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Sharp Bernstein-type inequalities for Cameron–Martin space associated with Brownian motion

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Abstract: Sharp Bernstein-type inequalities in the reproducing kernel Hilbert space associated with standard Brownian motion are studied. Reproducing kernel Hilbert spaces provide an important analytical framework by connecting geometric properties of Hilbert spaces with the pointwise behaviour of functions. As a result, they have become fundamental tools in several areas of mathematics including approximation theory, probability theory, operator theory, stochastic analysis and mathematical physics. In the Brownian setting considered in this work, the corresponding reproducing kernel Hilbert space coincides with the classical Cameron–Martin space, which plays a central role in the study of Gaussian processes and the analysis of Brownian motion. The main objective of this article is to establish precise estimates that relate pointwise values of functions to their global Hilbert space norm and to determine the exact constants for which these estimates remain valid. Unlike the smoother reproducing kernel settings where derivative evaluations often dominate, the Brownian framework naturally leads to inequalities formulated in terms of function values themselves. To address this problem, the analysis combines reproducing kernel methods with an explicit orthogonal decomposition argument in the derivative space. Using this approach, a sharp Bernstein-type inequality is obtained together with the exact characterisation of the corresponding extremal functions. As a consequence, an optimal uniform estimate is derived for the Brownian reproducing kernel Hilbert space. In addition, a sharp global embedding inequality from the Cameron–Martin space into $L^2(0,1)$ is established using variational and spectral methods, and the corresponding extremisers are identified explicitly. Overall, the results provide a complete local and global description of norm control in the Cameron–Martin space and illustrate how reproducing kernel techniques can be used to obtain exact inequalities and explicit extremal structures.

Keywords: reproducing kernel Hilbert space, Brownian motion, Cameron–Martin space, Bernstein-type inequality, Gaussian processes, sharp embedding inequalities

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Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

INTRODUCTION

Reproducing kernel Hilbert spaces (RKHSs) constitute one of the central frameworks connecting functional analysis, operator theory, approximation theory, stochastic processes and probability theory. The theory originated from the foundational work of Aronszajn [1], who established the general structure of Hilbert spaces of functions possessing continuous point evaluations and characterised them through positive definite reproducing kernels. Since then, RKHS methods have become fundamental in both pure and applied mathematics, providing an explicit representation of evaluation functionals and translating geometric properties of function spaces into analytic inequalities. General treatments of reproducing kernel theory and its applications can be found in the literature [2,3].

A distinguishing feature of an RKHS is that evaluation at a point is represented by the inner product with the corresponding kernel section. More precisely, if $\mathcal{H}$ is an RKHS with reproducing kernel K, then for every x in the underlying domain, the function $K_x(\cdot) = K(\cdot, x)$ belongs to $\mathcal{H}$ and satisfies the reproducing property:

$$f(x) = \langle f, K_x \rangle_{\mathcal{H}}.$$

This identity transforms pointwise estimation problems into questions concerning orthogonal projections and evaluation functionals in Hilbert spaces. In particular, the norm of the evaluation functional is determined by the diagonal of the reproducing kernel since

$$| f(x) |^2 \leq K(x, x) \| f \|_{\mathcal{H}}^2.$$

Thus, reproducing kernel methods provide a natural framework for obtaining sharp pointwise inequalities and for identifying their extremal functions.

Bernstein-type inequalities have a long history in approximation theory and functional analysis. Classical Bernstein inequalities typically provide bounds for derivatives or pointwise values in terms of suitable global norms. In reproducing kernel settings, these inequalities acquire a geometric interpretation through the boundedness of evaluation functionals and the structure of kernel sections. Related RKHS formulations of Bernstein–Szegő inequalities have been investigated by Reyes and Artes [4]. In this setting, determining the exact constant is particularly important since the optimal constant is directly related to the norm of the corresponding evaluation operator and the extremal functions can often be described explicitly.

Recent developments have further emphasised the importance of explicit reproducing kernels and sharp norm inequalities in Sobolev-type Hilbert spaces. Yamada [5] obtained an explicit reproducing kernel for Sobolev spaces on finite intervals while Saitoh [6] investigated extension formulas and norm inequalities in Sobolev Hilbert spaces. The connection between reproducing kernel methods and Bernstein-type approximation is also illustrated by the work of Feng et al. [7] on approximating RKHS functions by Bernstein operators. More recently, sharp point evaluation problems have been studied in other reproducing kernel settings including Paley–Wiener spaces, where the optimal constants and extremal functions are determined explicitly [8]. Sharp higher-order Sobolev embedding inequalities and their extremal functions have also been investigated by Hindov et al. [9]. These recent contributions provide a broader functional-analytic context for the sharp pointwise and embedding inequalities considered in the present paper.

Among the most important examples of RKHS arising from stochastic processes is the space associated with standard Brownian motion. Brownian motion is one of the fundamental Gaussian processes and plays a central role in probability theory, stochastic analysis and the theory of Gaussian random functions. The general connection between Gaussian processes, covariance

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

structures and functional-analytic methods is discussed, for example, in the work of Lifshits [10] and Revuz and Yor [11]. Spectral and variational methods associated with differential operators and Gaussian structures also provide an important perspective on covariance and energy spaces; see, for example, Dym and Kravitsky [12].

