

Full Paper

Statistical continuity of functions with values in uniform spaces

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Abstract: This paper comprehensively investigates the statistical continuity of functions taking values in uniform spaces. It aims to demonstrate how classical continuity can be characterised and extended by replacing traditional sequence convergence with statistically structured sequences defined via natural density. Thus, the concept of continuity is reinterpreted through statistical convergence, offering a more profound and comprehensive theoretical contribution to the study of continuity in uniform spaces.

Keywords: statistical continuity, statistical ward continuity, uniform space, statistical convergence, statistical quasi-Cauchy sequences

INTRODUCTION AND PRELIMINARIES

The study of continuous functions constitutes one of the fundamental pillars of mathematical analysis and topology. The concept of continuity bridges these two disciplines, facilitating the examination of various structural properties including compactness, completeness and convergence. A classical result asserts that every continuous function defined on a compact metric space is uniformly continuous, a phenomenon systematically investigated by Atsuji [1]. This line of research led to the introduction of Atsuji (UC) metric spaces, which lie strictly between compact and complete metric spaces and are characterised by the property that every continuous real-valued function is uniformly continuous. Extensions of these results to compact and generalised metric spaces have been studied in detail [2, 3].

Uniform spaces provide a natural and flexible framework for studying uniform continuity beyond metric structures. Atsuji [4] demonstrated that several characterisations valid in metric spaces fail in the uniform setting; in particular, real-valued functions are no longer sufficient to determine uniform continuity. Subsequent investigations addressed uniform continuity in sequentially uniform spaces [5] and clarified the relationship between uniform convergence and uniform continuity in generalised metric spaces [3]. Further refinements include the study of

products of uniformly continuous functions [6] and strengthened notions such as strong uniform continuity [7]. Comprehensive treatments of uniform spaces and their foundational role in general topology can be found in standard monographs [8-10].

In recent years statistical convergence has emerged as a powerful tool for refining traditional notions of convergence in a more flexible and nuanced manner. Statistical convergence was introduced independently by Fast [11] and Steinhaus [12] with conceptual roots dating back to Zygmund's influential monograph [13]. Defined via the natural density of subsets of \mathbb{N} , statistical convergence captures limiting behaviour that holds 'for most indices' rather than for all sufficiently large indices. Since its introduction, it has been extensively developed with essential contributions in summability theory, analysis and approximation theory [14-18].

Generalisations of statistical convergence soon emerged to address broader asymptotic behaviours. In particular, Kostyrko et al. [19, 20] introduced I-convergence, which unifies convergence types through ideals of \mathbb{N} and reduces to statistical convergence when the ideal consists of zero-density sets. Further developments by Lahiri and Das [21], Maddox [22], Das and Ghosal [23] and others enriched the theory in topological and uniform spaces. Additionally, I-limit and I-cluster point structures for more abstract settings were studied [24] while more general topological treatments of ideal-based convergence were presented [25, 26].

A significant advancement was achieved by Di Maio and Kočinac [27], who formulated statistical convergence in topological and uniform spaces, revealing deep relationships with uniformity, selection principles and hyperspaces. Later, Bilalov and Nazarova [28] and Bilalov and Sadigova [29] clarified the connections between statistical Cauchyness, completeness and the structure of uniform spaces. Ideal convergence of nets and function sequences in uniform structures has also been studied in detail by Megaritis [30].

Since classical continuity can be characterised by the preservation of convergent or Cauchy sequences, extending this concept to the statistical framework is a natural progression. Statistical continuity was first investigated for real-valued functions by Soybaş [31] and later generalised by Osmanoğlu and Dündar [32]. Moreover, Çakallı introduced statistical quasi-Cauchy sequences and statistical ward continuity [33, 34], thereby expanding the scope of continuity concepts relevant in discrete modelling, computational sciences and information theory. Further studies involving uniformly statistical convergence of function sequences were presented [35].

Recent developments also include structural interactions of ideals and convergence in function spaces [25], as well as extended surveys on the role of statistical and ideal convergences in general topology [36]. These works demonstrate the growing importance and applicability of generalised convergence methods in modern analysis.

Motivated by these developments, the present paper investigates different types of statistical continuity, namely statistical continuity, statistical Cauchy continuity and statistical ward continuity, for functions taking values in uniform spaces. We explore their fundamental properties, examine logical relationships among them and identify conditions under which these generalised continuities coincide with classical continuity. The need to move beyond the classical understanding of continuity, coupled with the flexibility offered by statistical convergence, forms the principal motivation for this study. Investigating the relationship between continuity and statistical convergence within the general framework of uniform spaces contributes to generalising classical analysis and enables deeper connections between statistical methods and topological structures.

