Maejo Int. J. Sci. Technol. **2016**, *10*(03), 346-353

Maejo International
Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

On converse theorems for the discrete Bürmann power series method of summability

Sefa Anil Sezer [1] and Ibrahim Canak [2,*]

[1] Department of Mathematics, Istanbul Medeniyet University, Istanbul 34720, Turkey
[2] Department of Mathematics, Ege University, Izmir 35100, Turkey
* Correspondingauthor, e-mail: ibrahimcanak@yahoo.com

Received: 20 May 2015 / Accepted: 2 December 2016 / Published: 19 December 2016

Abstract: In this communication we recover ordinary convergence of a series from its summability by the discrete Bürmann power series method under certain conditions.

Keywords: Tauberian theorem, Bürmann power series, discrete power series

INTRODUCTION

Let $\sum_{n=0}^{\infty} a_n$ be a series of real or complex numbers with partial sums (s_n). If a series is convergent, then it is summable to the same sum by a regular summability method. The converse of this statement is not always true. However, under certain supplementary condition(s) the converse does hold. Such condition(s) is called Tauberian condition(s) with respect to the summability method in question and the resulting theorem is said to be a Tauberian theorem.

A Bürmann series is a series representation of the form

$$f(z) = \sum_{k=0}^{\infty} b_k [h(z)]^k \tag{1}$$

where f and h are any given functions. It was presented by Whittaker and Watson [1] in their study of analytic function theory. Later, this notion was extended to the probability theory by King [2]. Besides, King [3] used Bürmann series to define summability matrices that satisfy the Silverman-Toeplitz theorem conditions of regularity.

On the other hand, Patterson et al. [4] extended the discrete power series method of Watson [5] by considering the Bürmann series (1), where (b_k) is a sequence of non-negative numbers such that $b_0 > 0$ and

$$B_n = \sum_{k=0}^{n} b_k \to \infty \quad \text{as } n \to \infty.$$

Maejo Int. J. Sci. Technol. **2016**, *10*(03), 346-353

Definition 1. Suppose that

$$f_s(z_n) = \frac{1}{f(z_n)} \sum_{k=0}^{\infty} s_k b_k [h(z_n)]^k$$

exists for each $n \geq 0$, where $f(z_n) = \sum_{k=0}^{\infty} b_k [h(z_n)]^k$ and $|h(z_n)| < 1$. If

$$\lim_{n \to \infty} f_s(z_n) = \xi ,$$

then we say that (s_n) is summable to ξ by the discrete Bürmann power series (P_B) method and we write $s_n \to \xi (P_B)$.

Some of the most important summability methods such as discrete power series, logarithmic power series, Abel and (J, p) methods [6] can be obtained as special cases of the Bürmann power series method. Additional results on the Bürmann power series can be found in the work of Sansone and Gerretsen [7] and Patterson and Sen [8].

Throughout this paper we denote a positive constant by *M*, which is possibly different at each occurrence. The symbols $a_n = o(1)$ and $a_n = O(1)$ mean respectively that $a_n \to 0$ as $n \to \infty$ and (a_n) is bounded.

In this work our goal is to find conditions under which the convergence follows from summability by the discrete Bürmann power series method. In general, one might expect to require Tauberian conditions on both (b_n) and $(|h(z_n)|)$. To answer this question we prove the following two Tauberian theorems. Our main results are inspired by the results by Ishiguro [9].

Theorem 1. Let the series $\sum_{k=0}^{\infty} a_k$ be summable to ξ by the discrete Bürmann power series method. If the conditions

(i) $\dfrac{B_n}{|f(z_n)|} = O(1)$,

(ii) $\dfrac{b_n}{1 - |h(z_n)|} = O(1)$,

(iii) $(|h(z_n)|)$ decreases monotonically,

and

(iv) $a_k = o\left(\dfrac{b_k}{B_k}\right)$ as $k \to \infty$

are satisfied, then $\sum_{k=0}^{\infty} a_k = \xi$.

