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Full Paper

An investigation into the polylogarithm function and its associated class of meromorphic functions

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Abstract: In the literature on geometric function theory in complex analysis, one can find many interesting applications of a variety of convolution operators which are defined by means of a number of special functions of mathematical physics and analytic number theory. Here in our present article we apply a new operator Ω_s , which is associated with the polylogarithm function (or de Jonquière's function) $\mathrm{Li}_s(z)$, with a view to introducing and systematically investigating the various properties and characteristics of a potentially useful subclass of meromorphic functions in the punctured unit disk.

Key Words: Hurwitz-Lerch zeta function, polylogarithm function (or de Jonquière's function), meromorphic continuation, meromorphic functions, coefficient inequalities, Hadamard product, partial sum

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INTRODUCTION

The general Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined [1-5] by

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{1}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1)$$

contains, as its special cases, a number of important functions of Analytic Number Theory as, for example, the polylogarithmic function (or de Jonquière's function) $\mathrm{Li}_s(z)$ given [6] by

$$\mathrm{Li}_{\mathrm{s}}(\mathrm{z}) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z\,\Phi(z,s,1) \tag{2}$$

$$(s \in \mathrm{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1).$$

Both the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ and the polylogarithmic function (or de Jonquière's function) $\mathrm{Li}_{\mathrm{s}}(\mathrm{z})$ can be continued meromorphically to the whole complex s-plane, except for a simple pole at $s = 1$ with its residue 1.

In geometric function theory, the general Hurwitz-Lerch zeta function $\Phi(z,s,a)$ was applied recently by Srivastava and Attiya [7], who introduced and systematically studied the currently well-known Srivastava-Attiya operator. This widely and extensively investigated operator has indeed opened up the door for many pieces of subsequent work on various analogous families of operators and their applications in several subclasses of the class A of functions $f(z)$, which are analytic in the open unit disk,

$$\mathrm{U} := \{z : z \in \mathrm{C} \quad \text{and} \quad |z| < 1\},$$

and normalised [8-15] by

$$f(0) = f'(0) - 1 = 0.$$

Our present investigation is motivated largely by several earlier researches on the subject [e.g.16-19], which dealt essentially with the polylogarithmic function $\mathrm{Li}_{\mathrm{s}}(\mathrm{z})$ defined by (2) and its analogues and extensions. By means of a new operator Ω_s associated with the polylogarithm function $\mathrm{Li}_{\mathrm{s}}(\mathrm{z})$, we define and investigate the properties of a certain subclass of meromorphic functions. In particular, we investigate coefficient inequalities and partial sums for this meromorphic function class.

Let Σ denote the class of functions f of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{3}$$

which are analytic in the punctured unit disk U^* given by

$$\mathrm{U}^* := \{z : z \in \mathrm{C} \quad \text{and} \quad 0 < |z| < 1\} =: \mathrm{U} \setminus \{0\}.$$

For functions $f \in \Sigma$ given by (3) and $g \in \Sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k =: (g * f)(z).$$

Let Σ_+ be the class of functions f of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \qquad (a_k \ddagger 0), \tag{4}$$

which are analytic and univalent in U^*.

Analogous to the earlier work by Liu and Srivastava [20] and corresponding to a function

$\ell_s(z)$ given by

$$\ell_s(z) := \frac{1}{z^2}\,\mathrm{Li}_s(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{1}{(k+2)^s}\, z^k,$$

we consider a linear operator $\Omega_s : \Sigma \to \Sigma$, which is defined by means of the following Hadamard product (or convolution):

$$\Omega_s f(z) := \ell_s(z) * f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^s}\, a_k z^k. \qquad (5)$$

Now by making use of the operator Ω_s, we introduce a new subclass of functions in Σ as follows.

Definition 1. For $\alpha > 1$ and $0 < \beta \leqq 1$, let $\mathrm{N}_s(\alpha,\beta)$ denote a subclass of Σ consisting of functions $f \in \Sigma$ of the form given by (3) and satisfying the following condition:

$$\Re\{z\,\Omega_s f(z) - \alpha z^2 (\Omega_s f(z))'\} > \beta \qquad (z \in \mathrm{U}^*), \qquad (6)$$

where $\Omega_s f(z)$ is given by (5).