A uniform space is a pair (X, \mathcal{U}) consisting of a set X and a uniformity \mathcal{U} on the set X . The elements of \mathcal{U} are called entourages. An entourage $U \in \mathcal{U}$ is called symmetric if $U^{-1} = U$. For every $U \in \mathcal{U}$ and $x_0 \in X$, we use the following notation:

$$U[x_0] = \{x \in X: (x_0, x) \in U\}.$$

For every uniform space (X, \mathcal{U}) , the uniform topology $\tau_{\mathcal{U}}$ on X is a family consisting of an empty set and all subsets $O \subseteq X$ such that for each $x \in O$, there exists $U \in \mathcal{U}$ with $U[x] \subseteq O$.

A mapping f from a topological space X into a uniform space (Y, \mathcal{V}) is called continuous at $x_0 \in X$ if for each $V \in \mathcal{V}$, there exists an open neighbourhood O_{x_0} of x_0 such that

$$f(O_{x_0}) \subseteq V[f(x_0)],$$

or equivalently, $(f(x_0), f(x)) \in V$ for every $x \in O_{x_0}$. The mapping f is called continuous if it is continuous at every point of X .

If $A \subseteq \mathbb{N}$, then $A(n)$ denotes the set

$$A(n) = \{k \in A: k \leq n\}.$$

The natural (or asymptotic) density of A is given by

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{|A(n)|}{n},$$

if it exists. The density takes values in the interval $[0,1]$. Clearly, finite subsets of \mathbb{N} have natural density zero. A subset $A \subseteq \mathbb{N}$ is said to be statistically dense if $\delta(A) = 1$. Also, for any $A \subseteq \mathbb{N}$, we have

$$\delta(\mathbb{N} \setminus A) = 1 - \delta(A).$$

A sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space X is said to be statistically convergent to $x \in X$ if for every neighborhood U of x ,

$$\delta(\{n \in \mathbb{N}: x_n \notin U\}) = 0.$$

If X is the first countable space, then this is equivalent to the existence of a subset $A \subseteq \mathbb{N}$ with $\delta(A) = 1$ such that the subsequence $(x_n)_{n \in A}$ converges to x [27].

A sequence $(x_n)_{n \in \mathbb{N}}$ in a uniform space (X, \mathcal{U}) is said to be statistically Cauchy if for each $U \in \mathcal{U}$, there exists $m \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N}: (x_n, x_m) \notin U\}) = 0.$$

Every statistically convergent sequence in a uniform space is statistically Cauchy [27].

STATISTICALLY CONTINUOUS

In this section the pair (X, τ) denotes a topological space and (Y, \mathcal{V}) a uniform space. For simplicity, they will occasionally be referred to simply as X and Y respectively. Additionally, throughout the article, let the set of all functions from topological space X to uniform space Y be denoted by Y^X .

Definition 1. A function $f \in Y^X$ is called

- *strongly statistically continuous* if the sequence $(f(x_n))_{n \in \mathbb{N}}$ in Y is convergent to $f(x)$ whenever a sequence $(x_n)_{n \in \mathbb{N}}$ in X is statistically convergent to x .

- *statistically continuous* if the sequence $(f(x_n))_{n \in \mathbb{N}}$ in Y is statistically convergent to $f(x)$ whenever a sequence $(x_n)_{n \in \mathbb{N}}$ in X is statistically convergent to x .

- *weakly statistically continuous* if the sequence $(f(x_n))_{n \in \mathbb{N}}$ in Y is statistically convergent to $f(x)$ whenever a sequence $(x_n)_{n \in \mathbb{N}}$ in X is convergent to x .

The definitions above have been given [31] for the topological space Y , possibly under different names. For consistency throughout the current paper, we redefine them by assuming that Y is a uniform space. In the theorems below, we provide new proofs under different conditions.

The proposition below follows easily from the fact that every convergent sequence is statistically convergent and from Definition 1.

Proposition 1. Let $f \in Y^X$ be a function. Strong statistical continuity of f implies its statistical continuity, which in turn ensures that f is weakly statistically continuous.

Theorem 1. If $f \in Y^X$ is continuous in the usual sense, then it is also statistically continuous.

Proof. Let $f \in Y^X$ be continuous and $x_n \xrightarrow{st} x$. Then for every neighbourhood U of x ,

$$\delta(\{n \in \mathbb{N}: x_n \notin U\}) = 0.$$

Since f is continuous, let us take $U = f^{-1}(V[f(x)])$ with respect to the uniform topology for every $V \in \mathcal{V}$. Hence

$$\delta(\{n \in \mathbb{N}: f(x_n) \notin f(U) = V[f(x)]\}) = 0.$$

Therefore, $f(x_n) \xrightarrow{st} f(x)$. ■

Corollary 1. If $f \in Y^X$ is continuous, it follows that f is weakly statistically continuous.