Theorem 2. Let the series $\sum_{k=0}^{\infty} a_k$ be summable to ξ by the discrete Bürmann power series method. If the conditions

(i) $\dfrac{B_n}{|f(z_n)|} = O(1)$,

(ii) $\dfrac{1}{1 - |h(z_n)|} = O(1)$,

(iii) (b_n) decreases monotonically,

and

(iv) $a_k = o\left(\dfrac{b_k}{B_k}\right)$ as $k \to \infty$

are satisfied, then $\sum_{k=0}^{\infty} a_k = \xi$.

DEFINITIONS AND NOTATIONS

Assume that (p_n) is a sequence of non-negative numbers with $p_0 > 0$ and

$$P_n = \sum_{k=0}^{n} p_k \to \infty, \; n \to \infty. \tag{2}$$

Definition 2. If

$$\sigma_{n,p} = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k \to \xi$$

as $n \to \infty$, then we say that (s_n) is summable to ξ by the weighted mean method $(\overline{N}, p)$ and we write $s_n \to \xi\,(\overline{N}, p)$.

Definition 3. Assume that the power series $p(x) = \sum_{k=0}^{\infty} p_k x^k$ has radius of convergence $=1$ and

$$p_s(x) = \frac{1}{p(x)}\sum_{k=0}^{\infty} p_k s_k x^k$$

exists for each $x \in (0,1)$. If

$$\lim_{x \to 1^-} p_s(x) = \xi \, ,$$

then we say that (s_n) is summable to ξ by the power series method (P) and we write $s_n \to \xi\,(P)$.

If a sequence is convergent, then it is summable to the same limit by a regular summability method. The above-mentioned $(\overline{N}, p)$ and (P) methods are regular if (2) is satisfied. The discrete power series method (P_λ) corresponding to (P) has been introduced by Watson [5] as follows:

Definition 4. Let the sequence (λ_n) be a strictly increasing sequence of real numbers such that $\lambda_0 \geq 1$ and $\lambda_n \to \infty$ as $n \to \infty$ and $x_n := 1 - \frac{1}{\lambda_n}$. Suppose that $p_s(x_n)$ exists for each $n \geq 0$. If

$$\lim_{n \to \infty} p_s(x_n) = \xi \, ,$$

then we say that (s_n) is summable to ξ by the discrete power series method (P_λ) and we write $s_n \to \xi\,(P_\lambda)$.

Note that (P_λ) includes (P) and (P_B) includes (P_λ). Eventually, (P_λ) and (P_B) inherit regularity from the underlying power series and discrete power series method respectively.

Summability methods for power series and discrete power series have been extensively studied by Armitage and Maddox [10], Watson [11], Osikiewicz [12] and Çanak and Totur [13, 14]. Lately, Sezer and Çanak [15] have obtained conditions under which power series and discrete power series methods are equivalent. Besides, some summability methods for series of real numbers and fuzzy numbers have been studied by Et et al. [16] and Esi [17].

Note that our results extend those by Watson [11] for the discrete Bürmann power series method. This clearly follows by choosing $h\,(z_n) = z_n$ and taking $z_n = 1 - \frac{1}{\lambda_n}$ in Definition 1.

Maejo Int. J. Sci. Technol. **2016**, *10*(03), 346-353

PROOFS OF TAUBERIAN THEOREMS

Proof of Theorem 1. We verify this theorem by showing that the following difference tends to zero as $n \to \infty$:

$$\begin{aligned} s_n - f_s(z_n) &= s_n - \frac{1}{f(z_n)} \sum_{k=0}^{\infty} s_k b_k [h(z_n)]^k \\ &= \frac{1}{f(z_n)} \sum_{k=0}^{\infty} s_n b_k [h(z_n)]^k - \frac{1}{f(z_n)} \sum_{k=0}^{\infty} s_k b_k [h(z_n)]^k \\ &= \frac{1}{f(z_n)} \sum_{k=0}^{\infty} (s_n - s_k) b_k [h(z_n)]^k \\ &= \frac{1}{f(z_n)} \sum_{k=0}^{n-1} (s_n - s_k) b_k [h(z_n)]^k + \frac{1}{f(z_n)} \sum_{k=n+1}^{\infty} (s_n - s_k) b_k [h(z_n)]^k \\ &= I + J. \end{aligned}$$