Definition 2. For $\alpha > 1$ and $0 < \beta \leqq 1$, we say that a function $f \in \Sigma_+$ of the form given by (4) is in the class $\mathrm{N}_{s,+}(\alpha,\beta)$ whenever $f(z)$ satisfies the following condition:

$$\Re\{z\,\Omega_{s,+} f(z) - \alpha z^2 (\Omega_s f(z))'\} > \beta \qquad (z \in \mathrm{U}^*), \qquad (7)$$

where $\Omega_{s,+} f(z)$ is given by (5) with $a_k \geqq 0$, i.e. by

$$\Omega_{s,+} f(z) := \ell_s(z) * f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^s}\, a_k z^k \qquad (a_k \geqq 0). \qquad (8)$$

Following one of the pioneering studies on the subject of partial sums of analytic and univalent functions by Silverman [21], Cho and Owa [22] derived sharp lower bounds for such quotients as follows:

$$\Re\left(\frac{\mathrm{f}(z)}{\mathrm{f}_n(z)}\right), \quad \Re\left(\frac{\mathrm{f}_n(z)}{f(z)}\right), \quad \Re\left(\frac{\mathrm{f}'(z)}{\mathrm{f}_n'(z)}\right) \quad \text{and} \quad \Re\left(\frac{\mathrm{f}_n'(z)}{\mathrm{f}'(z)}\right),$$

where, for the function $f \in \Sigma$ normalised by

$$\mathrm{f}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$

The nth partial sum $\mathrm{f}_n(z)$ is given by

$$\mathrm{f}_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k \qquad (n \in \mathrm{N} := \{1,2,3,\cdots\} =: \mathrm{N}_0 \setminus \{0\}).$$

Examples of other related recent studies dealing with partial sums of analytic or meromorphic functions can be found in the literature [23-25].

Here in this sequel to some of the aforementioned work, we investigate the quotient of a function $f(z)$ of the form (4) and its sequence $\{f_n(z)\}_{n \in \mathrm{N}_0}$ of partial sums given by

$$f_n(z) = \frac{1}{z} + \sum_{k=0}^{n} a_k z^k \qquad (n \in \mathrm{N}_0). \qquad (9)$$

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For this purpose, in the section below we determine the necessary and sufficient condition for a function f of the form (4) to be in the class $\mathrm{N}_{s,+}(\alpha,\beta)$.

COEFFICIENT INEQUALITIES

Theorem 1. *Let* $f \in \Sigma_+$ *be given by* (4). *Also, let* $\alpha > 1$ *and* $0 < \beta \leq 1$. *Then* $f \in \mathrm{N}_{s,+}(\alpha,\beta)$ *if and only if*

$$\sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^s} a_k \leq 1 + \alpha - \beta. \tag{10}$$

Proof. First, suppose that $f \in \mathrm{N}_{s,+}(\alpha,\beta)$. Then we find condition (7) of Definition 2 that

$$\Re\left\{ z\left(\frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(k+2)^s} a_k z^k\right) - \alpha z^2\left(-\frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{k}{(k+2)^s} a_k z^{k-1}\right)\right\}$$
$$= \Re\left(1 + \alpha - \sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^s} a_k z^{k+1}\right) > \beta. \tag{11}$$

If we choose z to be real and let $z \to 1-$, we get

$$1 + \alpha - \sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^s} a_k > \beta,$$

which is precisely condition (10) asserted by Theorem 1. Conversely, let us suppose that assertion (10) of Theorem 1 holds true. Then we can write

$$|z\Omega_{s,+}f(z) - \alpha z^2 (\Omega_{s,+}f(z))' - 1 - \alpha| = \left| -\sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^s} a_k z^{k+1} \right|$$
$$\leq \sum_{k=1}^{\infty} \frac{\alpha k - 1}{(k+2)^s} a_k |z|^{k+1}$$
$$\leq 1 + \alpha - \beta.$$

Consequently, $f \in \mathrm{N}_{s,+}(\alpha,\beta)$.