The covariance kernel of standard Brownian motion on the interval $[0,1]$ is

$$K(x,y) = \min\{x,y\}, \qquad x,y \in [0,1].$$

The corresponding reproducing kernel Hilbert space is the classical Cameron–Martin space

$$\mathcal{H}_B = \{f \in AC[0,1]: f(0) = 0,\ f' \in L^2(0,1)\},$$

equipped with the inner product

$$\langle f, g\rangle_{\mathcal{H}_B} = \int_0^1 f'(t) g'(t)\, dt.$$

Thus, the Brownian RKHS provides a particularly transparent example in which the reproducing kernel, the Hilbert space norm and the underlying Gaussian process are explicitly related. The Cameron–Martin space is also fundamental in the analysis of Wiener measure. The classical Cameron–Martin theorem describes the quasi-invariance of Wiener measure under translations by elements of the Cameron–Martin space. Standard treatments of this theory and its connections with stochastic analysis and Wiener-space methods are given by Matsumoto and Taniguchi [13] and Nualart [14]. Related questions concerning extensions of the Cameron–Martin theorem to random shifts of Brownian motion have also been discussed [15]. In particular, the Cameron–Martin framework forms a natural bridge between deterministic finite-energy functions and Gaussian measures on path spaces. Related developments have extended Cameron–Martin type quasi-invariance ideas to more general processes. Deng and Schilling [16], for example, established a Cameron–Martin type quasi-invariance theorem for subordinate Brownian motion and investigated associated analytical consequences. The reproducing kernel structure of Cameron–Martin spaces has also been used in the study of conditional Wiener integrals and Gaussian processes by Choi [17]. These developments emphasise the close relationship between reproducing kernels, Brownian motion, Gaussian processes and Cameron–Martin spaces.

The Brownian RKHS differs substantially from the smoother reproducing kernel models arising from higher-order Sobolev spaces or analytic function spaces. In many classical RKHS settings derivative evaluation is a bounded functional and consequently derivative inequalities arise naturally. In the Brownian case, however, the Cameron–Martin norm is given by the L^2-norm of the first derivative while the fundamental reproducing kernel is the covariance kernel $K(x,y) = \min\{x,y\}$. Point evaluation is therefore the natural bounded functional to consider. This leads to sharp Bernstein-type inequalities whose constants are determined by the geometry of the Brownian Cameron–Martin space.

The operator-theoretic viewpoint is also relevant to the present setting. Reproducing kernels and Hilbert space projections are closely connected with operator-theoretic structures including those arising in the theory of Hankel operators [18]. More generally, the functional analytic framework of Hille and Phillips [19] provides classical tools for studying operators, variational problems and norm estimates in function spaces. In the present paper the Hilbert space structure allows the sharp pointwise inequality to be obtained directly from an orthogonal decomposition of the derivative. For the global embedding problem, the optimal constant is instead characterised through a variational principle and the first eigenvalue of an associated differential operator. This

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

provides a natural connection between reproducing kernel methods, Hilbert-space geometry and Sturm–Liouville theory.

The purpose of the present paper is to establish sharp Bernstein-type inequalities for the reproducing kernel Hilbert space associated with standard Brownian motion and to identify the corresponding extremal functions. The first main result gives the sharp pointwise estimate

$$|f(x)|^2 \leq x \, \| f \|^2_{\mathcal{H}_B}, \qquad x \in [0,1],$$

for every $f \in \mathcal{H}_B$. The constant x is optimal for each $x \in [0,1]$ and the equality cases are characterised explicitly in terms of the corresponding kernel sections. The proof is based on an orthogonal decomposition of the derivative in $L^2(0,1)$, which yields an exact identity underlying the pointwise inequality.

The second main result concerns the global embedding of the Brownian Cameron–Martin space into $L^2(0,1)$. I prove the sharp inequality

$$\int_0^1 |f(x)|^2 \, dx \leq \frac{4}{\pi^2} \| f \|^2_{\mathcal{H}_B},$$

where $\frac{4}{\pi^2}$ is the optimal constant. The constant is obtained from the first eigenvalue of the associated variational problem. The extremal functions are precisely the non-zero multiples of $\sin\left(\frac{\pi x}{2}\right)$, which solve the corresponding Sturm–Liouville problem with the boundary conditions $f(0) = 0$ and $f'(1) = 0$. In particular, the free endpoint at $x = 1$ naturally produces the Neumann boundary condition in the variational formulation.

I use the term ‘Bernstein-type inequality’ in the present paper to emphasise the sharp control of pointwise function values by a global energy norm. This terminology is intended to distinguish my results from classical Bernstein inequalities for derivatives of trigonometric polynomials and entire functions. Here, the derivative information is incorporated into the Cameron–Martin norm

$$\| f \|^2_{\mathcal{H}_B} = \int_0^1 |f'(t)|^2 \, dt$$

while the quantity being controlled is the pointwise value $|f(x)|$. The results therefore illustrate how classical ideas concerning sharp inequalities and extremal functions can be naturally formulated in the reproducing kernel Hilbert space associated with Brownian motion.