It suffices to show that $I, J \to 0$ as $n \to \infty$. Since $|h(z_n)|^k < 1$, for I we have

$$\begin{aligned} |I| &= \frac{1}{|f(z_n)|} \left| \sum_{k=0}^{n-1} (s_n - s_k) b_k [h(z_n)]^k \right| \\ &\le \frac{1}{|f(z_n)|} \sum_{k=0}^{n-1} |s_n - s_k| b_k |h(z_n)|^k \\ &< \frac{1}{|f(z_n)|} \sum_{k=0}^{n-1} |s_n - s_k| b_k \\ &= \frac{1}{|f(z_n)|} \left\{ |a_1 + a_2 + \ldots + a_n| b_0 + |a_2 + a_3 + \ldots + a_n| b_1 + \ldots + |a_n| b_{n-1} \right\} \\ &\le \frac{1}{|f(z_n)|} \left\{ |a_1| b_0 + |a_2| (b_0 + b_1) + \ldots + |a_n| (b_0 + b_1 + \ldots + b_{n-1}) \right\} \\ &= \frac{B_n}{|f(z_n)|} \frac{1}{B_n} \sum_{k=1}^{n} |a_k| B_{k-1} \\ &= \frac{B_n}{|f(z_n)|} \left(\frac{1}{B_n} \sum_{k=1}^{n} b_k \frac{|a_k| B_{k-1}}{b_k} \right). \end{aligned}$$

Since

$$\frac{|a_k| B_{k-1}}{b_k} \to 0$$

by (iv), and the weighted mean method $(\overline{N}, p)$ is regular, we have $I \to 0$ as $n \to \infty$ by (i). Now, by fixing $\varepsilon > 0$, J is considered. By (iv), there exists N_0 such that for $k > N_0$,

$$|a_k| \le \varepsilon \frac{b_k}{B_k}.$$

Assume that $k > n > N_0$. Then by (ii) and (iii), we have

Maejo Int. J. Sci. Technol. **2016**, *10*(03), 346-353

$$
\begin{aligned}
|s_n - s_k| &= |a_{n+1} + a_{n+2} + \ldots + a_k| \\
&\le \varepsilon\left(\frac{b_{n+1}}{B_{n+1}} + \frac{b_{n+2}}{B_{n+2}} + \ldots + \frac{b_k}{B_k}\right) \\
&\le \frac{\varepsilon}{B_n}\left(b_{n+1} + b_{n+2} + \ldots + b_k\right) \\
&\le \frac{\varepsilon M}{B_n}\left[\left(1-|h(z_{n+1})|\right) + \left(1-|h(z_{n+2})|\right) + \ldots + \left(1-|h(z_k)|\right)\right] \\
&\le \frac{\varepsilon M}{B_n}(k+1)\left(1-|h(z_n)|\right).
\end{aligned}
$$

Using (ii) and (iii), we have

$$
\begin{aligned}
|J| &\le \frac{1}{|f(z_n)|}\frac{\varepsilon M}{B_n}\left(1-|h(z_n)|\right)\sum_{k=n+1}^{\infty}(k+1)b_k|h(z_n)|^k \\
&\le \frac{1}{|f(z_n)|}\frac{\varepsilon M^2}{B_n}\left(1-|h(z_n)|\right)^2\sum_{k=n+1}^{\infty}(k+1)|h(z_n)|^k \\
&\le \frac{B_n}{|f(z_n)|}\frac{\varepsilon M^2}{B_n^2}\left(1-|h(z_n)|\right)^2\sum_{k=0}^{\infty}(k+1)|h(z_n)|^k \\
&= \frac{B_n}{|f(z_n)|}\frac{\varepsilon M^2}{B_n^2} \\
&\le \varepsilon M.
\end{aligned}
$$

Therefore, $J \to 0$ as $n \to \infty$. This completes the proof. □

Proof of Theorem 2. As in the proof of Theorem 1, writing

$$s_n - f_s(z_n) = I + J,$$

we obtain $\lim_n I = 0$ by (i) and (iv).

