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On statistical convergence of order α of difference sequence of functions

Muhammed Cinar [1,*] and Mikail Et [2]

[1] Department of Mathematics, Mus Alparslan University, Mus-Turkey
[2] Department of Mathematics, Firat University, 23119, Elazig-Turkey
* Corresponding author, e-mail: muhammedcinar23@gmail.com

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Abstract: In this paper we introduce the concepts of uniform and pointwise Δ^m-statistical convergence of order α of sequence of functions. We also give notions of uniform and pointwise Δ^m-statistically Cauchy sequence of order α of sequence of functions and show that pointwise Δ^m-statistically Cauchy sequence (uniform Δ^m-statistical convergence) is equivalent to pointwise Δ^m-statistical convergence (uniform Δ^m-statistically Cauchy sequence) of order α. Moreover, some relations between pointwise Δ^m-statistical convergence of order α and strongly pointwise Δ_q^m-Cesàro summable of order α of sequence of functions are given.

Key words: statistical convergence, sequence of functions, Cesàro summability

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INTRODUCTION

Fast [1] and Schoenberg [2] independently introduced the notion of statistical convergence. The idea hinges on the density of subsets of the set $\mathbb{N}$. The density of E, a subset of $\mathbb{N}$, is defined by $\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$, provided that the limit exists, where χ_E is the characteristic function of E. A sequence $x=(x_k)$ is statistically convergent to a number L if, for every $\varepsilon>0$, $\delta\{k\in\mathbb{N} : |x_k-L|\geq\varepsilon\}=0$. The statistical convergence has been studied by Basar [3], Connor [4], Cinar et al. [5], Et et al. [6-7], Duman and Ankara [8], Fridy [9], Gokhan and Gungor [10], Gungor et al. [11], Gungor and Gokhan [12, 13], Mohiuddine et al. [14], Mursaleen [15], Salat [16] and many others.

Recently Gadjiev and Orhan [17] introduced the order of statistical convergence of a sequence of numbers and later Colak [18] studied statistical convergence of order α and strong $q-$Cesàro summability of order α and defined the $\alpha-$*density* of a subset E of $\mathbb{N}$ as

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$$\delta_\alpha(E) = \lim_n \frac{1}{n^\alpha} |\{k \le n \; : \; k \in E\}|, \text{ provided that the limit exists and } \alpha \in (0,1],$$

where $|\{k \le n \; : \; k \in E\}|$ denotes the number of elements of E not exceeding n.

If $x = (x_k)$ is a sequence such that x_k satisfies the property $P(k)$ for all k except a set of zero $\alpha - density$, then we say that x_k satisfies the property $P(k)$ for '*almost all k according to α*' and we abbreviate this by '$a.a.k\ (\alpha)$'. It can be shown that any finite subset of $\mathbb{N}$ has zero α density and $\delta_\alpha(E^c) = 1 - \delta_\alpha(E)$ does not hold for $\alpha \in (0,1]$. If $\alpha = 1$, then $\alpha - density$ reduces to the natural density.

Kizmaz [19] introduced the difference sequence spaces and Et and Colak [20] generalised the notion afterwards as follows:

$$\Delta^m(X) = \{x = (x_k) \; : \; (\Delta^m x_k) \in X\},$$

for $X = \ell_\infty$, c or c_0, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and so $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. Recently, Basar and Altay [21], Altay and Basar [22], Altin et al. [23], Et [24], Altinok and Mursaleen [25], Bektas et al. [26], Colak et al. [27], Djolovic and Malkowsky [28], Gungor and Et [29], Malkowsky et al. [30] and Mursaleen et al. [31] have studied difference sequences spaces.

MAIN RESULTS

In this section we give the main results of this article.

Definition 1. Let A be a subset of $\mathbb{R}$ and $\{f_k\}$ be a sequence of real valued functions defined on A. The pointwise $\Delta^m -$convergent sequence $\{f_k\}$ converges to f if there exists a natural number $N = N(x, \varepsilon)$ for $x \in A$ and $\varepsilon > 0$ such that $|\Delta^m f_k(x) - f(x)| < \varepsilon$ for every $k \ge N$. The notation $N = N(x, \varepsilon)$ means that the natural number N depends on x and ε. In this case we write $\lim_k \Delta^m f_k(x) = f(x)$ for every $x \in A$. The set of all pointwise $\Delta^m -$convergent sequences of functions is denoted by $c(\Delta^m, F(p))$, where $\Delta^m f_k(x) = \sum_{v=0}^{m} (-1)^v \binom{m}{v} f_{k+v}(x)$.