Example 1. The converse of Theorem 1 does not hold in general. To demonstrate this, consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

This is the classical Dirichlet function, which is nowhere continuous in the usual sense since every open interval in \mathbb{R} contains both rational and irrational numbers.

Now let (x_n) be a sequence in \mathbb{R} that is statistically convergent to some point $x \in \mathbb{R}$. Then the sequence $(f(x_n))$ takes values in the set $\{0,1\}$. If the set $\{n \in \mathbb{N}: x_n \in \mathbb{Q}\}$ has natural density 1, then $(f(x_n))$ is statistically convergent to $0 = f(x)$. Similarly, if $\{n \in \mathbb{N}: x_n \notin \mathbb{Q}\}$ has density 1, then $(f(x_n))$ is statistically convergent to $1 = f(x)$.

In either case we observe that the image sequence $(f(x_n))$ is statistically convergent to $f(x)$. Therefore, f is statistically continuous at every point in \mathbb{R} , although it is nowhere continuous in the usual sense. This shows that statistical continuity does not imply classical continuity.

Theorem 2. Let X be the first countable space and $f \in Y^X$ be a function. Then the following are equivalent:

1. f is continuous.
2. f is statistically continuous.
3. f is weakly statistically continuous.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) follows from Proposition 1 and Theorem 1.

(3) \Rightarrow (1): Suppose, for contradiction, that f is not continuous. Then there exists an open set $V \subseteq Y$ (with respect to the uniform topology on Y) such that $f^{-1}(V)$ is not open in X . Since X is first countable, the non-openness of $f^{-1}(V)$ implies that it is not sequentially open. Therefore, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $X \setminus f^{-1}(V)$ that converges (in the usual sense) to a point $x \in f^{-1}(V)$. It follows that $f(x_n) \in Y \setminus V$ for all n and since $Y \setminus V$ is closed, the sequence $(f(x_n))$ cannot converge to $f(x) \in V$. This contradicts the assumption that f is weakly statistically continuous because classical convergence $x_n \rightarrow x$ should imply statistical convergence $f(x_n) \xrightarrow{st} f(x)$. Hence f must be continuous. ■

Corollary 2. Let X be the first countable space and $f \in Y^X$ be strongly statistically continuous. Then f is continuous.

The notion of strong statistical continuity cannot be added as an equivalence under the conditions of Theorem 2. To illustrate this, let us consider the following example.

Example 2. Consider the identity function $I: \mathbb{R} \rightarrow \mathbb{R}$ defined by $I(x) = x$. It is clear that I is continuous in the usual sense. Moreover, I is statistically continuous. However, I is not strongly statistically continuous. To see this, let us define a sequence (x_n) as follows:

$$x_n = \begin{cases} x, & \text{if } n \notin D, \\ x + (-1)^n, & \text{if } n \in D, \end{cases}$$

where $\delta(D) = 0$. Then $\{x_n\}$ is statistically convergent to x , i.e. $x_n \xrightarrow{st} x$, but it is not classically convergent to x . Since $I(x_n) = x_n$, the sequence $(I(x_n))$ does not converge to $I(x)$. Therefore, I is not strongly statistically continuous.

STATISTICALLY WARD CONTINUOUS

This section considers the spaces (X, \mathcal{U}) and (Y, \mathcal{V}) as uniform spaces.

Definition 2. A function $f \in Y^X$ is called *statistically Cauchy continuous* if $(f(x_n))_{n \in \mathbb{N}}$ is a statistically Cauchy sequence in Y whenever $(x_n)_{n \in \mathbb{N}}$ is a statistically Cauchy sequence in X . That is, f preserves statistically Cauchy sequences.

The following theorem formalises that statistical Cauchy continuity is stronger than statistical continuity.

Theorem 3. If $f \in Y^X$ is statistically Cauchy continuous, then it is statistically continuous.

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X that statistically converges to $x \in X$. To show that f is statistically continuous, we need to prove that $(f(x_n))_{n \in \mathbb{N}}$ statistically converges to $f(x)$ in Y . Since $(x_n)_{n \in \mathbb{N}}$ is statistically convergent to x , for every entourage U ,

$$\delta(\{n \in \mathbb{N} \mid (x_n, x) \notin U\}) = 0.$$

Now consider the sequence

$$z = (x_1, x, x_2, x, x_3, x, \dots).$$

This new sequence z is statistically convergent to x because for every entourage U , the set of indices where $(x_n, x) \notin U$ forms a finite set. By Theorem 2.15 [27], z is also a statistically Cauchy sequence in X . Since f is statistically Cauchy continuous, applying f to the sequence z results in

$$f(z) = (f(x_1), f(x), f(x_2), f(x), f(x_3), f(x), \dots),$$

which is statistically Cauchy in Y by the definition of statistical Cauchy continuity. Within this sequence, we can extract the constant subsequence $(f(x), f(x), f(x), \dots)$, which clearly converges to $f(x)$. Since the original sequence $(f(x_1), f(x_2), f(x_3), \dots)$ is a subsequence of the statistically Cauchy sequence $f(z)$ and contains a subsequence converging to $f(x)$, we conclude that the original sequence $(f(x_n))_{n \in \mathbb{N}}$ statistically converges to $f(x)$. Therefore, f is statistically continuous. ■

