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***Full Paper***

# Convergence analysis of Picard's three-step iteration for Suzuki-type non-expansive mappings in Banach spaces

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**Abstract:** We investigate the weak and strong convergence behaviours of Picard's three-step iteration method applied to Suzuki generalised non-expansive mappings in the context of uniformly convex Banach spaces. By employing essential tools such as the Opial condition, the demi-closedness principle, and asymptotic centre theory, we establish several convergence theorems. Our results demonstrate that under appropriate assumptions, the iteration sequence converges strongly and weakly to fixed points of Suzuki generalised non-expansive operators. The proposed iteration method enhances the approximation process in comparison to classical methods, offering improved convergence performance. This study extends the fixed point theory by proposing a flexible and efficient framework suitable for solving non-linear problems in Banach spaces. In addition, numerical experiments and theoretical analysis are presented to demonstrate the efficiency and robustness of the proposed method. The results reveal that the method outperforms several well-known iteration methods in terms of convergence speed and accuracy.

**Keywords:** fixed point, Picard's three-step iteration method, Suzuki-type mappings, strong convergence, weak convergence

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## INTRODUCTION

Fixed point theory is one of the fundamental areas of non-linear analysis and has numerous applications in optimisation, differential equations, control theory, neural networks, dynamical systems and engineering. The origins of iteration approximation methods can be traced back to the pioneering work of Picard [1] while Banach's contraction principle [2] provides the theoretical foundation for modern fixed point theory. Subsequently, several classical iteration methods including the Mann [3], Krasnosel'skii [4], Ishikawa [5] and Noor [6] iteration methods, were introduced and extensively studied for approximating fixed points of non-linear operators.

To overcome the limitations of classical one-step methods and improve convergence efficiency, various multistep, hybrid and accelerated iteration methods have been proposed. Important developments include the three-step iteration methods of Thakur et al. [7] and Karakaya et al. [8], M-type iteration methods [9-11], S-iteration method [12], iteration methods for non-linear integral equations [13] and multistep approximation frameworks [14, 15]. Hybrid approaches such as the Picard–Mann iteration method [16], Kirk-SP iteration method [17] and Picard-type hybrid iteration methods [18] have further expanded the applicability of iteration approximation techniques and improved their convergence behaviour.

In addition to the development of iteration methods, considerable attention has been devoted to the convergence analysis of non-expansive and generalised non-expansive mappings in Banach spaces. Classical convergence results for non-expansive mappings were established by Opial [19], Senter and Dotson [20], Khan et al. [21] and Suzuki [22] while comprehensive treatments can be found in the monograph of Agarwal et al. [23]. More recently, the literature has focused on accelerated and hybrid iteration methods for generalised non-expansive and $\alpha-$non-expansive mappings. Notable contributions include the Picard-S hybrid iteration method [24], iteration methods for G-non-expansive mappings [25], Picard–Thakur hybrid iteration methods [26], generalised $(\alpha,\beta)-$non-expansive iteration methods [27], SM-iteration method [28] and the recently proposed Picard-SP and Picard-CR iteration methods [29, 30]. These developments have significantly enriched the theory of iteration approximation and provided more efficient tools for the study of fixed points of generalised non-linear mappings.

In recent years Suzuki-type generalised non-expansive mappings have attracted considerable attention due to their broader applicability and rich fixed point structure. Various convergence results and iteration approximation methods have been developed for this class of mappings. These include the pioneering iteration method of Thakur et al. [31], three-step iteration methods with visualisation techniques [32], Suzuki-type generalised contractions [33], F-Kannan–Suzuki-type mappings [34], multivalued Suzuki generalised non-expansive mappings [35] and convergence theorems for Suzuki generalised non-expansive mappings in Banach and metric spaces [36–38]. More recently, several accelerated and hybrid iteration methods have been proposed for Suzuki-type and Reich–Suzuki-type non-expansive mappings including Picard–Abbas iteration method [39], faster iteration approximation method [40], new iteration methods for Reich–Suzuki [41–44] and iteration methods for enriched Suzuki non-expansive mappings [45–47]. These studies have substantially advanced the convergence theory of Suzuki-type mappings and provided effective iteration frameworks for fixed point approximation in non-linear settings.

Motivated by these observations, this paper investigates the weak and strong convergence properties of Picard's three-step iteration method for Suzuki generalised non-expansive mappings in uniformly convex Banach spaces. By employing asymptotic centre theory, the Opial condition and the demi-closedness principle, we establish several new convergence theorems under suitable assumptions. The obtained results extend and complement existing studies on Suzuki-type mappings, Picard-type iteration methods and accelerated approximation methods. In addition, numerical experiments are presented to compare the proposed method with several classical iteration methods including Picard, Mann, Ishikawa, Noor, M, M* and K iteration methods. An application to a non-linear differential equation is also provided to demonstrate the practical applicability and effectiveness of the proposed approach.

**PRELIMINARIES**

This section introduces the fundamental definitions and concepts that form the basis for the results presented in the subsequent sections.

**Definition 1** [13]**.** Let $E$ be a metric or Banach space and let $T: E \to E$ be a mapping. For an initial element $u_1 \in E$, which is the initial point of the iteration, the sequence $\{u_n\}$ is generated by the following recurrence relations:

$$\begin{cases} u_{n+1} = Tv_n \\ v_n = Tw_n \\ w_n = Tu_n \,, \ n \in \mathbb{Z}_+ \,. \end{cases} \tag{1}$$

The method in (1) is called Picard's three-step iteration method or, in short, Picard3.