MAIN LEMMAS AND STRUCTURAL PROPERTIES OF BROWNIAN RKHSs

Before establishing the main Bernstein-type inequalities, it is useful to examine in detail the internal geometric structure of the reproducing kernel Hilbert space generated by the Brownian covariance kernel. The key observation is that the point evaluation in this space is represented by orthogonal projection in the derivative space $L^2(0,1)$. This viewpoint not only clarifies the reproducing mechanism, but also leads naturally to sharp inequalities and extremal characterisations.

It can be recalled that the reproducing kernel under consideration is

$$K(x,y) = \min\{x,y\}, \qquad x,y \in [0,1],$$

and the associated reproducing kernel Hilbert space is

$$\mathcal{H}_B = \{f \in AC[0,1]: f(0) = 0,\ f' \in L^2(0,1)\},$$

equipped with inner product

$$\langle f,g \rangle_{\mathcal{H}_B} = \int_0^1 f'(t) g'(t) \, dt.$$

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

It can be observed that the geometry of $\mathcal{H}_B$ is entirely determined by derivatives. Unlike classical Hilbert spaces where the norm depends directly on function values, here the norm measures accumulate variation. Consequently, the evaluation of $f(x)$ must be reconstructed from the derivative information. The following result shows that the kernel sections encode this reconstruction exactly.

Lemma 1 (Reproducing property). For each fixed $x \in [0,1]$, the kernel section K_x is defined by

$$K_x(t) = K(t,x) = \min\{t,x\}.$$

Then $K_x \in \mathcal{H}_B$ and

$$\langle f, K_x\rangle_{\mathcal{H}_B} = f(x), \qquad \forall f \in \mathcal{H}_B.$$

Proof. Fix $x \in [0,1]$. Since

$$K_x(t) = \begin{cases} t, & 0 \le t \le x \\ x, & x < t \le 1 \end{cases},$$

its derivative exists almost everywhere and is given by

$$K_x'(t) = \mathbf{1}_{[0,x]}(t) \ \ \text{a.e. on } (0,1).$$

Since $\mathbf{1}_{[0,x]} \in L^2(0,1)$,

$$\int_0^1 |K_x'(t)|^2 \ dt = \int_0^1 \mathbf{1}_{[0,x]}(t)\, dt = x < \infty,$$

and clearly $K_x(0) = 0$. Therefore, $K_x \in \mathcal{H}_B$.

Now let $f \in \mathcal{H}_B$. By the definition of the inner product, we have

$$\langle f, K_x\rangle_{\mathcal{H}_B} = \int_0^1 f'(t) K_x'(t)\, dt.$$

Substituting the expression for K_x' gives

$$\langle f, K_x\rangle_{\mathcal{H}_B} = \int_0^1 f'(t)\mathbf{1}_{[0,x]}(t)\, dt = \int_0^x f'(t)\, dt.$$

Since the functions in $\mathcal{H}_B$ are absolutely continuous and $f(0) = 0$, we have

$$f(x) - f(0) = \int_0^x f'(t)\, dt \Longrightarrow f(x) = \int_0^x f'(t)\, dt.$$

Hence

$$\langle f, K_x\rangle_{\mathcal{H}_B} = f(x),$$

which proves the reproducing property.

The preceding lemma has an important geometric interpretation. Point evaluation is not merely an external operation on the function space; rather, it is represented by an inner product with an element of the Hilbert space. Consequently, pointwise estimates can be interpreted as geometric estimates involving kernel sections and orthogonal projections.

Proposition 1 (Boundedness of point evaluation). For each fixed $x \in [0,1]$, define the evaluation functional:

$$E_x : \mathcal{H}_B \to \mathbb{R},\ E_x(f) = f(x).$$

Then E_x is linear and bounded. Moreover, its operator norm satisfies

$$\| E_x \| = \sqrt{x}.$$

Proof. Using the reproducing identity, we can write

$$| f(x) | = | \langle f, K_x\rangle_{\mathcal{H}_B} |.$$

Applying the Cauchy–Schwarz inequality, we obtain

$$| f(x) | \le \| f \|_{\mathcal{H}_B} \| K_x \|_{\mathcal{H}_B}.$$

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

Since

$$\| K_x \|_{\mathcal{H}_B}^2 = \int_0^1 | K_x'(t) |^2 \, dt = x,$$

it follows that

$$| f(x) | \leq \sqrt{x} \ \| f \|_{\mathcal{H}_B},$$

which implies

$$\| E_x \| \leq \sqrt{x}.$$

Taking $f = K_x$ yields equality, and hence

$$\| E_x \| = \sqrt{x}.$$

This completes the proof.

The preceding proposition already suggests the sharp inequality to come. However, a stronger statement is possible. Rather than obtaining only an upper bound, one may decompose the full energy exactly into a point-evaluation component and an orthogonal remainder.