Example 3. The converse of Theorem 3 is not true in general. Consider the uniform spaces $X = (0,1)$ (with the subspace uniformity inherited from \mathbb{R}) and $Y = \mathbb{R}$ (with the usual uniformity). Let $f: (0,1) \rightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x}$.

Then f is continuous on $(0,1)$ and therefore, by Theorem 1, statistically continuous.

Let (x_n) be the sequence in $(0,1)$ defined by $x_n = 1/n$ for all $n \in \mathbb{N}$. Since the sequence (x_n) converges to zero in \mathbb{R} , it is Cauchy and hence also statistically Cauchy in the uniform space $(0,1)$. However, for this sequence we have

$$f(x_n) = n, \quad n \in \mathbb{N}.$$

We show that $(f(x_n))$ is not statistically Cauchy in \mathbb{R} . Indeed, take $\varepsilon = 1$. For any index $m \in \mathbb{N}$, the set

$$A_m(\varepsilon) = \{n \in \mathbb{N} : |f(x_n) - f(x_m)| \geq \varepsilon\}$$

is considered. Since $f(x_n) = n$ and $f(x_m) = m$, we have $|f(x_n) - f(x_m)| \geq 1$ for every $n > m$. Hence

$$A_m(\varepsilon) \supseteq \{n \in \mathbb{N} : n > m\},$$

which has natural density 1. Therefore, $\delta(A_m(\varepsilon)) = 1 \neq 0$ for every $m \in \mathbb{N}$. By the definition of statistically Cauchy sequence, this shows that $(f(x_n))$ is not statistically Cauchy. Consequently, f is statistically continuous but not statistically Cauchy continuous. This provides a concrete counter example demonstrating that the converse of Theorem 3 does not hold in general.

Considering Theorem 2, the following result follows immediately.

Corollary 3. Let X be the first countable uniform space and $f \in Y^X$ be statistically Cauchy continuous. Then f is continuous in the usual sense.

Now to provide a more general perspective on the concept of Cauchy sequences in uniform spaces, we present the definition of a *quasi-Cauchy* sequence. This concept was originally introduced for metric spaces by Cakalli [33].

Definition 3. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called statistically quasi-Cauchy if for each $U \in \mathcal{U}$, we have

$$\delta(\{n \in \mathbb{N} : (x_n, x_{n+1}) \notin U\}) = 0.$$

Theorem 4. If a sequence $(x_n)_{n \in \mathbb{N}}$ in a uniform space (X, \mathcal{U}) is statistically Cauchy, then it is statistically quasi-Cauchy.

Proof. Suppose that (x_n) is statistically Cauchy. Then for every entourage $U \in \mathcal{U}$, there exists $n_0 \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : (x_n, x_{n_0}) \notin U\}) = 0.$$

Since (x_n) is statistically Cauchy, we also have

$$\delta(\{n \in \mathbb{N} : (x_n, x_{n_0}) \notin V\}) = 0 \quad \text{and} \quad \delta(\{n \in \mathbb{N} : (x_{n+1}, x_{n_0}) \notin V\}) = 0.$$

Now observe that if both $(x_n, x_{n_0}) \in V$ and $(x_{n+1}, x_{n_0}) \in V$, then by the definition of $V \circ V$, we have

$$(x_n, x_{n+1}) \in V \circ V \subseteq U.$$

Therefore, the set

$$\{n \in \mathbb{N} : (x_n, x_{n+1}) \notin U\}$$

is a subset of the union of the two sets:

$$\{n \in \mathbb{N} : (x_n, x_{n_0}) \notin V\} \cup \{n \in \mathbb{N} : (x_{n+1}, x_{n_0}) \notin V\}.$$

Since each of these sets has natural density zero, their union also has natural density zero. Thus,

$$\delta(\{n \in \mathbb{N} : (x_n, x_{n+1}) \notin U\}) = 0,$$

which means that (x_n) is statistically quasi-Cauchy. ■

Example 4. The converse of Theorem 4 does not hold in general. Consider the sequence (x_n) in \mathbb{R} defined by

$$x_n = \begin{cases} 1, & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$, the set A_ε is defined as

$$A_\varepsilon = \{n \in \mathbb{N} : |x_{n+1} - x_n| \geq \varepsilon\}.$$

Since non-zero differences occur only at indices $n = k^2 - 1$ and $n = k^2$, the set A_ε is sparse and $\delta(A_\varepsilon) = 0$. Hence (x_n) is statistically quasi-Cauchy.