**Definition 2 (Generalised Suzuki non-expansive mapping)** [22]**.** Let $E$ be a Banach space, $D$ be a non-empty subset of $E$ and $T: D \to D$ be a mapping. The mapping $T$ is said to satisfy Suzuki's condition (C) if, for all $u, v \in D$,

$$\frac{1}{2}\|u - Tu\| \leq \|u - v\| \Longrightarrow \|Tu - Tv\| \leq \|u - v\|.$$

This condition, introduced by Suzuki, is a generalisation of non-expansiveness and is weaker than the classical non-expansive condition. A mapping satisfying condition (C) is called a Suzuki generalised non-expansive mapping. Every non-expansive mapping satisfies condition (C), although the converse does not necessarily hold.

**Definition 3 (Demi-closed property)** [21]**.** Let $D$ be a non-empty subset of Banach space $E$ and $T: D \to D$ be a mapping. If $Tu = v$ when the sequence $\{u_n\}$ in $D$ weakly converges to the point $u$ and the sequence $\{Tu_n\}$ strongly converges to the point $v$ for $\forall u \in D$ and $v \in E$, then the mapping $T$ is called demi-closed at the point $v$.

**Definition 4 (Opial condition)** [19]**.** A Banach space $E$ is said to satisfy the Opial condition if, for every sequence $\{u_n\} \subset E$ that converges weakly to $u$, the inequality

$$\limsup_{n\to\infty}\|u_n - u\| < \limsup_{n\to\infty}\|u_n - v\|$$

holds for every $v \in E$ such that $v \neq u$.

**Definition 5 (Property (I))** [20]**.** Let $D$ be a non-empty convex subset of a Banach space $E$ and let $T: D \to D$ be a mapping such that $F(T) \neq \emptyset$, where $F(T)$ denotes the set of fixed points of $T$. Define $d\big(u, F(T)\big) = inf\{d(u, q) : \ q \in F(T)\}$. A mapping $T$ is said to satisfy property (I) if there exists a non-decreasing function $f: [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, such that the inequality

$$\|u - Tu\| \geq f\big(d(u, F(T))\big)$$

holds for all $u \in D$.

**Definition 6 (Asymptotic radius and asymptotic centre)** [23]**.** Let $D$ be a non-empty, closed and convex subset of a Banach space $E$ and let $\{u_n\}$ be a bounded sequence in $E$. For $u \in D$, consider the function $R(\cdot, \{u_n\}) = \limsup_{n\to\infty}\|u_n - u\|$, which defines a mapping $R(\cdot, \{u_n\}): E \to \mathbb{R}_+$.

The infimum of the function $R(\cdot,\{u_n\})$ over the set $D$ is called the asymptotic radius of the sequence $\{u_n\}$ with respect to $D$ and is denoted by $R(D,\{u_n\})$, i.e.

$$R(D,\{u_n\}) = inf\{R(u,\{u_n\}) : u \in D\}.$$

If $t \in D$ satisfies

$$R(t,\{u_n\}) = inf\{R(u,\{u_n\}) : u \in D\} = R(D,\{u_n\}),$$

then the point $t \in D$ is called an asymptotic centre of the sequence $\{u_n\}$ with respect to $D$. The set of all asymptotic centres of $\{u_n\}$ with respect to $D$ is denoted by $M(D,\{u_n\})$, i.e.

$$M(D,\{u_n\}) = \{t \in D : \ R(t,\{u_n\}) = R(D,\{u_n\})\}.$$

**Remark 1** [23]**.** In general the set $M(D,\{u_n\})$ may be empty, singleton or contain multiple elements. In the theorems that follow, it is shown that in a uniformly convex Banach space, the set $M(D,\{u_n\})$ is a singleton.

**Theorem 1** [23]**.** Let $E$ be a uniformly convex Banach space, $D$ be a non-empty closed convex subset of $E$ and $\{u_n\} \subset E$ be a bounded sequence. Then $M(D,\{u_n\})$ is a single-element set.

**Theorem 2** [23]**.** Let $E$ be a uniformly convex Banach space and $D$ be a non-empty closed convex subset of $E$. Every bounded sequence $\{u_n\}$ in $E$ has a unique asymptotic centre with respect to $D$. That is, $M(D,\{u_n\}) = \{t\}$, which implies that for all $u \in D$ with $u \neq t$, the following inequality holds:

$$\limsup_{n\to\infty}\|u_n - t\| < \limsup_{n\to\infty}\|u_n - u\|.$$

**Proposition 1** [22]**.** Let $D$ be a non-empty subset of Banach space $E$ and $T: D \to D$ be a mapping.

i) If $T$ is a non-expansive mapping, the condition (C) is satisfied for the mapping $T$;

ii) Every mapping that has a fixed point and satisfies condition (C) is a quasi non-expansive mapping;

iii) If condition (C) is satisfied for $T$ mapping, then the inequality

$$\|u - Tv\| \leq 3\|Tu - u\| + \|u - v\|$$

holds for every $u, v \in D$.

**Lemma 1** [22]**.** Let $D$ be a weakly compact and convex subset of a uniformly convex Banach space and let $T: D \to D$ be a mapping satisfying condition (C). Then the iteration sequence $\{u_n\}$ converges to a fixed point of $T$.