Definition 2. Let A be a subset of $\mathbb{R}$, $\{f_k\}$ be a sequence of real valued functions defined on A, and α be any real number such that $0 < \alpha \le 1$. The sequence $\{f_k\}$ is said to be pointwise $\Delta^m -$ statistically convergent of order α to f if there exists a natural number $N = N(x, \varepsilon)$ for $x \in A$ and $\varepsilon > 0$ such that

$$|\Delta^m f_k(x) - f(x)| < \varepsilon \quad a.a.k(\alpha);$$

that is, for $x \in A$ and $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} \left| \{k \le n \; : \; |\Delta^m f_k(x) - f(x)| \ge \varepsilon\} \right| = 0.$$

In this case we write $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ on A. The set of all pointwise $\Delta^m -$statistically convergent sequences of functions of order α is denoted by $S^\alpha(\Delta^m, F(p))$.

It can be shown that the pointwise $\Delta^m -$statistical convergence of order α for sequence of

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functions is well defined for $\alpha \in (0,1]$, but not well defined for $\alpha > 1$. For this, let $\{f_k\}$ be defined as follows:

$$f_k(x) = \begin{cases} 1 & k = 2n \\ x^k & k \neq 2n \end{cases} \quad n = 1,2,3...,x \in \left[0,\tfrac{1}{2}\right].$$

Then we calculate $\Delta f_k(x)$ as

$$\Delta f_k(x) = \begin{cases} 1 - x^{k+1} & k = 2n \\ x^k - 1 & k \neq 2n \end{cases} \quad n = 1,2,3...,x \in \left[0,\tfrac{1}{2}\right].$$

In this case

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left| \left\{ k \le n \ : \ \left| \Delta f_k(x) - \left(x^k - 1\right) \right| \ge \varepsilon, \text{ for every } x \in A \right\} \right| = \lim_{n\to\infty} \frac{n}{2n^{\alpha}} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \left| \left\{ k \le n \ : \ \left| \Delta f_k(x) - \left(1 - x^{k+1}\right) \right| \ge \varepsilon, \text{ for every } x \in A \right\} \right| = \lim_{n\to\infty} \frac{n}{2n^{\alpha}} = 0$$

for $\alpha > 1$. Therefore, $S^{\alpha} - \lim \Delta f_k(x) = 1$ and $S^{\alpha} - \lim \Delta f_k(x) = -1$, which is impossible.

Theorem 1. Let $\alpha \in (0,1]$, and $\{f_k\}$ and $\{g_k\}$ be two sequences of real valued functions defined on *A*.

(i) If $S^{\alpha} - \lim \Delta^m f_k(x) = f(x)$ and $c \in \mathrm{R}$, then $S^{\alpha} - \lim c\Delta^m f_k(x) = cf(x)$.

(ii) If $S^{\alpha} - \lim \Delta^m f_k(x) = f(x)$ and $S^{\alpha} - \lim \Delta^m g_k(x) = g(x)$, then
$S^{\alpha} - \lim \Delta^m (f_k(x) + g_k(x)) = f(x) + g(x)$.

Proof. Let $\{f_k\}$ and $\{g_k\} \in S^{\alpha}\left(\Delta^m, F(p)\right)$ with $S^{\alpha} - \lim \Delta^m f_k(x) = f(x)$ and $S^{\alpha} - \lim \Delta^m g_k(x) = g(x)$ on $A \subset \mathrm{R}$, and $c \in \mathrm{R}$. Then one can easily see by these assumptions that

$$S^{\alpha} - \lim \Delta^m (f_k(x) + g_k(x)) = f(x) + g(x),$$
$$S^{\alpha} - \lim c\Delta^m f_k(x) = cf(x),$$

as desired.