Lemma 2 (Orthogonal decomposition identity). For every $f \in \mathcal{H}_B$ and every $x \in (0,1]$,

$$\| f \|_{\mathcal{H}_B}^2 = \frac{| f(x) |^2}{x} + \int_0^1 | f'(t) - \frac{f(x)}{x} \mathbf{1}_{[0,x]}(t) |^2 \, dt.$$

Proof. From Lemma 1, we have

$$f(x) = \int_0^1 f'(t) \mathbf{1}_{[0,x]}(t) \, dt.$$

Let $e_x = \mathbf{1}_{[0,x]} \in L^2(0,1)$. Then $\| e_x \|_{L^2}^2 = x$. Since $f(x) = \langle f', e_x \rangle_{L^2}$, the orthogonal projection of f' onto the one-dimensional subspace generated by e_x in $L^2(0,1)$ is given by

$$P_x f' = \frac{\langle f', e_x \rangle}{\| e_x \|_{L^2}^2} e_x = \frac{f(x)}{x} \mathbf{1}_{[0,x]}.$$

By the property of orthogonal projections, we can decompose f' into parallel and orthogonal components: $f' = P_x f' + (f' - P_x f')$. Applying the Pythagorean theorem in $L^2(0,1)$ yields

$$\| f' \|_{L^2}^2 = \| P_x f' \|_{L^2}^2 + \| f' - P_x f' \|_{L^2}^2.$$

Computing the norm of the projection component, we get

$$\| P_x f' \|_{L^2}^2 = \left(\frac{f(x)}{x} \right)^2 \int_0^1 \mathbf{1}_{[0,x]}(t) dt = \frac{| f(x) |^2}{x^2} \cdot x = \frac{| f(x) |^2}{x}.$$

Since

$$\| f \|_{\mathcal{H}_B} = \| f' \|_{L^2},$$

substituting this value back into the Pythagorean relation establishes the identity.

The decomposition above has an intuitive meaning. The quantity $\frac{|f(x)|^2}{x}$ represents the minimum derivative energy required to attain the prescribed value $f(x)$ at the point x while the remaining integral measures the energy orthogonal to the evaluation direction. Thus, every function in the Brownian RKHS admits a unique decomposition into a kernel component accounting for the prescribed point evaluation and an orthogonal remainder.

This geometric interpretation is precisely what makes the sharp Bernstein-type inequality immediate in the next section.

MAIN RESULTS

The preliminary results developed in the previous section reveal that the Brownian reproducing kernel Hilbert space possesses a remarkably transparent geometric structure. In

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

particular, Lemma 1 shows that evaluation at a point is represented through the reproducing kernel, Proposition 1 establishes that point evaluation is a bounded functional whose norm is explicitly computable, and Lemma 2 provides an exact orthogonal decomposition of the Hilbert norm. These observations now allow us to derive the main inequalities in this paper.

The first theorem establishes a sharp Bernstein-type estimate that controls pointwise values of functions entirely through their Hilbert-space energy.

Theorem 1 (Sharp Bernstein-type inequality). For every $f \in \mathcal{H}_B$ and every fixed $x \in [0,1]$,

$$|f(x)|^2 \leq x \, \| f \|^2_{\mathcal{H}_B}.$$

Moreover, the coefficient x is optimal for each fixed $x \in [0,1]$. Specifically, for $x \in (0,1]$, the coefficient x cannot be replaced by any smaller value while for $x = 0$, the optimal constant is identically 0.

Proof. We distinguish two cases.

Case 1: Let $x = 0$. By definition of the space $\mathcal{H}_B$, every function satisfies $f(0) = 0$. Thus, the inequality becomes

$$|f(0)|^2 = 0 \leq 0 \cdot \| f \|^2_{\mathcal{H}_B},$$

which holds trivially. Moreover, since $f(0) = 0$ for every $f \in \mathcal{H}_B$, the inequality

$$|f(0)|^2 \leq C \, \| f \|^2_{\mathcal{H}_B}$$

holds for all $f \in \mathcal{H}_B$ if and only if $C \geq 0$. Thus, the smallest admissible constant is $C = 0$. Hence the coefficient $x = 0$ is optimal.

Case 2: Assume now that $x \in (0,1]$. From Lemma 2, we obtain the orthogonal decomposition

$$\| f \|^2_{\mathcal{H}_B} = \frac{|f(x)|^2}{x} + \int_0^1 \left| f'(t) - \frac{f(x)}{x} \mathbf{1}_{[0,x]}(t) \right|^2 dt.$$

The second term is non-negative because it is L^2-norm squared. Therefore,

$$\| f \|^2_{\mathcal{H}_B} \geq \frac{|f(x)|^2}{x}.$$

Multiplying both sides by x gives

$$|f(x)|^2 \leq x \, \| f \|^2_{\mathcal{H}_B},$$

which proves the inequality.

To prove optimality for $x \in (0,1]$, consider the kernel section introduced in Lemma 1, namely $f = K_x$. By the reproducing property,

$$K_x(x) = K(x,x) = x.$$

Moreover, also by the reproducing property,

$$\| K_x \|^2_{\mathcal{H}_B} = \langle K_x, K_x \rangle_{\mathcal{H}_B} = K_x(x) = K(x,x) = x.$$

Substituting this into the inequality gives

$$|K_x(x)|^2 = x^2 = x \, \| K_x \|^2_{\mathcal{H}_B}.$$

Thus, equality is attained, meaning no constant smaller than x can satisfy the inequality for all functions when $x \in (0,1]$.