Next we estimate J. From (iv), we obtain

$$
\begin{aligned}
|s_n - s_k| &= |a_{n+1} + a_{n+2} + \ldots + a_k| \\
&\le \varepsilon\left(\frac{b_{n+1}}{B_{n+1}} + \frac{b_{n+2}}{B_{n+2}} + \ldots + \frac{b_k}{B_k}\right) \\
&\le \frac{\varepsilon}{B_n}\left(b_{n+1} + b_{n+2} + \ldots + b_k\right) \\
&= \frac{\varepsilon}{B_n}\left(B_k - B_n\right) \\
&\le \varepsilon\frac{B_k}{B_n}.
\end{aligned}
$$

Hence using (i) and (iii), we have

Maejo Int. J. Sci. Technol. **2016**, *10*(03), 346-353

$$\begin{aligned}\left|J\right| &\le \frac{1}{|f(z_n)|}\frac{\varepsilon}{B_n}\sum_{k=n+1}^{\infty} B_k b_k\,|h(z_n)|^k \\ &\le \frac{B_n}{f(z_n)}\frac{\varepsilon b_n}{B_n^2}\sum_{k=n+1}^{\infty} B_k |h(z_n)|^k \\ &\le \varepsilon M\frac{b_n}{B_n^2}\sum_{k=n+1}^{\infty} B_k |h(z_n)|^k.\end{aligned}$$

Here, if we set

$$\begin{aligned}R_k &= \sum_{v=k}^{\infty} |h(z_n)|^v \\ &= \sum_{v=0}^{\infty} |h(z_n)|^v - \sum_{v=0}^{k-1} |h(z_n)|^v \\ &= \frac{1}{1-|h(z_n)|} - \frac{1-|h(z_n)|^k}{1-|h(z_n)|} \\ &= \frac{|h(z_n)|^k}{1-|h(z_n)|},\end{aligned}$$

then we have

$$\begin{aligned}\sum_{k=n+1}^{\infty} B_k\,|h(z_n)|^k &= \sum_{k=n+1}^{\infty} B_k\left(R_k - R_{k+1}\right) \\ &= B_{n+1}R_{n+1} + \sum_{k=n+2}^{\infty} R_k\left(B_k - B_{k-1}\right) \\ &= B_{n+1}R_{n+1} + \sum_{k=n+2}^{\infty} b_k R_k \\ &= B_{n+1}\frac{|h(z_n)|^{n+1}}{1-|h(z_n)|} + \sum_{k=n+2}^{\infty} b_k \frac{|h(z_n)|^k}{1-|h(z_n)|}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|J\right| &\le \varepsilon M\,\frac{b_n}{B_n^2}B_{n+1}\frac{|h(z_n)|^{n+1}}{1-|h(z_n)|} + \varepsilon M\,\frac{b_n}{B_n^2}\sum_{k=n+2}^{\infty} b_k\,\frac{|h(z_n)|^k}{1-|h(z_n)|} \\ &= S_1 + S_2.\end{aligned}$$