**Lemma 2** [22]**.** Let $T: D \to D$ be a mapping on a non-empty subset $D$ of a Banach space $E$ such that:

i) $E$ satisfies the Opial condition,

ii) $T$ also satisfies condition (C),

iii) The sequence $\{u_n\} \subset D$ converges weakly to $t \in D$ and $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$.

Then it follows that $Tt = t$, i.e. $I - T$ is demi-closed at zero.

## MAIN RESULTS

In this section we present the principal convergence results for Picard3. The proofs utilise structural properties of uniformly convex Banach spaces and assumptions on the involved mappings.

**Lemma 3.** Let $D$ be a non-empty, closed convex subset of a uniformly convex Banach space $E$ and suppose that $T: D \to D$ is a Suzuki generalised non-expansive mapping with a non-empty fixed point set $F(T)$. If the sequence $\{u_n\}$ is generated by Picard3, then the sequence $\{\|u_n - q\|\}$ is convergent for all $q \in F(T)$.

*Proof.* Let $q \in F(T)$, so $Tq = q$ and $u \in D$. Since $T$ satisfies condition (C), it follows from Proposition 1 (ii) that $T$ is quasi-non-expansive. Applying the iteration rule and using the properties of $T$, we obtain:

$$\|Tu - q\| \leq \|u - q\|.$$

Using Picard3,

$$\|w_n - q\| = \|Tu_n - q\| \leq \|u_n - q\|, \tag{2}$$

and

$$\|v_n - q\| = \|Tw_n - q\| \leq \|w_n - q\|, \tag{3}$$

$$\|u_{n+1} - q\| = \|Tv_n - q\| \leq \|v_n - q\|. \tag{4}$$

By combining (2)–(4), we deduce that $\|u_{n+1} - q\| \leq \|u_n - q\|$ for all $n \in \mathbb{N}$. Hence the sequence $\{\|u_n - q\|\}$ is non-increasing. In addition, since norms are always non-negative, the sequence is bounded below by zero. It follows that it converges. Consequently the limit $\lim_{n\to\infty} \|u_n - q\|$ exists for every $q \in F(T)$.

**Lemma 4.** Let $D$ be a non-empty, closed, convex subset of a uniformly convex Banach space $E$ and let $T: D \to D$ be a Suzuki generalised non-expansive mapping. Let $\{u_n\}$ be a sequence defined by Picard3 and let $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$. Then for $F(T) \neq \emptyset$, the necessary and sufficient condition is that $\{u_n\}$ is bounded.

*Proof.* ($\Rightarrow$) Let $F(T) \neq \emptyset$ be assumed. By Lemma 3 $\lim_{n\to\infty} \|u_n - q\|$ exists and $\{u_n\}$ is bounded.

($\Leftarrow$) Now conversely, let $\{u_n\}$ be bounded, $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$ and $q \in M(D, \{u_n\})$. From Proposition 1 (iii) and the notion of asymptotic centre,

$$\begin{aligned} R(Tq, \{u_n\}) &= \limsup_{n\to\infty} \|u_n - Tq\| \\ &\leq \limsup_{n\to\infty} (3\|Tu_n - u_n\| + \|u_n - q\|) \\ &= \limsup_{n\to\infty} \|u_n - q\| \\ &= R(q, \{u_n\}) \\ &= R(D, \{u_n\}). \end{aligned}$$

Hence $Tq \in M(D, \{u_n\})$ is obtained. Since $E$ is uniformly convex, $M(D, \{u_n\})$ has a single element. Then it is

$$Tq = q.$$

**Theorem 3.** Let $D$ be a non-empty, closed, convex subset of a uniformly convex Banach space $E$. Let $T: D \to D$ be a Suzuki generalised non-expansive mapping with $F(T) \neq \emptyset$ and assume that the Opial condition holds. Then the iteration sequence $\{u_n\}$, constructed via Picard3, converges weakly to a point in $F(T)$.

*Proof.* Since $q \in F(T)$, by Lemma 3 there exists $\lim_{n\to\infty} \|u_n - q\|$. We shall prove that the sequence $\{u_n\}$ has at most one weak subsequential limit in $F(T)$. Let $\{u_{n_j}\}$ and $\{u_{n_k}\}$ be subsequences of $\{u_n\}$ and their weak limits be $u$ and $v$ respectively. From Lemma 4 we have $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$ and by Lemma 2 the operator $I - T$ is demi-closed at zero. This implies that $(I - T)u = 0$ and

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hence $u = Tu$. Similarly, it can be seen that $Tv = v$. Now let us show that this limit is unique. For this, let us assume $u \neq v$. From the Opial condition,

$$\begin{aligned}\lim_{n\to\infty}\|u_n - u\| &= \lim_{n_j\to\infty}\left\|u_{n_j} - u\right\| \\ &< \lim_{n_j\to\infty}\left\|u_{n_j} - v\right\| \\ &= \lim_{n\to\infty}\|u_n - v\| \\ &= \lim_{n_k\to\infty}\left\|u_{n_k} - v\right\| \\ &< \lim_{n_k\to\infty}\left\|u_{n_k} - u\right\| \\ &= \lim_{n\to\infty}\|u_n - u\|.\end{aligned}$$

This is a contradiction. In this case $u = v$. Therefore, it follows that the sequence $\{u_n\}$ converges weakly to some element of the fixed point set $F(T)$.