The significance of Theorem 1 extends beyond the inequality itself. It identifies exactly how much Hilbert-space energy is required to produce a prescribed value at a given point. The

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

coefficient x is observed to depend on the location. At the origin where $x = 0$, every function in $\mathcal{H}_B$ vanishes by definition and hence the optimal evaluation constant is zero. As x approaches the left endpoint, the minimum energy required to attain a prescribed value at x decreases whereas evaluation near $x = 1$ requires comparatively more energy.

This interpretation follows directly from Proposition 1, where the norm of the evaluation operator is shown to satisfy

$$\| E_x \| = \sqrt{x}.$$

Consequently, Theorem 1 can be rewritten as

$$| E_x(f) | \leq \| E_x \| \| f \|_{\mathcal{H}_B},$$

meaning that the Bernstein-type inequality is exactly the Cauchy–Schwarz inequality applied to evaluation functionals.

Remark (Terminology and relation to classical Bernstein inequalities). It is useful to clarify the terminology used here. Classical Bernstein inequalities typically provide upper bounds on derivatives in terms of global function norms. In the present RKHS setting, Theorem 1 provides a sharp pointwise estimate that bounds the value $(| f(x) |)$ in terms of the global Hilbert-space energy $\left(\| f' \|_{L^2(0,1)}\right)$. We therefore refer to Theorem 1 as a Bernstein-type inequality. The derivative structure is incorporated into the Cameron–Martin norm, so the estimate relates pointwise function values to the total derivative energy rather than directly bounding derivatives by function values.

Corollary 1 (Sharp L^∞-embedding and extremisers). Every $f \in \mathcal{H}_B$ satisfies

$$\| f \|_{L^\infty(0,1)} \leq \| f \|_{\mathcal{H}_B}.$$

The constant 1 is optimal. Moreover, equality in the pointwise estimate

$$| f(x) |^2 \leq x \| f \|_{\mathcal{H}_B}^2$$

at a fixed point $x \in (0,1]$ occurs if and only if

$$f(t) = c\min\{t, x\}$$

for some $c \in \mathbb{R}$. In particular, equality in the global L^∞-estimate occurs if and only if

$$f(t) = ct$$

for some $c \in \mathbb{R}$.

Proof. The proof is derived directly from the embedding and pointwise estimates established in Theorem 1, together with the reproducing kernel properties from Lemma 2.

Step 1: Sharp embedding

By Theorem 1, every $f \in \mathcal{H}_B$ satisfies the continuous embedding

$$\| f \|_{L^\infty(0,1)} \leq \| f \|_{\mathcal{H}_B}$$

with an embedding constant of at most 1. To see that this constant is sharp, we test the linear function $f(t) = t$. A direct computation of its norms gives

$$\| f \|_{L^\infty(0,1)} = 1 \text{ and } \| f \|_{\mathcal{H}_B} = 1.$$

Since the ratio of these norms is exactly 1, the embedding constant cannot be improved.

Step 2: Pointwise extremisers

The pointwise inequality

$$| f(x) |^2 \leq x \| f \|_{\mathcal{H}_B}^2$$

is equivalent to the bound

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

$$| f(x) | \leq \sqrt{x} \parallel f \parallel_{\mathcal{H}_B}$$

given in Theorem 1. According to the Cauchy–Schwarz inequality applied to the reproducing property

$$f(x) = \langle f, K_x \rangle_{\mathcal{H}_B}$$

from Lemma 2, equality holds if and only if f and K_x are linearly dependent. Since Lemma 2 identifies the kernel as

$$K_x(t) = \min\{t, x\},$$

the pointwise extremisers at a fixed point $x \in (0,1]$ are precisely of the form

$$f(t) = c \min\{t, x\}, \qquad c \in \mathbb{R}.$$

Step 3: Global extremisers

For a non-zero function to achieve equality in the global estimate

$$\parallel f \parallel_{L^\infty(0,1)} = \parallel f \parallel_{\mathcal{H}_B},$$

there must exist some point $x \in [0, 1]$ such that

$$| f(x) | = \parallel f \parallel_{\mathcal{H}_B}.$$

From the chain of inequalities provided by Theorem 1, we have

$$\parallel f \parallel_{\mathcal{H}_B} = | f(x) | \leq \sqrt{x} \parallel f \parallel_{\mathcal{H}_B} \leq \parallel f \parallel_{\mathcal{H}_B},$$

and the factor $\sqrt{x}$ must equal 1 at that point. Since $x \in [0, 1]$, this implies $x = 1$. Thus, the global norm can only be attained at the endpoint $x = 1$. Evaluating the pointwise extremiser condition from Step 2 at $x = 1$ yields

$$f(t) = c \min\{t, 1\} = ct.$$

Conversely, a direct computation of the norms shows that every function of the form $f(t) = ct$ satisfies

$$\parallel f \parallel_{L^\infty(0,1)} = \parallel f \parallel_{\mathcal{H}_B} = |c|.$$

Therefore, the embedding constant 1 is sharp and the global extremisers are precisely the functions of the form

$$f(t) = ct, \qquad c \in \mathbb{R}.$$

The previous result completely characterises local extremisers. We now turn to a global question. Instead of controlling a single point value, we seek the strongest possible estimate for the entire L^2-mass of functions in the Brownian RKHS.