Now by taking $\varepsilon = 1/2$ and fixing $m = k^2$, then $x_m = 1$ and the set

$$S_m = \{n \in \mathbb{N} : |x_n - x_m| \geq 1/2\} = \{n \in \mathbb{N} : x_n = 0\}$$

has density $\delta(S_m) = 1$. Thus, (x_n) is not statistically Cauchy. Consequently, a statistically quasi-Cauchy sequence need not be statistically Cauchy and the converse of Theorem 4 fails in general.

Corollary 4. If a sequence $(x_n)_{n \in \mathbb{N}}$ in a uniform space (X, \mathcal{U}) is statistically convergent, then it is a statistically quasi-Cauchy sequence.

The following definition was originally introduced for statistically quasi-Cauchy sequences in the set of real numbers \mathbb{R} by Cakalli [34]. Below, we revise this definition for uniform spaces. Thus, we provide a more general and comprehensive approach that extends the classical setting to a broader topological context.

Definition 4. A function $f \in Y^X$ is called statistically ward continuous if $(f(x_n))_{n \in \mathbb{N}}$ is a statistically quasi-Cauchy sequence in Y whenever $(x_n)_{n \in \mathbb{N}}$ is a statistically quasi-Cauchy sequence in X . That is, f preserves statistically quasi-Cauchy sequences.

As shown in Theorem 3, statistical Cauchy continuity is stronger than statistical continuity. The following theorem considers statistically ward continuous functions and establishes that they are also statistically continuous.

Theorem 5. If $f \in Y^X$ is statistically ward continuous, then it is statistically continuous.

Proof. Since every statistically Cauchy sequence is statistically quasi-Cauchy, the proof is similar to that of Theorem 3. ■

Example 5. The converse of Theorem 5 is not true in general. If we consider the same sequence and function given in Example 3, namely

$$f: (0,1) \rightarrow \mathbb{R}, \quad f(x) = \frac{1}{x}, \quad \text{and} \quad x_n = \frac{1}{n} \quad (n \in \mathbb{N}),$$

then we obtain a counter example showing that statistical continuity does not imply statistically ward continuity. For each $\varepsilon > 0$,

$$|x_{n+1} - x_n| = \frac{1}{n(n+1)} < \varepsilon$$

for all sufficiently large n . Hence the set

$$\{n \in \mathbb{N} : |x_{n+1} - x_n| \geq \varepsilon\}$$

is finite and thus has natural density 0. Therefore (x_n) is a statistically quasi-Cauchy sequence in X . Since $f(x_n) = n$, for $\varepsilon = 1/2$ we have

$$|f(x_{n+1}) - f(x_n)| = |(n+1) - n| = 1 \geq 1/2$$

for all $n \in \mathbb{N}$. Consequently,

$$\{n \in \mathbb{N} : |f(x_{n+1}) - f(x_n)| \geq 1/2\} = \mathbb{N},$$

whose natural density equals 1. Thus, $(f(x_n))$ fails to be statistically quasi-Cauchy. In conclusion, although f is statistically continuous (as shown in Example 3), it is not statistically ward continuous. Hence the converse of Theorem 5 does not hold in general.

Considering Theorem 2, the following result follows immediately.

Corollary 5. Let X be the first countable uniform space and $f \in Y^X$ be statistically ward continuous. Then f is continuous in the usual sense.

Lemma 1. Let X be a complete uniform space with a countable base. Then every statistically quasi-Cauchy sequence in X has a statistically convergent subsequence.

Proof. Let (x_n) be a statistically quasi-Cauchy sequence in X . Since X has a countable base $(U_k)_{k \in \mathbb{N}}$ such that

$$U_{k+1} \circ U_{k+1} \subseteq U_k \quad \text{for all } k,$$

we construct a subsequence (x_{n_k}) inductively by choosing $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$ and

$$(x_{n_k}, x_{n_{k+1}}) \in U_k.$$

This is possible since for each k , the set $\{n \in \mathbb{N}: (x_n, x_{n+1}) \notin U_k\}$ has natural density zero and hence infinitely many values of n satisfy $(x_n, x_{n+1}) \in U_k$. The sequence (y_k) is defined by $y_k = x_{n_k}$. Then $(y_k, y_{k+1}) \in U_k$ for all k . Let $m > n$; then

$$(y_n, y_m) \in U_n \circ U_{n+1} \circ \cdots \circ U_{m-1} \subseteq U_n,$$

using the base condition. Hence (y_k) is a Cauchy sequence in X . Since X is complete, there exists $x \in X$ such that $y_k \rightarrow x$. To prove statistical convergence, fix any entourage U . Then there exists $k_0 \in \mathbb{N}$ such that $U_k \subseteq U$ for all $k \geq k_0$. Since $(y_k, x) \in U_k \subseteq U$ for all sufficiently large k , the set $\{k \in \mathbb{N}: (y_k, x) \notin U\}$ is finite and hence has density zero. Therefore $y_k \xrightarrow{st} x$. Thus, (y_k) is a statistically convergent subsequence of (x_n) . ■