**Theorem 4.** Let $D$ be a non-empty, closed, convex subset of a uniformly convex Banach space $E$ and let $T: D \to D$ be a Suzuki generalised non-expansive operator with $F(T) \neq \emptyset$. Suppose that $\{u_n\}$ is generated by Picard3. Define $d\big(u_n, F(T)\big) = \inf\{\|u_n - q\| : \ q \in F(T)\}$. Then a necessary and sufficient condition for the sequence $\{u_n\}$ to converge to a point in $F(T)$ is that $\liminf_{n\to\infty} d\big(u_n, F(T)\big) = 0$.

*Proof.* ($\Rightarrow$): The result follows directly from the definition.

($\Leftarrow$): Assume that $\liminf_{n\to\infty} d\big(u_n, F(T)\big) = 0$. Since $T$ satisfies condition (C) and $F(T) \neq \emptyset$, it follows from Proposition 1(ii) that $T$ is quasi-non-expansive. Hence for any $q \in F(T)$, we have

$$\|u_{n+1} - q\| \leq \|u_n - q\|.$$

Taking the infimum over all $q \in F(T)$, we obtain

$$d\big(u_{n+1}, F(T)\big) \leq d\big(u_n, F(T)\big)$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\big\{d\big(u_n, F(T)\big)\big\}$ is non-increasing and bounded below by 0, so it is convergent. Since $\liminf_{n\to\infty} d\big(u_n, F(T)\big) = 0$, we conclude that

$$\lim_{n\to\infty} d\big(u_n, F(T)\big) = 0.$$

We now show that $\{u_n\}$ is a Cauchy sequence in $D$. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} d\big(u_n, F(T)\big) = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\left(u_{n_0}, F(T)\right) < \frac{\varepsilon}{4}.$$

By the definition of distance from a point to a set, there exists $p \in F(T)$ such that

$$\left\|u_{n_0} - p\right\| < d\left(u_{n_0}, F(T)\right) + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.$$

Since $p$ is a fixed point and the sequence $\|u_n - p\|$ is non-increasing by Lemma 3, it follows that for all $n \geq n_0$, we have $\|u_n - p\| \leq \left\|u_{n_0} - p\right\| < \frac{\varepsilon}{2}$. Thus, for all $m, n \geq n_0$,

$$\|u_n - u_m\| \leq \|u_n - p\| + \|u_m - p\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{u_n\}$ is a Cauchy sequence in $D$. Since $D$ is closed and $E$ is complete, there exists $q \in D$ such that $u_n \to q$. Finally, since $\lim_{n\to\infty} d\big(u_n, F(T)\big) = 0$ and $u_n \to q$, it follows that

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$$d(q, F(T)) = 0.$$

Because $F(T)$ is closed, we obtain $q \in F(T)$. Therefore,

$$u_n \to q \in F(T).$$

This completes the proof.

**Theorem 5.** Let $D$ be a non-empty, compact, convex subset of a uniformly convex Banach space $E$ and let $T: D \to D$ be a Suzuki generalised non-expansive mapping. Suppose that $\{u_n\}$ is the sequence defined by Picard3. Then $\{u_n\}$ converges strongly to a fixed point of $T$.

*Proof.* From Lemma 1 we see that $F(T) \neq \emptyset$ and from Lemma 4 we see that $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$. Since $D$ is compact, there exists a subsequence $\{u_n\}$ such that the strong convergence $u_{n_j} \to q$ exists for some $q \in D$. From Proposition 1 (iii), we have

$$\left\|u_{n_j} - Tq\right\| \leq 3 \left\|Tu_{n_j} - u_{n_j}\right\| + \left\|u_{n_j} - q\right\|, \quad \forall j \geq 1 .$$

If this last inequality is taken to the limit, we obtain

$$\lim_{n\to\infty} \left\|u_{n_j} - Tq\right\| = 0 .$$

Thus, $Tq = q$, i.e. $q \in F(T)$. Also, since $\lim_{n\to\infty} \|u_n - q\|$ exists from Lemma 3, the sequence $\{u_n\}$ strongly converges to $q$.

**Theorem 6.** Let $D$ be a non-empty, closed and convex subset of a uniformly convex Banach space $E$ and suppose that $F(T) \neq \emptyset$. Let $T: D \to D$ be a Suzuki generalised non-expansive mapping that satisfies property (I) from Definition 5. Then the sequence $\{u_n\}$ defined by Picard3 converges strongly to a fixed point of $T$.

*Proof.* From Lemma 3 for all $q \in F(T)$, the limit $\lim_{n\to\infty} \|u_n - q\|$ exists. Hence the limit $\lim_{n\to\infty} d(u_n, F(T))$ also exists. Assume that $\lim_{n\to\infty} \|u_n - q\| = \gamma$ for some $\gamma > 0$. If $\gamma = 0$, then the desired conclusion $\lim_{n\to\infty} d(u_n, F(T)) = 0$ follows immediately.