Theorem 2 (Sharp global L^2-embedding). For every function $f \in \mathcal{H}_B$, the inequality

$$\int_0^1 | f(x) |^2 \, dx \leq \frac{4}{\pi^2} \parallel f \parallel_{\mathcal{H}_B}^2$$

holds. The constant $\frac{4}{\pi^2}$ is optimal. Equality holds if and only if

$$f(x) = c \sin\left(\frac{\pi x}{2}\right)$$

for some constant $c \in \mathbb{R}$.

Proof. Unlike Theorem 1, which is based on point evaluation, Theorem 2 concerns the global L^2-magnitude of functions in the Cameron–Martin space $\mathcal{H}_B$ relative to their derivative energy. The optimal constant is determined by the Rayleigh quotient

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

$$\lambda_1 = \inf_{\substack{f \in \mathcal{H}_B \\ f \neq 0}} \frac{\int_0^1 | f'(x) |^2 \ dx}{\int_0^1 | f(x) |^2 \ dx}.$$

Since the quotient is homogeneous, we may restrict our attention to the set

$$\mathcal{A} = \left\{ f \in \mathcal{H}_B : \int_0^1 | f(x) |^2 \ dx = 1 \right\}.$$

Let $\{f_n\} \subset \mathcal{A}$ be a minimising sequence. Then

$$\int_0^1 | f_n'(x) |^2 \ dx \longrightarrow \lambda_1.$$

Consequently, the sequence $\{f_n\}$ is bounded in $H^1(0,1)$. Indeed, since

$$\| f_n \|_{L^2(0,1)} = 1,$$

we have

$$\| f_n \|_{H^1(0,1)}^2 = \| f_n \|_{L^2(0,1)}^2 + \| f_n' \|_{L^2(0,1)}^2 = 1 + \| f_n' \|_{L^2(0,1)}^2.$$

Therefore, after passing to a subsequence, there exists $u \in H^1(0,1)$ such that

$$f_n \rightharpoonup u \text{ weakly in } H^1(0,1).$$

By the Rellich–Kondrachov compactness theorem,

$$f_n \longrightarrow u \text{ strongly in } L^2(0,1).$$

Since $f_n(0) = 0$ for every n, the continuity of the trace operator implies

$$u(0) = 0.$$

Hence $u \in \mathcal{H}_B$. Moreover, the strong convergence in $L^2(0,1)$ gives

$$\int_0^1 | u(x) |^2 \ dx = 1.$$

On the other hand, by the weak lower semi-continuity of the L^2-norm,

$$\int_0^1 | u'(x) |^2 \ dx \leq \liminf_{n \to \infty} \int_0^1 | f_n'(x) |^2 \ dx = \lambda_1.$$

Thus, u attains the infimum and

$$\lambda_1 = \int_0^1 | u'(x) |^2 \ dx$$

because $\| u \|_{L^2(0,1)} = 1$.

We now derive the Euler–Lagrange equation satisfied by the minimiser u. For every $v \in \mathcal{H}_B$, the first variation of the Rayleigh quotient yields

$$\int_0^1 u'(x) v'(x) \, dx = \lambda_1 \int_0^1 u(x) v(x) \, dx.$$

Thus, u is a weak solution of

$$-u''(x) = \lambda_1 u(x), \qquad 0 < x < 1.$$

By the one-dimensional regularity of weak solutions, $u \in H^2(0,1)$ and hence integration by parts gives

$$\int_0^1 u'(x) v'(x) \, dx = u'(1)v(1) - u'(0)v(0) - \int_0^1 u''(x) v(x) \, dx.$$

Every $v \in \mathcal{H}_B$ satisfies $v(0) = 0$, so the term at $x = 0$ vanishes. However, the value $v(1)$ is unrestricted because the endpoint $x = 1$ is free. Therefore,

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

$$\int_0^1 u'(x)v'(x)\,dx = u'(1)v(1) - \int_0^1 u''(x)v(x)\,dx.$$

Comparing this identity with the Euler–Lagrange equation and using the arbitrariness of $v(1)$, we obtain the natural boundary condition

$$u'(1) = 0.$$

Consequently, the minimiser satisfies the boundary value problem

$$\begin{cases} -u''(x) = \lambda_1 u(x), & 0 < x < 1, \\ u(0) = 0, \\ u'(1) = 0. \end{cases}$$