It can be shown that every pointwise $\Delta^m -$ convergent sequence of functions is pointwise $\Delta^m -$statistically convergent of order α $(0 < \alpha \le 1)$. However, the converse of this does not hold. Indeed if we take the sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1 & k = n^2 \\ \frac{kx}{1+k^2x^2} & otherwise \end{cases},$$

then we have

$$\Delta f_k(x) = \begin{cases} 1 - \frac{(k+1)x}{1+(k+1)^2x^2} & k = n^2 \\ \frac{kx}{1+k^2x^2} - 1 & k = n^2 - 1 \\ \frac{kx}{1+k^2x^2} - \frac{(k+1)x}{1+(k+1)^2x^2} & otherwise\ . \end{cases}$$

Therefore , $\{f_k\}$ is pointwise $\Delta -$statistically convergent of order α with $S^{\alpha} - \lim \Delta f_k(x) = 0$ for $\alpha > \frac{1}{2}$, but it is not pointwise $\Delta -$convergent.

Definition 3. Let A be a subset of R, $\{f_k\}$ be a sequence of real valued functions defined on A,

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and α be any real number such that $0<\alpha\leq 1$. The sequence $\{f_k\}$ is a pointwise Δ^m- statistically Cauchy sequence of order α provided that there exists a number $N=N(\varepsilon,x)$ for $x\in A$ and $\varepsilon>0$ such that

$$\left|\Delta^m f_k(x)-\Delta^m f_N(x)\right|<\varepsilon \quad a.a.k\,(\alpha);$$

that is, for $x\in A$ and $\varepsilon>0$,

$$\lim_n \frac{1}{n^{\alpha}}\left|k\leq n \ : \ \left|\Delta^m f_k(x)-\Delta^m f_N(x)\right|\geq\varepsilon\right|=0.$$

Using the techniques for proving Theorem 3.4 in Cinar et al. [5], we obtain the proof of the following theorem.

Theorem 2. Let $\{f_k\}$ be a sequence of functions defined on a set A. The following statements are equivalent:

(i) $\{f_k\}$ is a pointwise Δ^m-statistically convergent sequence of order α on A;

(ii) $\{f_k\}$ is a pointwise Δ^m-statistically Cauchy sequence of order α on A;

(iii) $\{f_k\}$ is a sequence of functions for which there is a pointwise convergent sequence of function $\{\Delta^m g_k\}$ such that $\Delta^m f_k(x)=\Delta^m g_k(x)$ $a.a.k\,(\alpha)$ for every $x\in A$.

Theorem 3. If $0<\alpha\leq\beta\leq 1$, then the inclusion $S^{\alpha}(\Delta^m,F(p))\subseteq S^{\beta}(\Delta^m,F(p))$ strictly holds.

Proof. Since the proof of the inclusion $S^{\alpha}(\Delta^m,F(p))\subset S^{\beta}(\Delta^m,F(p))$ is easy, we omit detail. To show the strictness of the inclusion, consider a sequence $\{f_k\}$ defined by

$$f_k(x)=\begin{cases}1, & k=n^2\\ \frac{k^2x}{1+k^3x^2}, & otherwise\end{cases} \quad n=1,2,3...,x\in[0,1] \ .$$

Then one can calculate $\Delta f_k(x)$ as follows:

$$\Delta f_k(x)=\begin{cases}1-\frac{(k+1)^2x}{1+(k+1)^3x^2}, & k=n^2\\ \frac{k^2x}{1+k^3x^2}-1, & k=n^2-1.\\ \frac{k^2x}{1+k^3x^2}-\frac{(k+1)^2x}{1+(k+1)^3x^2}, & otherwise\end{cases}\ .$$

Therefore, for $\frac{1}{2}<\alpha\leq 1$, we get

$$\frac{1}{n^{\alpha}}\left|\left\{k\leq n \ : \ \left|\Delta f_k(x)-0\right|\geq\varepsilon, \text{ for every } x\in[0,1]\right\}\right|$$
$$=\frac{1}{n^{\alpha}}\left|\left\{k\leq n \ : \ \left|\Delta f_k(x)\right|\geq\varepsilon, \text{ for every } x\in[0,1]\right\}\right|$$
$$\leq\frac{2\sqrt{n}}{n^{\alpha}}\to 0, \ as\ n\to\infty.$$

Then $S^{\beta}-\lim\Delta f_k(x)=0$, i.e. $x\in S^{\beta}(\Delta,f)$ for $\frac{1}{2}<\beta\leq 1$, but $x\notin S^{\alpha}(\Delta,f)$ for $0<\alpha\leq\frac{1}{2}$.