Lemma 2. In a uniform space (X, \mathcal{U}) , if a sequence $(x_n)_{n \in \mathbb{N}}$ has a co-final statistically quasi-Cauchy subsequence $(x_{n_k})_{k \in \mathbb{N}}$, then (x_n) is statistically quasi-Cauchy.

Proof. Let $U \in \mathcal{U}$ be any entourage. Choose a symmetric entourage $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. Since (x_{n_k}) is statistically quasi-Cauchy,

$$\delta(\{k \in \mathbb{N}: (x_{n_k}, x_{n_{k+1}}) \notin V\}) = 0.$$

For the indices not in the subsequence, the set is defined as

$$A = \{n \in \mathbb{N}: n \neq n_k \text{ for any } k\}.$$

By the co-finality assumption, $\delta(A) = 0$. Now the following is considered:

$$\{n \in \mathbb{N}: (x_n, x_{n+1}) \notin U\} \subseteq A \cup \{n_k \in \mathbb{N}: (x_{n_k}, x_{n_{k+1}}) \notin V\}.$$

Since both sets on the right have natural density zero, it follows that

$$\delta(\{n \in \mathbb{N}: (x_n, x_{n+1}) \notin U\}) = 0.$$

Therefore, (x_n) is statistically quasi-Cauchy. ■

Theorem 6. Let (X, \mathcal{U}) be a complete uniform space with a countable base. Then a function $f \in Y^X$ is statistically ward continuous if and only if it is statistically continuous.

Proof. According to Theorem 5, every statistically ward continuous function $f \in Y^X$ is statistically continuous.

Conversely, assume that $f \in Y^X$ is statistically continuous and let $(x_n)_{n \in \mathbb{N}}$ be a statistically quasi-Cauchy sequence in X . By Lemma 1, there exists a statistically convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n) . Since f is statistically continuous, the sequence $(f(x_{n_k}))_{k \in \mathbb{N}}$ is statistically convergent in Y and hence it is statistically quasi-Cauchy. As $(f(x_{n_k}))$ is a statistically quasi-

Cauchy subsequence of $(f(x_n))$, it follows by Lemma 2 that $(f(x_n))$ is statistically quasi-Cauchy. Therefore, f is statistically ward continuous. ■

Theorem 7. If $f \in Y^X$ is statistically ward continuous, then it is statistically Cauchy continuous.

Proof. Every statistically Cauchy sequence is also statistically quasi-Cauchy. Since statistically ward continuity preserves statistically quasi-Cauchy sequences, it follows that f preserves statistically Cauchy sequences as well. Hence f is statistically Cauchy continuous. ■

Example 6. The converse of Theorem 7 does not hold in general. Consider the sequence (x_n) in \mathbb{R} defined as in Example 4:

$$x_n = \begin{cases} 1, & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

As shown in Example 4 for $\varepsilon > 0$, the set A_ε is defined as

$$A_\varepsilon = \{n \in \mathbb{N} : |x_{n+1} - x_n| \geq \varepsilon\}$$

and is sparse with density $\delta(A_\varepsilon) = 0$, so (x_n) is statistically quasi-Cauchy. Now define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x_n) = x_n.$$

Then f preserves statistically quasi-Cauchy sequences, so f is statistically ward continuous. However, taking $\varepsilon = 1/2$ and $m = k^2$,

$$S_m = \{n \in \mathbb{N} : |x_n - x_m| \geq 1/2\} = \{n \in \mathbb{N} : x_n = 0\}$$

has density $\delta(S_m) = 1$. Therefore, (x_n) is not statistically Cauchy and f does not preserve statistically Cauchy sequences. Consequently, a statistically ward continuous function need not be statistically Cauchy continuous, showing that the converse of Theorem 7 fails in general.

Definition 5 [35]. A sequence $(f_n)_{n \in \mathbb{N}}$ in Y^X is said to be *uniformly statistically convergent* to a function f on X if for each $U \in \mathcal{U}$,

$$\delta(\{n \in \mathbb{N} : (f_n(x), f(x)) \notin U, \text{ for all } x \in X\}) = 0.$$

In other words, for each $U \in \mathcal{U}$,

$$\delta(\{n \in \mathbb{N} : f_n(x) \notin U[f(x)], \text{ for all } x \in X\}) = 0.$$

We denote it by $f_n \xrightarrow{st-u} f$.