It suffices to show that $S_1, S_2 \to 0$ as $n \to \infty$. For S_1, we get

$$\begin{aligned}\left|S_1\right| &= \varepsilon M\,\frac{b_n}{B_n^2}B_{n+1}\frac{|h(z_n)|^{n+1}}{1-|h(z_n)|} \\ &\le \varepsilon M\,\frac{b_n}{(n+1)b_n}\frac{B_{n+1}}{B_n}\frac{1}{1-|h(z_n)|} \\ &\le \frac{\varepsilon M^2}{n+1}\left(1+\frac{b_{n+1}}{B_n}\right) \\ &\le \frac{\varepsilon M^2}{n+1}\left(1+\frac{b_{n+1}}{(n+1)b_{n+1}}\right) \\ &\le \varepsilon M\end{aligned}$$

by (ii) and (iii). For S_2, using (ii) and (iii), we get

$$\begin{aligned}
|S_2| &= \varepsilon M \frac{b_n}{B_n^2} \sum_{k=n+2}^{\infty} b_k \frac{|h(z_n)|^k}{1-|h(z_n)|} \\
&\le \varepsilon M^2 \frac{b_n^2}{B_n^2} \sum_{k=n+2}^{\infty} |h(z_n)|^k \\
&\le \varepsilon M^2 \frac{b_n^2}{(n+1)^2 b_n^2} \sum_{k=0}^{\infty} |h(z_n)|^k \\
&\le \frac{\varepsilon M^2}{(n+1)^2} \frac{1}{1-|h(z_n)|} \\
&\le \varepsilon M.
\end{aligned}$$

Hence we have

$$|J| \le \varepsilon M.$$

Therefore, $J \to 0$ as $n \to \infty$. This completes the proof. □

REFERENCES

1. E. T. Whittaker and G. N. Watson, “A Course of Modern Analysis”, Cambridge University Press, Cambridge, **1927**.
2. J. P. King, “Burmann-series distributions and approximation operators”, *Studia Sci. Math. Hungar.*, **2005**, *42*, 355-369.
3. J. P. King, “Summability methods and Burmann series expansions”, *Indian J. Math.*, **2006**, *48*, 249-258.
4. R. F. Patterson, P. Sen and B. E. Rhoades, “A Tauberian theorem for a generalized power series method”, *Appl. Math. Lett.*, **2005**, *18*, 1129-1133.
5. B. Watson, “Discrete power series methods”, *Analysis*, **1998**, *18*, 97-102.
6. J. Boos, “Classical and Modern Methods in Summability”, Clarendon Press, Wotton-under-Edge, **2000**.
7. G. Sansone and J. Gerretsen, “Lectures on the theory of functions of a complex variable”, Noordhoff, Groningen, **1960**.
8. R. F. Patterson and P. Sen, “Characterization of the central limit theorem by the Bürmann power series”, *Int. J. Math. Stat.*, **2011**, *9*, 68-77.
9. K. Ishiguro, “Two Tauberian theorems for (J, p_n) summability”, *Proc. Japan Acad.*, **1965**, *41*, 40-45.
10. D. H. Armitage and I. J. Maddox, “Discrete Abel means”, *Analysis*, **1990**, *10*, 177-186.
11. B. Watson, “A Tauberian theorem for discrete power series methods", *Analysis*, **2002**, *22*, 361-365.
12. J. A. Osikiewicz, “Equivalence results for discrete Abel means”, *Int. J. Math. Math. Sci.*, **2002**, *30*, 727-731.
13. İ. Çanak and Ü. Totur, “A theorem for the (J, p) summability method”, *Acta Math. Hungar.*, **2015**, *145*, 220-228.
14. İ. Çanak and Ü. Totur, “Tauberian theorems for the (J, p) summability method”, *Appl. Math. Lett.*, **2012**, *25*, 1430-1434.
15. S. A. Sezer and İ. Çanak, “Conditions for the equivalence of power series and discrete power series methods of summability”, *Filomat*, **2015**, *29*, 2275-2280.

Maejo Int. J. Sci. Technol. **2016**, *10*(03), 346-353

16. M. Et, M. Cinar and M. Karakaş, "On asymptotic statistical equivalence of order alpha of generalized difference sequences", *Maejo Int. J. Sci. Technol.*, **2014**, *8*, 297-306.
17. A. Esi, "p-Absolutely summable sequences of fuzzy real numbers", *Maejo Int. J. Sci. Technol.*, **2013**, *7*, 107-112.