defined. Now let $F \subseteq \tilde{C}^{n-1} \cap W^{n-1}$ for some $n > 2$ and let $x^n = P_{\tilde{C}^{n-1} \cap W^{n-1}}(x^1)$. Now we have $F \subseteq \tilde{C}^n$ and for any $t \in F$, it follows from the property of projection operator that

$$\langle x^1 - x^n, t - x^n\rangle = \langle x^1 - P_{\tilde{C}^{n-1} \cap W^{n-1}}(x^1), t - P_{\tilde{C}^{n-1} \cap W^{n-1}}(x^1)\rangle \leq 0.$$

This implies that $t \in W^n$. Therefore, $F \subseteq \tilde{C}^n \cap W^n$ for any $n \geq 1$. Hence the sequence $\{x_n\}$ is well defined.

Step 2. The sequences $\{x^n\}_{n \in N}, \{y_i^n\}_{n \in N}$ and $\{z_i^n\}_{n \in N}$ are bounded.

Since $x^{n+1} = P_{C^n \cap W^n}(x^1)$, we have, for any $t \in C^n \cap W^n$,

$$\|x^{n+1} - x^1\| \leq \|t - x^1\|. \tag{21}$$

Hence $\{x^n\}_{n \in N}$ is bounded. From the definition of W^n, we have $x^n = P_{W^n}(x^1)$. Since $x^{n+1} \in W^n$, it follows from (7) that

$$\|x^{n+1} - x^n\|^2 + \|x^n - x^1\|^2 \leq \|x^{n+1} - x^1\|^2. \tag{22}$$

Thus, the sequence $\|x^n - x^1\|_{n \in N}$ is increasing and bounded, and hence convergent. Also,

$$lim_{n \to \infty} \|x^{n+1} - x^n\| = 0. \tag{23}$$

Since $x^{n+1} \in C_i^n$, we have

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$$\|z_i^n - x^n\| \le \|x^n - x^{n+1}\|, \tag{24}$$

and therefore,

$$lim_{n\to\infty}\|z_i^n - x^n\| = 0, for\ i = 1,2,\cdots,n. \tag{25}$$

Thus, $\{z_i^n\}$ is a bounded sequence for each $i = 1,2,\cdots,n$. It follows from (20) that

$$\|x^n - y_i^n\|^2 \left(1 - (\lambda_i^n k_1\alpha_1 + \beta_i^n k_2\alpha_2)^2\right) \le \|x^n - t\|^2 - \|z_i^n - t\|^2$$

$$\begin{aligned}\|x^n - y_i^n\|^2 &\le \left(1 - (\lambda_i^n k_1\alpha_1 + \beta_i^n k_2\alpha_2)^2\right)^{-1} (\|x^n - t\|^2 \\ &\quad -\|z_i^n - t\|^2) \\ &= \left(1 - (\lambda_i^n k_1\alpha_1 + \beta_i^n k_2\alpha_2)^2\right)^{-1} (\|x^n - t\| + \|z_i^n + t\|) \\ &\quad (\|x^n - t\| - \|z_i^n - t\|) \\ &\le \left(1 - (\lambda_i^n k_1\alpha_1 + \beta_i^n k_2\alpha_2)^2\right)^{-1}(\|x^n - z_i^n\|) \\ &\quad (\|x^n - t\| + \|z_i^n - t\|).\end{aligned} \tag{26}$$

Now using (25), condition (i), and the fact that $\{x^n\}_{n\in N}$ and $\{z_i^n\}_{n\in N}$ are bounded, we have

$$lim_{n\to\infty}\|x^n - y_i^n\| = 0, for\ all\ i = 1,2,\cdots,n. \tag{27}$$

Therefore, $\{y_i^n\}$ is also bounded.

Step 3. Any weak limit point of the sequences $\{x^n\}_{n\in N}, \{y_i^n\}_{n\in N}$ and $\{z_i^n\}_{n\in N}$ belongs to F.

Since $\{x^n\}$ is bounded, there exists a subsequence $\{x^{n_j}\}_{j\in N}$ of $\{x^n\}$ which converges weakly to x^*. So it follows from (27) that there exists a subsequence $\{y_i^{n_j}\}_{j\in N}$ of $\{y_i^n\}$ which converges to x^* for each $i = 1,2,\cdots,n$. Next, we show $x^* \in F$. Let T_i and T_j be the mappings defined by

$$T_i(v) = \begin{cases} A_i(v) + N_{K_i}(v) & v \in K_i; \\ \emptyset & v \notin K_i, \end{cases} \tag{28}$$

and

$$T_j(v) = \begin{cases} B_j(v) + N_{K_j}(v) & v \in K_j; \\ \emptyset & v \notin K_j, \end{cases} \tag{29}$$

where $N_{K_i}(v), N_{K_j}(v)$ are normal cones of K_i, K_j respectively at $v \in K_i \cap K_j$. Since A_i and B_j are maximal monotone mappings, then from Theorem 5 of Rockafellar [15], T_i, T_j are maximal monotone operators and $T_i^{-1}(0) \cap T_j^{-1}(0) = SLSVI(A_i, B_i, K_i)$.