Corollary 1. If a sequence of functions $\{f_k\}$ is Δ^m-statistically convergent of order α to f, then it is Δ^m-statistically convergent to f.

Definition 4. Let A be a subset of $\mathbb{R}$, $\{f_k\}$ be a sequence of real valued functions defined on A, and α be any real number such that $0<\alpha\leq 1$. A sequence of functions $\{f_k\}$ is said to be strongly pointwise Δ_q^m-Cesàro summable of order α for $x\in A$ and $\varepsilon>0$ if there exists a function f such

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that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}\left|\Delta^{m} f_k(x)-f(x)\right|^{q}=0.$$

In this case we write $w_q^{\alpha}-\lim\Delta^{m} f_k(x)=f(x)$ on A. The set of all strongly pointwise Δ_q^m −Cesàro summable sequences of functions of order α is denoted by $w_q^{\alpha}(\Delta^m, F(p))$. We write $w_{o,q}^{\alpha}(\Delta^m, F(p))$ in the case of $f(x)=0$.

Theorem 4. Let $\alpha,\beta\in(0,1]$ with $\alpha\le\beta$ and $0<q<\infty$. Then $w_q^{\alpha}(\Delta^m,F(p))\subseteq w_q^{\beta}(\Delta^m,F(p))$ and the inclusion is strict for some α and β such that $\alpha\le\beta$.

Proof. The inclusion part of the proof is easy, so we omit it. To show the strictness of the inclusion, consider a sequence $\{f_k\}$ defined by

$$f_k(x)=\begin{cases}\frac{kx}{1+kx} & k=n^2\\ 0 & otherwise\end{cases}\quad x\in[1,2].$$

Then we calculate $\Delta f_k(x)$ as

$$\Delta f_k(x)=\begin{cases}\frac{kx}{1+kx} & k=n^2\\ -\frac{(k+1)x}{1+(k+1)x} & k=n^2-1\\ 0 & otherwise\end{cases},$$

and so

$$\frac{1}{n^{\beta}}\sum_{k=1,\,x\in A}^{n}\left|\Delta f_k(x)-0\right|^{p}\le\frac{2\sqrt{n}}{n^{\beta}}=\frac{2}{n^{\beta-\frac{1}{2}}}\quad .$$

Since $2/\left(n^{\beta-\frac{1}{2}}\right)\to 0$ as $n\to\infty$, then $w_q^{\beta}-\lim\Delta f_k(x)=0$, which means that the sequence $\{f_k\}$ is strongly pointwise Δ_q −Cesàro summable of order α for $\frac{1}{2}<\beta\le 1$. But since

$$\frac{2\sqrt{n}}{2n^{\alpha}}\le\frac{1}{n^{\alpha}}\sum_{k=1,\,x\in A}^{n}\left|\Delta f_k(x)-0\right|^{p}$$

and $2\sqrt{n}/2n^{\alpha}\to\infty$ as $n\to\infty$, the sequence $\{f_k\}$ is not strongly Δ_q −Cesàro summable of order α for $0<\alpha<\frac{1}{2}$.

Corollary 2. Let q be a positive real number. Then the inclusion $w_q^{\alpha}(\Delta^m,F(p))\subseteq w_q(\Delta^m,F(p))$ strictly holds for some $\alpha\in(0,1]$.

Theorem 5. Let $\alpha,\beta\in(0,1]$ with $\alpha\le\beta$ and $0<q<\infty$. If a sequence of functions $\{f_k\}$ is strongly pointwise Δ_q^m −Cesàro summable of order α to f, then it is pointwise Δ^m −statistically convergent of order β to f.