In many analytical settings it is essential to determine whether continuity-type properties are preserved under limits of functions. In the statistical framework this naturally leads to whether the limit of a sequence of statistically ward continuous functions remains statistically ward continuous. The following theorem provides an affirmative answer.

Theorem 8. Let $(f_n)_{n \in \mathbb{N}}$ in Y^X be a sequence of statistically ward continuous functions and $f_n \xrightarrow{st-u} f$. Then f is also statistically ward continuous.

Proof. Let $(x_k)_{k \in \mathbb{N}}$ be a statistically quasi-Cauchy sequence in (X, \mathcal{U}) . Then for every entourage $U \in \mathcal{U}$,

$$\delta(\{k \in \mathbb{N} : (x_{k+1}, x_k) \notin U\}) = 0.$$

We aim to show that $(f(x_k))_{k \in \mathbb{N}}$ is also a statistically quasi-Cauchy sequence in (Y, \mathcal{V}) . Since each f_n is statistically ward continuous, it preserves statistically quasi-Cauchy sequences. Thus, for each $n \in \mathbb{N}$, the sequence $(f_n(x_k))_{k \in \mathbb{N}}$ is statistically quasi-Cauchy in Y . Then for every entourage $V \in \mathcal{V}$,

$$\delta(\{k \in \mathbb{N} : (f_n(x_{k+1}), f_n(x_k)) \notin V\}) = 0.$$

On the other hand, since $f_n \xrightarrow{st-u} f$, for any entourage $W \in \mathcal{V}$, there exists an entourage $V \in \mathcal{V}$ such that $V \circ V \circ V \subseteq W$ (by Lemma 2.5 [30]) and we have

$$\delta(\{n \in \mathbb{N}: (f_n(x), f(x)) \notin V, \text{ for all } x \in X\}) = 0.$$

Let

$$A = \{n \in \mathbb{N}: (f_n(x), f(x)) \in V, \text{ for all } x \in X\}.$$

Then $\delta(A) = 1$. Let $n_0 \in A$ be fixed. For every $k \in \mathbb{N}$, we have

$$(f(x_{k+1}), f(x_k)) \in V \circ (f_{n_0}(x_{k+1}), f_{n_0}(x_k)) \circ V.$$

Since $(f_{n_0}(x_k))_{k \in \mathbb{N}}$ is statistically quasi-Cauchy, the set

$$B = \{k \in \mathbb{N}: (f_{n_0}(x_{k+1}), f_{n_0}(x_k)) \notin V\}$$

has natural density zero, i.e. $\delta(B) = 0$. Moreover, since $f_{n_0} \in A$, the sets

$$C = \{k \in \mathbb{N}: (f_{n_0}(x_k), f(x_k)) \notin V\}, \quad D = \{k \in \mathbb{N}: (f_{n_0}(x_{k+1}), f(x_{k+1})) \notin V\}$$

also have natural density zero. Therefore, the set

$$E = \{k \in \mathbb{N}: (f(x_{k+1}), f(x_k)) \notin W\}$$

is a subset of $B \cup C \cup D$, so $\delta(E) = 0$. This shows that $(f(x_k))_{k \in \mathbb{N}}$ is statistically quasi-Cauchy.

Hence f is statistically ward continuous. ■

Theorem 9. If $f \in Y^X$ is uniformly continuous, then it is statistically ward continuous.

Proof. Let (x_n) be a statistically quasi-Cauchy sequence in (X, \mathcal{U}) . Then for every entourage $U \in \mathcal{U}$,

$$\delta(\{k \in \mathbb{N}: (x_k, x_{k+1}) \notin U\}) = 0.$$

Since f is uniformly continuous, for each entourage $V \in \mathcal{V}$ in (Y, \mathcal{V}) , there exists an entourage $U \in \mathcal{U}$ such that

$$(x, y) \in U \Rightarrow (f(x), f(y)) \in V.$$

Now the set

$$A = \{k \in \mathbb{N}: (f(x_k), f(x_{k+1})) \notin V\}$$

is defined. By uniform continuity,

$$(f(x_k), f(x_{k+1})) \notin V \Rightarrow (x_k, x_{k+1}) \notin U,$$

so $A \subseteq \{k \in \mathbb{N}: (x_k, x_{k+1}) \notin U\}$. Hence

$$\delta(A) = 0.$$

This shows that $(f(x_n))$ is also a statistically quasi-Cauchy sequence in Y . Therefore, f is statistically ward continuous. ■

Example 7. The converse of Theorem 9 does not hold in general. Let $X = \mathbb{R}, Y = \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$.