Let $(v, t_1) \in graph\, T_i$, $(v, t_2) \in graph\, T_j$ with $v \in K_i \cap K_j$. Let $p_i \in A_i(v)$ and $q_i \in B_i(v)$. Since $t_1 \in T_i(v) = A_i(v) + N_{K_i}(v)$, we get $f(t_1) - f(p_i) \in N_{K_i}(v)$ and for $t_2 \in T_j(v) = B_i(v) + N_{K_i}(v)$, we get $g(t_2) - g(q_i) \in N_{K_i}(v)$. Since $y_i^{n_j} \in K_i \cap K_j$, we obtain $\langle f(t_1) - f(p_i), v - y_i^{n_j}\rangle \ge 0$ and $\langle g(t_2) - g(q_i), v - y_i^{n_j}\rangle \ge 0$, and since we have $y_i^{n_j} = P_{K_i}(x^{n_j} - \lambda_i^{n_j} f(u_i^{n_j}) - \beta_i^{n_j} g(v_i^{n_j}))$, we obtain that

$$\begin{aligned} \langle x^{n_j} - \lambda_i^{n_j} f(u_i^{n_j}) - \beta_i^{n_j} g(v_i^{n_j}) - y_i^{n_j}, v - y_i^{n_j}\rangle &\le 0, \\ \langle x^{n_j} - y_i^{n_j}, v - y_i^{n_j}\rangle - \lambda_i^{n_j}\langle f(u_i^{n_j}), v - y_i^{n_j}\rangle - \beta_i^{n_j}\langle g(v_i^{n_j}), -y_i^{n_j}\rangle &\le 0, \\ \langle \frac{x^{n_j} - y_i^{n_j}}{\lambda_i^{n_j}\beta_i^{n_j}}, v - y_i^{n_j}\rangle - \langle f(u_i^{n_j}), v - y_i^{n_j}\rangle - \frac{\beta_i^{n_j}}{\lambda_i^{n_j}}\langle g(v_i^{n_j}), v - y_i^{n_j}\rangle &\le 0, \\ \langle \frac{x^{n_j} - y_i^{n_j}}{\lambda_i^{n_j}\beta_i^{n_j}}, v - y_i^{n_j}\rangle - \langle f(u_i^{n_j}), v - y_i^{n_j}\rangle - \langle g(v_i^{n_j}), v - y_i^{n_j}\rangle &\le 0. \end{aligned} \tag{30}$$

In view of the monotonicity of the mappings $A_i, B_i, i = 1,2,\cdots,n$, we see that

$$\begin{aligned}
\langle f(t_1) + g(t_2), v - y_i^{n_j} \rangle &\geq \langle f(p_i) + g(q_i), v - y_i^{n_j} \rangle \\
&\geq \langle f(p_i) + g(q_i), v - y_i^{n_j} \rangle + \langle \frac{x^{n_j} - y_i^{n_j}}{\lambda_i^{n_j} \beta_i^{n_j}}, v - y_i^{n_j} \rangle \\
&\quad - \langle f(u_i^{n_j}), v - y_i^{n_j} \rangle - \langle g(v_i^{n_j}), v - y_i^{n_j} \rangle \\
&= \langle f(p_i) - f(w_i^{n_j}), v - y_i^{n_j} \rangle + \langle f(w_i^{n_j}) - f(u_i^{n_j}), v - y_i^{n_j} \rangle \\
&\quad + \langle g(q_i) - g(s_i^{n_j}), v - y_i^{n_j} \rangle + \langle g(s_i^{n_j}) - g(v_i^{n_j}), v - y_i^{n_j} \rangle \\
&\quad + \langle \frac{x^{n_j} - y_i^{n_j}}{\lambda_i^{n_j} \beta_i^{n_j}}, v - y_i^{n_j} \rangle \\
&\geq \langle f(w_i^{n_j}) - f(u_i^{n_j}), v - y_i^{n_j} \rangle + \langle g(s_i^{n_j}) - g(v_i^{n_j}), v - y_i^{n_j} \rangle \\
&\quad + \langle \frac{x^{n_j} - y_i^{n_j}}{\lambda_i^{n_j} \beta_i^{n_j}}, v - y_i^{n_j} \rangle. \qquad (31)
\end{aligned}$$

From the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\langle f(t_1) + g(t_2), v - y_i^{n_j} \rangle \geq -(&\|f(w_i^{n_j}) - f(u_i^{n_j})\| \|v - y_i^{n_j}\| + \|g(s_i^{n_j}) - g(v_i^{n_j})\| \|v - y_i^{n_j}\| \\
&+ \frac{\|x^{n_j} - y_i^{n_j}\|}{ab} \|v - y_i^{n_j}\|). \qquad (32)
\end{aligned}$$

Since A_i, B_i, f and g are Lipschitz continuous, the above inequality becomes

$$\begin{aligned}
\langle f(t_1) + g(t_2), v - y_i^{n_j} \rangle &\geq -(\alpha_1 k_1 \|x^{n_j} - y_i^{n_j}\| + \alpha_2 k_2 \|x^{n_j} - y_i^{n_j}\| + \frac{\|x^{n_j} - y_i^{n_j}\|}{ab} (\|v - y_i^{n_j}\|) \\
&= -M(\alpha_1 k_1 \|x^{n_j} - y_i^{n_j}\| + \alpha_2 k_2 \|x^{n_j} - y_i^{n_j}\| + \frac{\|x^{n_j} - y_i^{n_j}\|}{ab}), \qquad (33)
\end{aligned}$$

where $M = sup_{n \in N} \{\|v - y_i^{n_j}\|\}$. Taking $j \to \infty$ and using the fact that $\{\|v - y_i^{n_j}\|\}_{j \in N}$ is bounded, we see that $\langle f(t_1) + g(t_2), v - x^* \rangle \geq 0$. Using maximal monotonicity of T_i, we have $x^* \in T_i^{-1}(0) = SLSVI(A_i, B_i, K_i)$. Hence $x^* \in F$.

Step 4. The sequences $\{x^n\}_{n \in N}, \{y_i^n\}_{n \in N}$ and $\{z_i^n\}_{n \in N}$ converge strongly to $P_F(x^1)$.

One can prove this by following the same arguments as used in claim 3.10 of Theorem 3.6 of Censor et al. [6] and by Kadec-Klee property. This completes the proof. ■

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