Proof. For any sequence of functions $\{f_k\}$ defined on A, we can write

$$\sum_{k=1,\,x\in A}^{n}\left|\Delta^m f_k(x)-f(x)\right|^{q}\ge\left|\left\{k\le n\ :\ \left|\Delta^m f_k(x)-f(x)\right|\ge\varepsilon,\ \text{for every } x\in A\right\}\right|.\varepsilon^{q},$$

which leads to the consequence that

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$$\frac{1}{n^{\alpha}}\sum_{k=1,\,x\in A}^{n}\left|\Delta^{m}f_{k}(x)-f(x)\right|^{q}\geq\frac{1}{n^{\alpha}}\left|\left\{k\leq n\ :\ \left|\Delta^{m}f_{k}(x)-f(x)\right|\geq\varepsilon,\ \text{for every } x\in A\right\}\right|.\varepsilon^{q}$$

$$\geq\frac{1}{n^{\beta}}\left|\left\{k\leq n\ :\ \left|\Delta^{m}f_{k}(x)-f(x)\right|\geq\varepsilon,\ \text{for every } x\in A\right\}\right|\varepsilon^{q}.$$

Corollary 3. Let α be a fixed real number such that $0<\alpha\leq 1$ and $0<q<\infty$. If a sequence of functions $\{f_k\}$ is strongly pointwise Δ_q^m-Cesàro summable of order α to f, then it is pointwise Δ^m-statistically convergent of order α to f.

Definition 5. Let A be a subset of $\mathbb{R}$, $\{f_k\}$ be a sequence of real valued functions defined on A, and α be a fixed real number such that $0<\alpha\leq 1$. Then $\{f_k\}$ is said to be uniformly and Δ^m-statistically convergent of order α to f on A if there exists a natural number $N=N(\varepsilon)$ for $x\in A$ and $\varepsilon>0$ such that

$$\left|\Delta^{m}f_{k}(x)-f(x)\right|<\varepsilon\qquad a.a.k(\alpha)\text{ and for every } x\in A,$$

meaning that for $\varepsilon>0$,

$$\lim_{n}\frac{1}{n^{\alpha}}\left|\left\{k\leq n\ :\ \left|\Delta^{m}f_{k}(x)-f(x)\right|\geq\varepsilon\right\}\right|=0\text{ for every } x\in A.$$

In this case we write $S_u^{\alpha}-\lim\Delta^m f_k(x)=f(x)$ on A. The set of all uniform, Δ^m-statistically convergent sequences of function order α is denoted by $S^{\alpha}(\Delta^m,F(u))$. In this definition the natural number N depends only on ε. Therefore, if a sequence is uniformly and Δ^m-statistically convergent to f, then it is pointwise Δ^m-statistically convergent to f. However, the converse does not hold in general. To show this, consider a sequence $\{f_k\}$ defined by

$$f_k(x)=\begin{cases}2 & k=n^2\\ \frac{kx}{1+kx^2} & otherwise\end{cases}\ n=1,2,\ldots\text{and } x\in[0,1].$$

Then

$$\Delta f_k(x)=\begin{cases}2-\frac{(k+1)x}{1+(k+1)x^2} & k=n^2\\ \frac{kx}{1+kx^2}-2 & k=n^2-1\,n=1,2,\ldots\text{and } x\in[0,1].\\ \frac{kx}{1+kx^2}-\frac{(k+1)x}{1+(k+1)x^2} & otherwise\end{cases}$$

Therefore, $\{f_k\}$ is pointwise $\Delta-$statistically convergent of order α to $f(x)=0$ on $[0,1]$ for $0<\alpha<\frac{1}{2}$, but $\{f_k\}$ is not uniformly and $\Delta-$statistically convergent of order α in the following theorem since $\lim_{k\to\infty}c_k$ does not exist, where

$$c_k=\max_{x\in[0,1]}\left|\Delta f_k(x)-0\right|=\begin{cases}2 & k=n^2\\ \frac{\sqrt{k}}{2}-2 & k=n^2-1\,n=1,2,\ldots,\\ A & otherwise\end{cases}$$

in which $A=\frac{\left(\sqrt{6k^2+6k}\right)\left(\sqrt{-(2k+1)+\sqrt{16k^2+16k+1}}\right)}{\left(4k+5+\sqrt{16k^2+16k+1}\right)}-\frac{\left(\sqrt{6k^2+6k}\right)\left(\sqrt{-(2k+1)+\sqrt{16k^2+16k+1}}\right)}{\left(4k+5+\sqrt{16k^2+16k+1}\right)}.$

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Theorem 6. Let f and $\Delta^m f_k$ (for all $k \in \mathbb{N}$) be continuous functions on $A = [a,b] \subset \mathbb{R}$ and $0 < \alpha \leq 1$. Then $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ uniformly on A if and only if $S^\alpha - \lim c_k = 0$, where $c_k = \max_{x \in A} \left| \Delta^m f_k(x) - f(x) \right|$.