Assume $\gamma > 0$. From the hypothesis and Property (I) we obtain

$$f\big(d(u_n, F(T))\big) \leq \|Tu_n - u_n\|. \tag{5}$$

Since $F(T) \neq \emptyset$, Lemma 4 implies that

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty} f\big(d(u_n, F(T))\big) = 0.$$

Since $f$ is a non-decreasing function, we obtain

$$\liminf_{n\to\infty} d(u_n, F(T)) = 0.$$

Thus, for each $i \in \mathbb{N}$, we can find a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and a sequence $\{v_i\} \subset F(T)$ such that

$$\left\|u_{n_i} - v_i\right\| < \tfrac{1}{2^i}. \tag{6}$$

From (6) we have

$$\left\|u_{n_{i+1}} - v_i\right\| \leq \left\|u_{n_i} - v_i\right\| < \frac{1}{2^i},$$
$$\|v_{i+1} - v_i\| \leq \|v_{i+1} - u_{i+1}\| + \|u_{i+1} - v_i\|$$

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$$\leq \frac{1}{2^{i+1}} + \frac{1}{2^i} < \frac{1}{2^{i-1}} \to 0 \text{ as } i \to \infty.$$

Hence $\{v_i\}$ is a Cauchy sequence in $F(T)$ and thus converges to some point $q \in F(T)$. Consequently the subsequence $\{u_{n_i}\}$ converges strongly to $q$. Since $\lim_{n\to\infty} \|u_n - q\|$ exists, it follows that $\{u_n\} \to q$.

## NUMERICAL RESULTS

In this section we aim to demonstrate the efficiency and applicability of the proposed iteration method, Picard3, by means of numerical examples and comparisons. The selected problems are designed to cover a variety of functional forms and convergence behaviours, thereby providing a comprehensive evaluation of the performance of the method. Furthermore, comparisons with several well-known iteration methods are carried out to clearly highlight the advantages of the proposed approach in terms of convergence speed and accuracy.

In all numerical experiments the stopping criterion is taken as $\|u_{n+1} - u_n\| < 10^{-6}$. The choice of parameters is made to ensure stability and to allow a fair comparison with existing iteration methods. The results demonstrate that the proposed method converges faster and more smoothly compared to classical schemes.

**Example 1.** Let $E = \mathbb{R}$ be a set of real numbers. Let $T: \mathbb{R} \to \mathbb{R}$. Define the mapping $T(u) = \frac{u+2}{5}$, $\forall u \in \mathbb{R}$. We choose the control sequence as $\alpha_n = \frac{1}{n+1}$ with an initial guess $u_0 = 2.0$. We obtain the results that Picard3 converges faster to the unique fixed point of $T$ compared to Picard, Mann, Ishikawa and Noor iteration methods. Numerical computations show that Picard's three-step iteration converges to $u_* = 0.5$ in fewer steps than the other iteration schemes. The results are presented in Figure 1. We also conclude that Picard3 has a more efficient convergence rate than the M, M* and K iteration methods. The results are presented graphically in Figure 2.