Since $u \not\equiv 0$, we have $\lambda_1 > 0$. Let

$$\mu = \sqrt{\lambda_1}.$$

Then the general solution of

$$-u'' = \lambda_1 u$$

is

$$u(x) = A\sin(\mu x) + B\cos(\mu x).$$

The condition $u(0) = 0$ implies $B = 0$ and hence

$$u(x) = A\sin(\mu x).$$

Differentiating, we obtain

$$u'(x) = A\mu\cos(\mu x).$$

The natural boundary condition $u'(1) = 0$ therefore gives

$$A\mu\cos(\mu) = 0.$$

Since $u \not\equiv 0$, we have $A \neq 0$ and hence

$$\cos(\mu) = 0.$$

Therefore,

$$\mu = \frac{\pi}{2} + k\pi, \qquad k = 0,1,2,\ldots.$$

The smallest positive eigenvalue is thus

$$\lambda_1 = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}.$$

It follows that, for every $f \in \mathcal{H}_B$,

$$\frac{\int_0^1 |f'(x)|^2\,dx}{\int_0^1 |f(x)|^2\,dx} \geq \frac{\pi^2}{4}.$$

Equivalently,

$$\int_0^1 |f(x)|^2\,dx \leq \frac{4}{\pi^2}\int_0^1 |f'(x)|^2\,dx = \frac{4}{\pi^2}\|f\|_{\mathcal{H}_B}^2.$$

Hence $4/\pi^2$ is the optimal constant. Finally, equality holds if and only if f belongs to the eigenspace corresponding to the first eigenvalue $\lambda_1 = \pi^2/4$. Therefore,

$$f(x) = c\sin\left(\frac{\pi x}{2}\right), \qquad c \in \mathbb{R}.$$

Conversely, these functions attain equality. This completes the proof.

Remark. It is worth distinguishing between the optimal constant appearing in the squared integral inequality and the norm of the corresponding embedding operator. Theorem 2 establishes that the

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

optimal constant for the quadratic inequality is $\frac{4}{\pi^2}$. Consequently, the operator norm of the continuous global embedding $T: \mathcal{H}_B \to L^2(0,1)$ defined by $Tf = f$ is given precisely by

$$\| T \|_{\mathcal{H}_B \to L^2(0,1)} = \sup_{f \in \mathcal{H}_B, f \neq 0} \frac{\| f \|_{L^2}}{\| f \|_{\mathcal{H}_B}} = \frac{2}{\pi}.$$

While the spectral optimisation problem directly yields the sharp constant $\frac{4}{\pi^2}$ for the energy functional distribution, the operator norm of the embedding operator is its square root, $\frac{2}{\pi}$.

Theorems 1 and 2 together provide complementary descriptions of the Brownian RKHS. The first result controls local behaviour through point evaluation while the second governs global energy distribution through spectral optimisation. Together they describe the exact geometry of the embedding of the Brownian reproducing kernel Hilbert space into classical function spaces.

APPLICATION AND INTERPRETATION OF MAIN RESULTS

The inequalities established in the previous section provide more than isolated estimates. Together, Theorem 1 and Theorem 2 describe how functions in the Brownian reproducing kernel Hilbert space distribute their energy both locally and globally. The local inequality explains how large a function may become at an individual point while the global embedding result determines the maximal total mass of the function over the entire interval. These results lead to several applications in reproducing kernel analysis, functional inequalities and Gaussian process theory.

Local Pointwise Control and Stability of Evaluation

The first immediate consequence of Theorem 1 is a precise quantitative estimate for evaluating functions at individual points. Recalling that for every $f \in \mathcal{H}_B$, Theorem 1 gives

$$| f(x) |^2 \leq x \| f \|_{\mathcal{H}_B}^2, \qquad x \in [0,1].$$

Equivalently,

$$| f(x) | \leq \sqrt{x} \; \| f \|_{\mathcal{H}_B}.$$

This inequality shows that pointwise values cannot grow independently of the Hilbert-space energy. In particular, if two functions $f, g \in \mathcal{H}_B$ are close in the RKHS norm, then their values at every point must also remain close:

$$| f(x) - g(x) | \leq \sqrt{x} \; \| f - g \|_{\mathcal{H}_B}.$$

Thus, evaluation is not only continuous but uniformly stable. This observation is especially important because functions in $\mathcal{H}_B$ are characterised by their derivative energy rather than by direct pointwise constraints. The theorem guarantees that the finite derivative energy automatically prevents uncontrolled oscillation. Moreover, the estimate becomes stronger near the left endpoint:

$$| f(x) | \leq \sqrt{x} \; \| f \|_{\mathcal{H}_B} \to 0 \quad \text{as } x \to 0^+.$$

This behaviour reflects the boundary condition $f(0) = 0$, which is built into the structure of the Brownian RKHSs. Therefore, Theorem 1 provides an exact quantitative version of continuity at the origin.

Continuous Embedding into $L^\infty(0,1)$

A second consequence of Theorem 1 concerns the embedding of the Brownian RKHS into spaces of bounded functions. Since $0 \leq x \leq 1$, Theorem 1 immediately implies

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

$$|f(x)| \le \| f \|_{\mathcal{H}_B}$$

for every point x. Taking the supremum gives

$$\| f \|_{L^\infty(0,1)} \le \| f \|_{\mathcal{H}_B}.$$

Thus, the inclusion embedding $\mathcal{H}_B \hookrightarrow L^\infty(0,1)$is continuous. This result has an important interpretation: The norm in $\mathcal{H}_B$ measures only derivative energy,

$$\| f \|_{\mathcal{H}_B}^2 = \int_0^1 | f'(t) |^2 \, dt,$$

yet Theorem 1 shows that the derivative information alone determines an absolute bound on function values. Consequently, no sequence satisfying

$$\sup_n \| f_n \|_{\mathcal{H}_B} < \infty$$

can exhibit arbitrarily large spikes. This property is useful in approximation theory and variational problems because it guarantees compactness-like behaviour under norm control.