The function f is not uniformly continuous on \mathbb{R} because the derivative grows without bound; that is, for any $\delta > 0$, $|x - y| < \delta$ does not guarantee $|f(x) - f(y)| < \varepsilon$ for all $x, y \in \mathbb{R}$. However, f is statistically ward continuous in the sense that if (x_n) is statistically quasi-Cauchy (i.e. the set of indices with large differences is sparse), then $(f(x_n))$ is also statistically quasi-Cauchy. The reason is that large jumps in (x_n) occur rarely (density 0), so $(f(x_n))$ preserves the statistical quasi-Cauchy property.

Hence f is statistically ward continuous but not uniformly continuous, showing that the converse of Theorem 9 fails.

Definition 6. A uniform space X is said to be *statistically ward compact* if every sequence in X has a statistically quasi-Cauchy subsequence.

Proposition 2. Let $f \in Y^X$ be statistically ward continuous and X be statistically ward compact. Then $f(X)$ is statistically ward compact in Y .

Proof. Let (y_n) be a sequence in $f(X)$. Then for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $f(x_n) = y_n$. So we obtain a sequence (x_n) in X . Since X is statistically ward compact, the sequence (x_n) has a statistically quasi-Cauchy subsequence (x_{n_k}) . Now since f is statistically ward continuous, the sequence $(f(x_{n_k})) = (y_{n_k})$ is also statistically quasi-Cauchy. Thus, the sequence (y_n) in $f(X)$ has a statistically quasi-Cauchy subsequence. Hence $f(X)$ is statistically ward compact. ■

Finally, we investigate the relationship between statistical ward continuity and uniform continuity. In particular, we show that when the domain is statistically ward compact, statistical ward continuity of a function ensures its uniform continuity. This connection is established in the following theorem.

Theorem 10. Let X be a statistically ward compact uniform space with a countable base and $f \in Y^X$ be statistically ward continuous. Then f is uniformly continuous.

Proof. Assume, for contradiction, that f is not uniformly continuous. Then there exists a symmetric entourage $V \in V$ of Y such that for every entourage $U \in U$ of X , there exist points $x, y \in X$ with $(x, y) \in U$ but $(f(x), f(y)) \notin V$. Since U has a countable base $(U_n)_{n \in \mathbb{N}}$ satisfying

$$U_{n+1} \circ U_{n+1} \subseteq U_n,$$

for each n we choose points $(x_n, y_n) \in X \times X$ with

$$(x_n, y_n) \in U_n, \quad (f(x_n), f(y_n)) \notin V.$$

By statistical ward compactness of X , the sequence (x_n) has a statistically quasi-Cauchy subsequence (x_{n_k}) . We show that (y_{n_k}) is also statistically quasi-Cauchy. For any entourage $W \in U$, since (x_{n_k}) is statistically quasi-Cauchy,

$$(x_{n_{k+1}}, x_{n_k}) \in U_{n_k} \circ W \circ U_{n_k}$$

for sufficiently large k . Since

$$(y_{n_{k+1}}, y_{n_k}) \in U_{n_k} \circ (x_{n_{k+1}}, x_{n_k}) \circ U_{n_k},$$

it follows that

$$(y_{n_{k+1}}, y_{n_k}) \in U_{n_k} \circ W \circ U_{n_k}.$$

Therefore, (y_{n_k}) is statistically quasi-Cauchy. Consider the interleaved sequence $z = (x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, \dots)$. This sequence is statistically quasi-Cauchy since for any fixed entourage U_m ,

$$\delta\{k: (x_{n_k}, y_{n_k}) \notin U_m\} = 0,$$

and

$$\delta\{k: (y_{n_k}, x_{n_{k+1}}) \notin U_m\} = 0.$$

However, the image sequence $f(z) = (f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), \dots)$ is not statistically quasi-Cauchy because $(f(x_{n_k}), f(y_{n_k})) \notin V$ for all k , implying

$$\delta\{k: (f(z_{2k-1}), f(z_{2k})) \notin V\} \geq \frac{1}{2}.$$

This contradicts the statistical ward continuity of f . Hence f must be uniformly continuous. ■

CONCLUSIONS

We have developed a comprehensive framework that refines the traditional understanding of continuity within topological and uniform settings. Our findings reveal a hierarchical structure among statistical continuity types defined via statistically convergent, statistically Cauchy and statistically quasi-Cauchy sequences. We demonstrate that these notions coincide with classical continuity under topological conditions. Furthermore, we have shown that statistical ward continuity implies statistical Cauchy continuity and, under suitable assumptions such as compactness and the presence of a countable base, implies uniform continuity. Additionally, we prove that the class of statistically ward continuous functions is closed under uniform statistical convergence.

These results highlight the potential of statistical methods for generalising classical topological concepts and providing more nuanced analytical tools for investigating the behaviour of functions in abstract spaces.

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