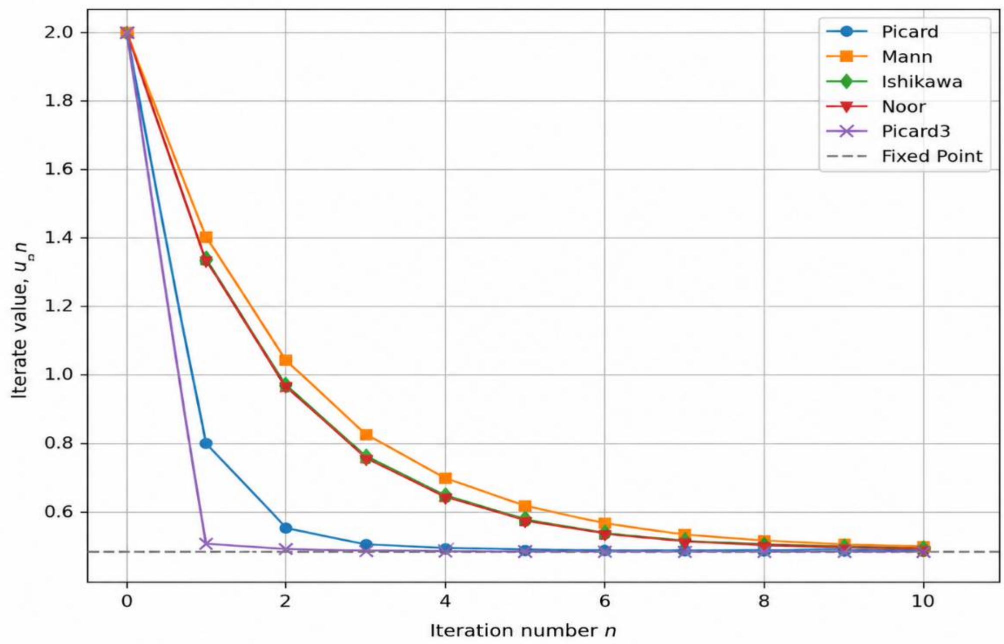



**Figure 1.** Comparison of Picard, Mann, Ishikawa and Noor iteration methods with Picard3

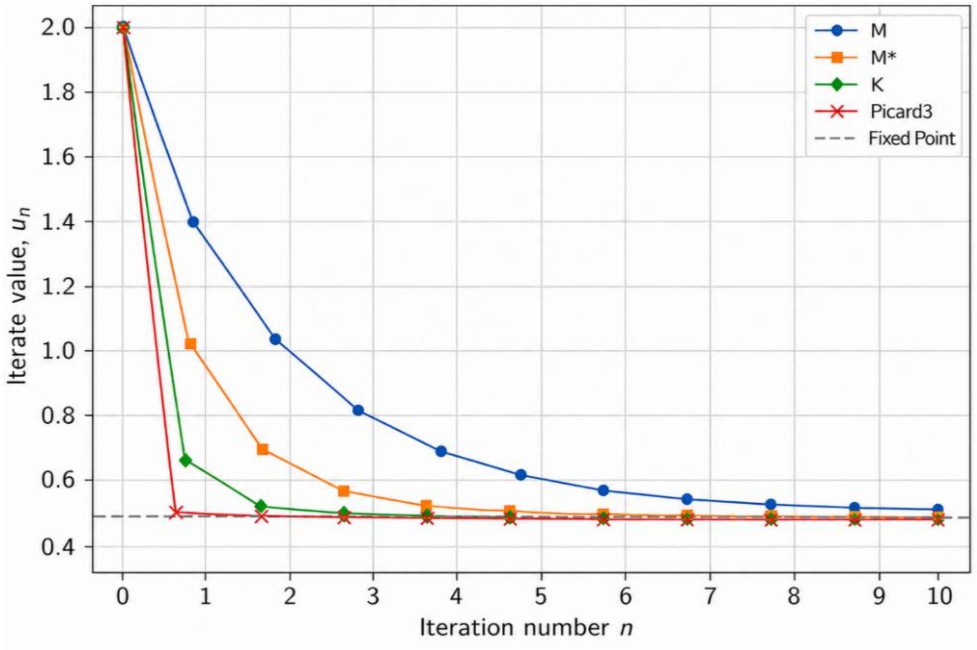



**Figure 2.** Comparison of M, M* and K iteration methods with Picard3

**Example 2.** Let $E = \mathbb{R}$ be a set of real numbers. Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $T(u) = cosu$ for all $u \in \mathbb{R}$. Choosing the control sequence as $a_n = 0.6$, $b_n = 0.5$, $c_n = 0.5$ with an initial guess $u_0 = 1.5$. We obtain that Picard3 has a more efficient convergence rate than Picard, Mann, Ishikawa and Noor iteration methods. The results are presented in Figure 3. We also conclude that Picard3 has a more efficient convergence rate than the M, M* and K iteration methods. The results are presented graphically in Figure 4.

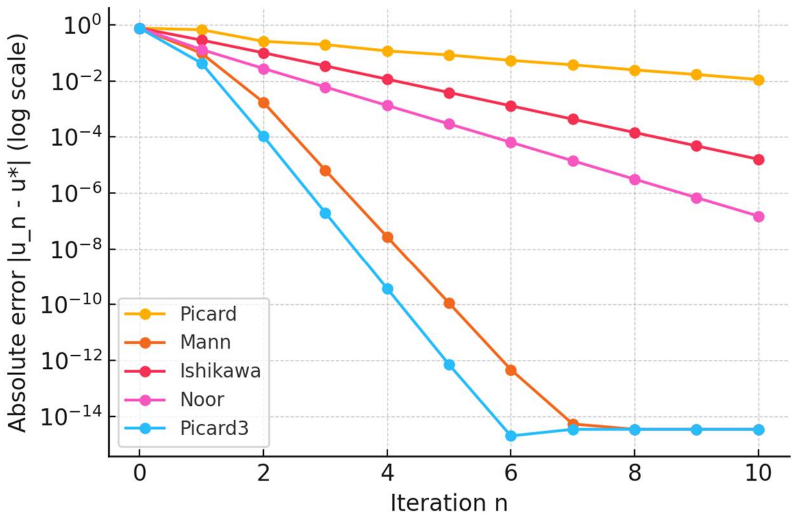



**Figure 3.** Comparison of Picard, Mann, Ishikawa and Noor iteration methods with Picard3

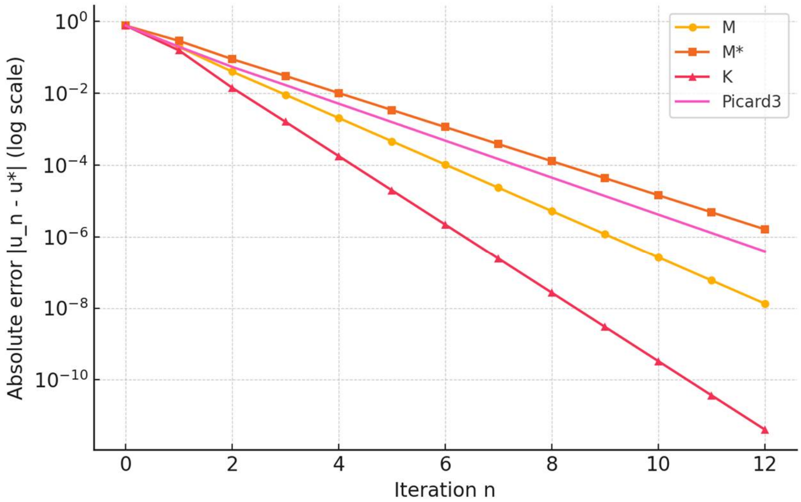



**Figure 4.** Comparison of M, M* and K iteration methods with Picard3

**Example 3 (Application to a non-linear differential equation).** Consider the non-linear initial value problem

$$u'(t) = -u(t) + cost\,,\ \ u(0) = 1, t \in [0,1]\,.$$

The exact solution of this problem is

$$u(t) = \frac{e^{-t}}{2} + \frac{sint}{2} + \frac{cost}{2}.$$

Rewriting the problem as an equivalent fixed point problem, we obtain the operator

$$(Tu)(t) = 1 + \int_0^t (-u(s) + coss)\, ds\,,$$

which defines a Suzuki-type mapping on $C[0,1]$ with the supremum norm. It can be verified that $T$ satisfies condition (C) and hence belongs to the class of Suzuki generalised non-expansive operators. We apply Picard3 with the initial guess $u_0(t) = 1$. For comparison, the classical Picard, Mann, Ishikawa and Noor iteration methods, as well as M, M* and K iteration methods, are implemented. The numerical results are summarised in Table 1, which shows the superior performance of Picard3. Figure 5 illustrates the error curves in logarithmic scale, confirming the faster convergence of the proposed method. In fact, Table 1 clearly illustrates that Picard's three-step iteration requires significantly fewer iterations to achieve the same level of accuracy compared with the other considered methods. This demonstrates a higher efficiency of the Picard3 scheme as it converges faster while maintaining stability.