Proof. Omitted.

It is trivial that if $\lim \Delta^m f_k(x) = f(x)$ is uniform on A, then $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ is uniform on A. However, the converse is not true in general. To show this, consider the sequence defined by

$$f_k(x) = \begin{cases} 1 & k = n^2 \\ x^k & otherwise \end{cases} \quad k = 1,2,3 ..., x \in [0,1] \, .$$

So we have

$$\Delta f_k(x) = \begin{cases} 1 - x^{k+1} & k = n^2 \\ x^k - 1 & k = n^2 - 1 \\ x^k - x^{k+1} & otherwise \end{cases} .$$

If $x \in [0,1]$ and $\alpha \in [\frac{1}{2}, 1]$, then $\{f_k\}$ is uniformly and $\Delta-$statistically convergent of order α to $f(x) = 0$ on $[0,1]$ since $S^\alpha - \lim c_k = 0$, where

$$c_k = \max_{x \in [0,1]} \left| \Delta f_k(x) - 0 \right| = \begin{cases} 1 & k = n^2 \\ 0 & k = n^2 - 1 \\ \left(\frac{k}{k+1} \right)^k \frac{1}{k+1} & otherwise \end{cases} .$$

However, $(\Delta f_k(x))$ is not uniformly convergent on $[0,1]$ since $\lim_{k \to \infty} c_k$ does not exist.

Corollary 3. (i) $\lim \Delta^m f_k(x) = f(x)$ uniformly on $A \Rightarrow$ $\lim \Delta^m f_k(x) = f(x)$ on $A \Rightarrow$ $S^\alpha - \lim \Delta^m f_k(x) = f(x)$ pointwise on A.

(ii) $S_u^\alpha - \lim \Delta^m f_k(x) = f(x)$ uniformly on $A \Rightarrow S^\alpha - \lim \Delta^m f_k(x) = f(x)$ pointwise on A.

(iii) If $0 < \alpha \leq \beta \leq 1$, then $S^\alpha(\Delta^m, F(u)) \subseteq S^\beta(\Delta^m, F(u))$.

Definition 6. Let A be a subset of $\mathbb{R}$, $\{f_k\}$ be a sequence of real valued functions defined on A, and $\alpha \in (0,1]$. Then $\{f_k\}$ is said to be uniform Δ^m-statistical Cauchy sequence of order α if there exists a natural number $N = N(\varepsilon)$ for any $\varepsilon > 0$ such that

$$\left| \Delta^m f_k(x) - \Delta^m f_N(x) \right| < \varepsilon \quad a.a.k(\alpha) \text{ and for every } x \text{ in } A,$$

meaning that, for $\varepsilon > 0$,

$$\lim_n \frac{1}{n^\alpha} \left| \left\{ k \leq n \; : \; \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon \right\} \right| = 0, \text{ for every } x \text{ in } A.$$

In this definition the natural number N only depends on ε. By $S_u^\alpha(\Delta^m, F(p))$, we denote the set of uniform Δ^m-statistical Cauchy sequences of functions of order α.

The proofs of the following two theorems are similar to those of Theorems 1 and 2. Thus, we give them without proofs.

Theorem 7. The set $S_u^\alpha(\Delta^m, F(p))$ is a vector space under the usual operations, which are addition and scalar multiplication of the sequences of functions.

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Theorem 8. Let $\alpha \in (0,1]$ and $\{f_k\}$ be a sequence of functions defined on a set A. The following statements are equivalent:

(i) $\{f_k\}$ is a uniform, $\Delta^m -$ statistically convergent sequence of order α on *A*;

(ii) $\{f_k\}$ is a uniform, $\Delta^m -$ statistically Cauchy sequence of order α on *A*;

(iii) $\{f_k\}$ is a sequence of functions for which there is a uniformly convergent sequence of functions $\{\Delta^m g_k\}$ such that $\Delta^m f_k(x) = \Delta^m g_k(x)$ $a.a.k.(\alpha)$ for all $x \in A$.

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