Global Energy Distribution and Sharp L^2-Control

While Theorem 1 describes local behaviour, Theorem 2 provides information at the opposite scale. The theorem states that

$$\int_0^1 | f(x) |^2 \, dx \le \frac{4}{\pi^2} \| f \|_{\mathcal{H}_B}^2.$$

This inequality determines the exact operator norm of the embedding $\mathcal{H}_B \hookrightarrow L^2(0,1)$. The quantity

$$\int_0^1 | f(x) |^2 \, dx$$

measures the total function energy over the interval. Thus, Theorem 2 implies that the total energy accumulation can never exceed $\frac{4}{\pi^2}$ times the derivative energy.

This result perfectly complements Theorem 1, which controls the value at a single point, whereas Theorem 2 controls the global L^2-mass over the interval. Together, they provide complete local-to-global norm estimates. An immediate consequence is

$$\| f \|_{L^2} \le \frac{2}{\pi} \| f \|_{\mathcal{H}_B}.$$

Therefore, every bounded sequence in the Brownian RKHS remains bounded in $L^2(0,1)$. This observation is crucial in optimisation problems and spectral approximations because convergence in the RKHS norm automatically implies convergence in the L^2 sense.

Extremisers and Optimal Energy Concentration

One of the most informative aspects of sharp inequalities is the explicit description of the equality cases. Theorem 1 shows that local equality occurs precisely for functions of the form

$$f(t) = c\min\{t, x\}.$$

These functions represent local extremisers. Geometrically, these local extremisers concentrate all available energy on producing the largest possible value at a prescribed point x. Their derivative is given by

$$f'(t) = c\mathbf{1}_{[0,x]}(t),$$

Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

which means that the energy is concentrated precisely where it is needed to achieve the prescribed point value. By contrast, Theorem 2 shows that equality in the global estimate occurs for the sine profile,

$$f(x) = c\sin\left(\frac{\pi x}{2}\right).$$

This function distributes energy smoothly across the entire interval. Unlike local extremisers, which maximise pointwise growth, the sine profile attains the maximal L^2-mass relative to the prescribed derivative energy. Therefore, the two extremal families reveal two distinct optimisation principles:

- Local optimization, i.e. $f(t) = c\min\{t, x\}$, maximises evaluation at a single point.
- Global optimization, i.e. $f(x) = c\sin\left(\frac{\pi x}{2}\right)$, maximises average energy distribution.

This clear distinction nicely illustrates the deep interaction between reproducing kernels and spectral structures.

Interpretation in Context of Brownian Motion

Since $\mathcal{H}_B$ is the Cameron–Martin space associated with Brownian motion, these inequalities also admit a beautiful probabilistic interpretation. Functions in the Cameron–Martin space represent deterministic finite-energy drift directions associated with Brownian motion. Theorem 1 quantifies how much local displacement can occur under finite energy constraints while Theorem 2 measures the maximal average displacement over time. From this perspective, the constants $\sqrt{x}$ and $\frac{2}{\pi}$ represent the exact constants governing local and global behaviour respectively. Accordingly, the Brownian RKHS exhibits a precise balance between pointwise growth and global regularity, and the sharp inequalities obtained in this work characterise that balance completely.

CONCLUSIONS

In this paper I investigate sharp Bernstein-type inequalities in the reproducing kernel Hilbert space associated with the Brownian motion covariance kernel. Using the reproducing property and an orthogonal decomposition identity, I establish a sharp pointwise inequality and determine the optimal constant for each fixed evaluation point. I identify the corresponding extremisers explicitly as scaled kernel sections. In addition, I establish a sharp global embedding inequality from the Cameron–Martin space into $L^2(0, 1)$ using variational and spectral methods. I obtain the optimal constant from the first eigenvalue of the associated Sturm–Liouville problem and characterise the corresponding extremisers explicitly. Together, these results provide a local-to-global description of the geometry of the Brownian RKHS and illustrate the natural connection between reproducing kernel methods, orthogonal projections, variational analysis and spectral theory.

The techniques developed in this paper may also be useful for studying sharp inequalities in the more general reproducing kernel Hilbert spaces. Possible directions for future research include Bernstein-type inequalities for fractional Brownian motion and other Gaussian covariance kernels, multidimensional and Sobolev-type RKHSs generated by elliptic operators, and extensions of the orthogonal decomposition approach to higher-order evaluations, derivatives and non-linear functionals. These problems may provide further connections between reproducing kernel theory, functional inequalities and stochastic analysis.

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Maejo Int. J. Sci. Technol. **2026**, *20*(03), 269-283

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