To further support the numerical performance of Picard's three-step iteration, we also employ a polynomiography-based visualisation approach. Polynomiography provides a graphical representation of the convergence behaviour of iteration methods applied to complex polynomial equations. In this approach each point in a selected region of the complex plane is used as an initial value and the corresponding pixel is coloured according to the number of iterations required to approximate a root within a prescribed tolerance. Therefore, the resulting images provide a visual comparison of the convergence speed and stability of different iteration methods [48, 49].

**Table 1.** Comparison of sequence of iterate values $u_n$ generated by Picard, Mann, Ishikawa, Noor, M, M* and K iteration methods with Picard3 ($n$ = iteration step number)

| $n$ | Picard | Mann | Ishikawa | Noor | M | M* | K | Picard3 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1 | 0.8776 | 0.8950 | 0.8870 | 0.8820 | 0.8600 | 0.8520 | 0.8450 | 0.8001 |
| 2 | 0.7803 | 0.8122 | 0.7900 | 0.7841 | 0.7655 | 0.7523 | 0.7461 | 0.7079 |
| 3 | 0.7035 | 0.7451 | 0.7156 | 0.7097 | 0.6950 | 0.6812 | 0.6760 | 0.6500 |
| 4 | 0.6420 | 0.6890 | 0.6560 | 0.6495 | 0.6405 | 0.6281 | 0.6240 | 0.6062 |
| 5 | 0.5922 | 0.6412 | 0.6070 | 0.6002 | 0.5981 | 0.5850 | 0.5823 | 0.5705 |
| 6 | 0.5510 | 0.6002 | 0.5660 | 0.5605 | 0.5601 | 0.5480 | 0.5460 | 0.5402 |
| 7 | 0.5165 | 0.5652 | 0.5321 | 0.5273 | 0.5280 | 0.5162 | 0.5140 | 0.5150 |
| 8 | 0.4870 | 0.5350 | 0.5029 | 0.4990 | 0.4999 | 0.4890 | 0.4880 | 0.4932 |
| 9 | 0.4610 | 0.5080 | 0.4770 | 0.4741 | 0.4755 | 0.4650 | 0.4641 | 0.4740 |
| 10 | 0.4380 | 0.4841 | 0.4542 | 0.4512 | 0.4531 | 0.4430 | 0.4430 | 0.4570 |