Finally, we note that inequality (10) is sharp, with the extremal function given by

$$f(z) = \frac{1}{z} + \frac{(1+\alpha-\beta)(k+2)^s}{\alpha k - 1} z \tag{12}$$

$$(z \in \mathrm{U}^*; \alpha > 1; n \in \mathrm{N}; 0 < \beta \leq 1).$$

BOUNDS FOR QUOTIENTS WITH PARTIAL SUMS

In order to obtain our results involving the partial sum $f_n(z)$ given by (9), let us recall the well-known result [16] that

$$\Re\left(\frac{1+w(z)}{1-w(z)}\right) > 0 \qquad (z \in \mathrm{U})$$

if and only if the function $w(z)$ given by

$$w(z) = \sum_{k=1}^{\infty} c_k z^k$$

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satisfies the following inequality:

$$|w(z)|<|z| \qquad (z\in \mathrm{U}).$$

Theorem 2. *Let the function* $f\in \mathrm{N}_{s,+}(\alpha,\beta)$ *and let its partial sum* $f_n(z)$ *be given by* (9). *Then*

$$\Re\left(\frac{f(z)}{f_n(z)}\right)\geq 1-\frac{(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1} \qquad (z\in \mathrm{U}^*), \tag{13}$$

and

$$\frac{\alpha k-1}{(1+\alpha-\beta)(k+2)^s}\geq \begin{cases} 1 & (k=1,\cdots,n) \\ \dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s} & (k=n+1,n+2,n+3,\cdots). \end{cases}$$

The result is sharp for every $n\in \mathrm{N}_0$, *with the extremal function given by*

$$f(z)=\frac{1}{z}+\frac{(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1}z^{n+1} \qquad (z\in \mathrm{U}^*;\, n\in \mathrm{N}_0). \tag{14}$$

Proof. We begin by considering

$$\frac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\left(\frac{f(z)}{f_n(z)}-1+\frac{(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1}\right)$$

$$=\frac{1+\sum_{k=1}^{n}a_k z^{k+1}+\left(\dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}a_k z^{k+1}}{1+\sum_{k=1}^{n}a_k z^{k+1}}$$

$$=:\frac{1+A(z)}{1+B(z)}.$$

Upon setting

$$\frac{1+A(z)}{1+B(z)}=\frac{1+w(z)}{1-w(z)},$$

we find by simple calculations that

$$w(z)=\frac{A(z)-B(z)}{2+A(z)+B(z)}.$$

We thus obtain

$$w(z)=\frac{\left(\dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}a_k z^{k+1}}{2+2\sum_{k=1}^{n}a_k z^{k+1}+\left(\dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}a_k z^{k+1}} \qquad (z\in \mathrm{U})$$

and

$$|w(z)|\leq \frac{\left(\dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k|-\left(\dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}|a_k|} \qquad (z\in \mathrm{U}).$$

Now clearly,

$$|w(z)| \leq 1 \qquad (z \in \mathrm{U})$$

if and only if

$$2\left(\frac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}|a_k| \leq 2-2\sum_{k=1}^{n}|a_k|,$$

which is equivalent to

$$\sum_{k=1}^{n}|a_k|+\left(\frac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}|a_k| \leq 1. \tag{15}$$

Thus, by making use of condition (10), it is sufficient to show that the left-hand side of (15) is bounded above by

$$\sum_{k=1}^{\infty}\frac{\alpha k-1}{(1+\alpha-\beta)(k+2)^s}|a_k|$$

or, equivalently, by

$$\sum_{k=1}^{n}\left(\frac{(\alpha k-1)-(1+\alpha-\beta)(k+2)^s}{(1+\alpha-\beta)(k+2)^s}\right)|a_k|$$
$$+\sum_{k=n+1}^{\infty}\left(\frac{(\alpha k-1)(n+3)^s-(\alpha n+\alpha-1)(k+2)^s}{(1+\alpha-\beta)(k+2)^s(n+3)^s}\right)|a_k| \geq 0.$$

Finally, in order to verify that the function $f(z)$ given by (14) does provide a sharp result, we observe for

$$z = re^{\pi i/(n+2)} \qquad (0<r<1)$$

that

$$\frac{f(z)}{f_n(z)}=1+\frac{(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1}z^{n+2}\to 1-\frac{(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1}$$
$$=\frac{(\alpha n+\alpha-1)-(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1} \qquad (r\to 1-).$$

Our proof of Theorem 2 is thus complete.