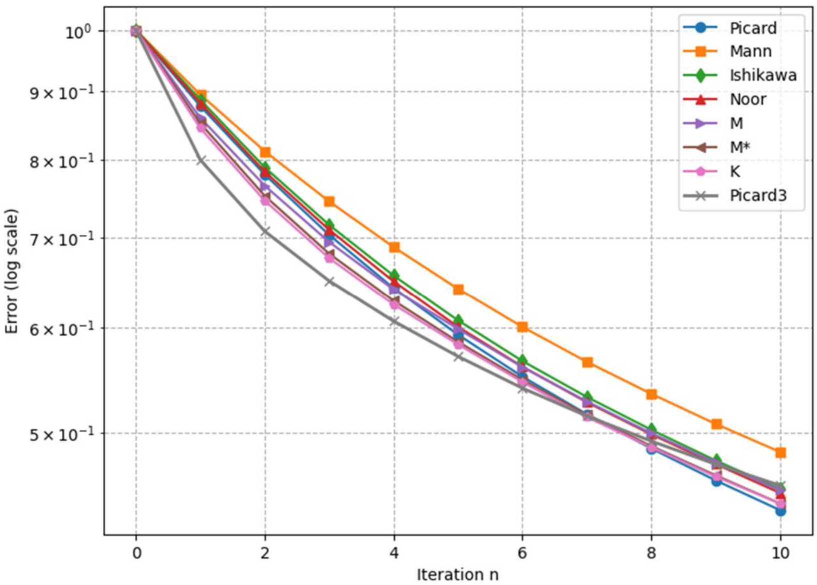



**Figure 5.** Error curves for Example 3

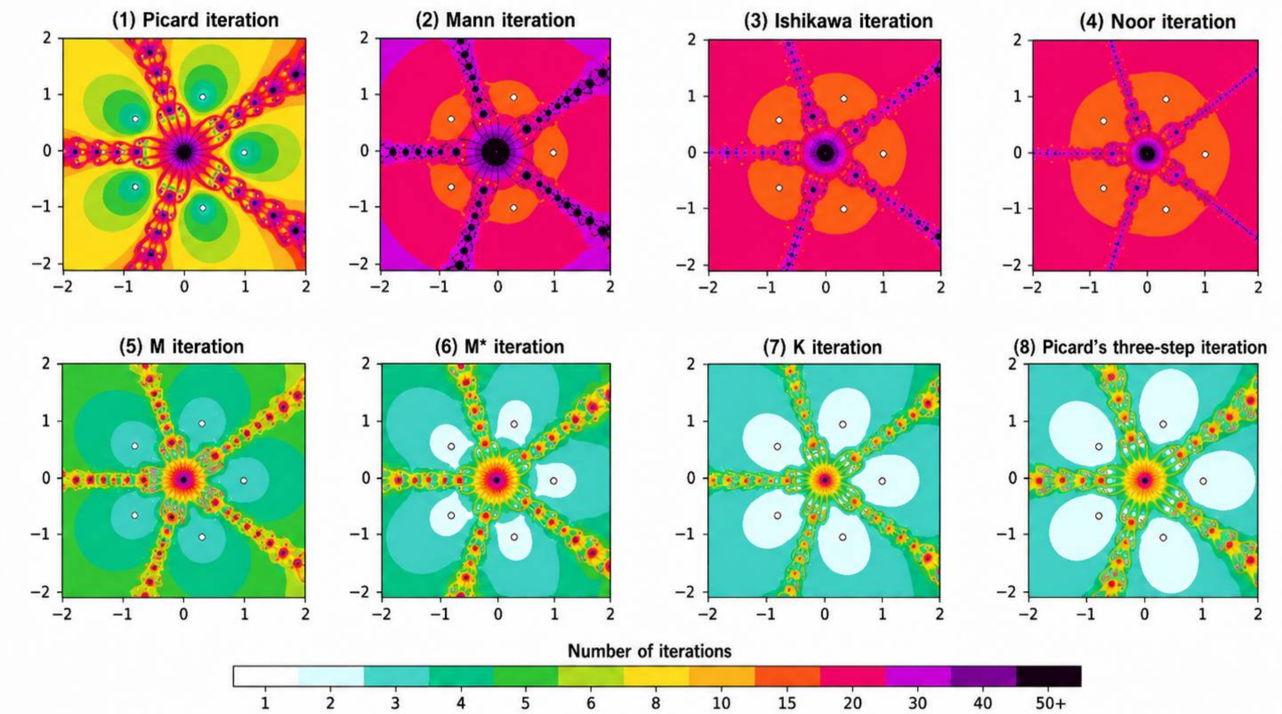
**Example 4 (Polynomiography-based comparison).** We consider the polynomial

$$P(z) = z^5 - 1,$$

whose roots are uniformly distributed on the unit circle in the complex plane. The associated Newton operator is used to generate the iteration sequences. The region $[-2, 2] \times [-2, 2]$ of the complex plane is selected as the computational domain and each iteration method is applied from every initial point in this region. The iteration process is terminated whenever the approximation satisfies the prescribed tolerance or when the maximum number of iterations is reached.

We compare the performance of Picard's three-step iteration with several well-known iteration methods, i.e. Picard, Mann, Ishikawa, Noor, M, M* and K iteration methods. The resulting polynomiographs reveal the attraction basins associated with the roots of the polynomial and provide visual information about the convergence characteristics of each method. In particular, larger attraction regions and lighter colour distributions indicate faster convergence, whereas darker regions correspond to slower convergence or non-convergence.

The numerical experiments show that Picard's three-step iteration produces wider regions of stable convergence, smoother attraction basins and generally requires fewer iterations to reach the roots when compared with several classical iteration methods. The corresponding polynomiographs and average numbers of iteration are presented in Figure 6.




**Key Observation:** Picard's three-step iteration exhibits the largest attraction regions, smoother boundaries and the lightest colours over most of the domain, indicating the fastest convergence among tested methods.

White circles indicate the five roots of $P(z) = z^5 - 1$. Domain: $[-2, 2] \times [-2, 2]$. Tolerance: $10^{-6}$. Maximum iterations: 50.

Average Number of Iterations

| Picard | Mann | Ishikawa | Noor | M | M* | K | Picard's three-step |
|---|---|---|---|---|---|---|---|
| 13.84 | 19.76 | 16.21 | 22.47 | 9.38 | 7.49 | 6.88 | 4.72 |

**Figure 6.** Polynomiographic comparison of Picard, Mann, Ishikawa, Noor, M, M* and K iteration methods with Picard's three-step iteration

The polynomiographic visualisations and average number of iteration values demonstrate that Picard's three-step iteration converges faster and exhibits more stable attraction basins than the other methods considered. These observations are consistent with the numerical results presented in the previous examples and provide additional graphical evidence supporting the efficiency and stability of the proposed iteration process.

## CONCLUSIONS

Compared to conventional fixed point methods, the iteration method studied here offers improved performance in terms of convergence behaviours and applicability. In particular, it provides a broader framework capable of handling mappings that do not necessarily fulfill strict contraction conditions, thus widening the scope of fixed point theory.

Beyond its theoretical value, the proposed method holds potential for numerical applications, particularly in solving non-linear equations that arise in various fields such as functional analysis, differential equations and optimisation. These findings suggest that the Picard's three-step iteration method can serve as a robust alternative to classical iteration techniques.

Despite its advantages, the proposed method may involve a higher computational cost per iteration compared to simpler iteration methods due to its multi-step structure. In addition, its efficiency may depend on the choice of initial values and parameters and in some cases, the method may not significantly outperform classical approaches for problems with simple contraction properties. Therefore, further investigation is needed to better understand its performance in large-scale or highly complex problems.

Future studies may extend the methodology to other classes of mappings including multivalued or random operators, or adapt the technique to more generalised functional spaces such as b-metric or modular function spaces. Furthermore, its application to non-linear models involving delay, fractional or integral equations remains an open and promising area of research.

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