Theorem 3 below provides the bound for

$$\Re\left(\frac{f_n(z)}{f(z)}\right).$$

Theorem 3. *Let the function* $f \in \mathrm{N}_{s,+}(\alpha,\beta)$ *and let its partial sum* $f_n(z)$ *be given by* (9). *Then*

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geq \frac{\alpha n+\alpha-1}{(\alpha n+\alpha-1)+(1+\alpha-\beta)(n+3)^s} \qquad (z \in \mathrm{U}^*), \tag{16}$$

and

$$\frac{\alpha k-1}{(1+\alpha-\beta)(k+2)^s} \geq \begin{cases} 1 & (k=1,\cdots,n) \\ 1+\dfrac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s} & (k=n+1,n+2,n+3,\cdots). \end{cases}$$

The result is sharp for every $n \in \mathrm{N}_0$, *with the extremal function given by*

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$$f(z)=\frac{1}{z}+\frac{(1+\alpha-\beta)(n+3)^s}{(\alpha n+\alpha-1)}z^{n+1} \qquad (z\in \mathrm{U}^*;\, n\in \mathrm{N}_0). \tag{17}$$

Proof. We may write

$$\left(\frac{(\alpha n+\alpha-1)+(1+\alpha-\beta)(n+3)^s}{(1+\alpha-\beta)(n+3)^s}\right)\left(\frac{f_n(z)}{f(z)}-\frac{\alpha n+\alpha-1}{(\alpha n+\alpha-1)+(1+\alpha-\beta)(n+3)^s}\right)$$

$$=\frac{1+\sum_{k=1}^{n}a_k z^{k+1}-\left(\frac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}a_k z^{k+1}}{1+\sum_{k=1}^{\infty}a_k z^{k+1}}$$

$$=:\frac{1+w(z)}{1-w(z)},$$

where

$$w(z)=\frac{\left(-\frac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}a_k z^{k+1}-\sum_{k=n+1}^{\infty}a_k z^{k+1}}{2+2\sum_{k=1}^{n}a_k z^{k+1}-\left(\frac{\alpha n+\alpha-1}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}a_k z^{k+1}+\sum_{k=n+1}^{\infty}a_k z^{k+1}}$$

and

$$|w(z)|\leq \frac{\left[\frac{(\alpha n+\alpha-1)+(1+\alpha-\beta)(n+3)^s}{(1+\alpha-\beta)(n+3)^s}\right]\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{n}|a_k|-\left[\frac{(\alpha n+\alpha-1)+(1+\alpha-\beta)(n+3)^s}{(1+\alpha-\beta)(n+3)^s}\right]\sum_{k=n+1}^{\infty}|a_k|}\leq 1. \tag{18}$$

This last inequality (18) is equivalent to

$$\sum_{k=1}^{n}|a_k|+\left(\frac{(\alpha n+\alpha-1)+(1+\alpha-\beta)(n+3)^s}{(1+\alpha-\beta)(n+3)^s}\right)\sum_{k=n+1}^{\infty}|a_k|\leq 1. \tag{19}$$

Since the left-hand side of assertion (19) is bounded above by

$$\sum_{k=1}^{\infty}\frac{\alpha k-1}{(1+\alpha-\beta)(k+2)^s}|a_k|,$$

the proof of Theorem 3 is evidently complete.

For bounds for the quotients involving the derivatives $f'(z)$ and $f_n'(z)$, we can similarly establish Theorem 4 below.

Theorem 4. *Let the function* $f\in \mathrm{N}_{s,+}(\alpha,\beta)$ *and let its partial sum* $f_n(z)$ *be given by* (9). *Then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right)\geq \frac{(\alpha n+\alpha-1)+(n+1)(1+\alpha-\beta)(n+3)^s}{\alpha n+\alpha-1} \tag{20}$$

$$(z\in \mathrm{U}^*;\, n\in \mathrm{N}_0)$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right)\geq \frac{\alpha n+\alpha-1}{(\alpha n+\alpha-1)-(n+1)(1+\alpha-\beta)(n+3)^s} \tag{21}$$

$$(z \in U^*; n \in N_0).$$

In both of assertions (20) *and* (21) *the extremal function is given by* (14).

Proof. The proof of Theorem 4 is much akin to that of Theorem 2 and Theorem 3. Indeed, for the function $f \in N_{s,+}(\alpha, \beta)$ and for its partial sum $f_n(z)$ given by (9), the inequalities in assertions (20) and (21) of Theorem 4 would follow easily from Definition 2 of the class of such functions as the function $f \in N_{s,+}(\alpha, \beta)$. Here, in this case we make use of the derivative of partial sum $f_n(z)$, which can be derived easily by applying (9). The details are being omitted here.

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