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Full Paper

Semi-r-clean rings

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Abstract: We introduce a class of semi-r-clean rings, defined by the property that every element can be expressed as the sum of a periodic element and a von Neumann regular element. We establish fundamental properties of these rings, including their behaviour under homomorphic images, direct products and various ring-theoretic constructions. We prove that polynomial rings over commutative rings are never semi-r-clean and investigate the semi-r-clean property in triangular matrix rings and ideal extensions. Additionally, we explore the relationship between semi-r-clean rings and the previously studied class of semi-clean rings.

Keywords: clean rings, r-clean rings, periodic elements, regular elements, matrix rings

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INTRODUCTION

Decomposition properties of elements have played an important role in the development of modern ring theory. A fundamental example is the notion of a clean ring introduced by Nicholson [1] in connection with exchange rings. An element a of a ring R is called clean if it can be written in the form $a = e + u$, where e is an idempotent and u is a unit; a ring is clean if all of its elements admit such a decomposition. The relation between clean and exchange rings, as well as the lifting of idempotents, was further developed by Nicholson and Warfield [2] while subsequent work established important connections between clean rings and semi-perfect rings [3]. The behaviour of cleanness under matrix-ring constructions was studied by Han and Nicholson [4]. Further developments include cleanness of endomorphism rings and linear transformation rings [5, 6], strongly clean rings and their connection with strongly π regular rings [7], and uniquely clean rings [8, 9]. Other directions have connected clean-ring theory with topology and with various classes of rings defined through special decompositions [10, 11]. These developments demonstrate the usefulness of additive decompositions in understanding the structure and permanence properties of rings.

A natural way to extend the clean decomposition is to modify one of its two components while retaining the other. One such direction replaces the idempotent by a more general type of element. In particular, strongly clean rings require the two components in the decomposition to commute [7], whereas weakly clean rings allow an element to be expressed as either a sum or a difference of a unit and an idempotent [12]. In nil-clean rings, the unit is replaced by a nilpotent element [13], while ART rings provide decompositions involving regular, tripotent and nilpotent elements [14]. Of particular relevance to the present work is the replacement of the idempotent by a periodic element. Ye [15] introduced semi-clean rings, in which every element is the sum of a periodic element and a unit. Among other results, it was shown that every clean ring is semi-clean, that finite matrix rings over semi-clean rings remain semi-clean, and that polynomial rings over commutative rings do not possess the semi-clean property. This line of investigation was subsequently extended to almost semi-clean rings by Anderson and Bisht [16], where elements are expressed as sums of a non-zero-divisor and a periodic element. Their results include the preservation of this property under polynomial and power-series extensions and finite direct products. Bisht [17] introduced semi-nil clean rings, characterised by decomposition into nilpotent and periodic elements, and investigated their structural properties and extension behaviour. More recently, quasi-semi-clean rings were considered by Remzi et al. [18], who studied decompositions involving units and quasi-periodic elements. These developments establish periodic elements as an important substitute for idempotents in decomposition-based ring classes.

A second major direction of generalisation retains the idempotent component but replaces the unit by a von Neumann regular element (in the sense of Goodearl [19]). Recall that an element $a \in R$ is von Neumann regular if there exists $x \in R$ such that $a = axa$. Since every unit is regular, this replacement leads to a class larger than the class of clean rings. Ashrafi and Nasibi [20, 21] introduced r-clean rings, in which each member is expressible as the sum of a regular element and an idempotent. They established several structural properties of r-clean rings and investigated their relationship with clean and exchange rings. In particular, the two notions coincide for abelian rings while additional conditions on the idempotents yield exchange-type properties. Their work also led to investigations concerning regular ideals and lifting phenomena associated with r-clean decompositions. Thus, regular elements provide another natural enlargement of the class of units occurring in additive decompositions.

Several related classes have subsequently been developed around the regular-element viewpoint. Precious rings were studied by Ashrafi et al. [22] while Chen and Sheibani [23] investigated related additive decompositions. Bisht [24] introduced r-precious rings by combining regular, idempotent and nilpotent components, and strongly r-precious rings were subsequently considered by Ghumde et al. [25]. Weakly r-clean rings, in which an element is allowed to be either the sum or the difference of a regular element and an idempotent, were introduced by Sharma and Basnet [26]. Other related investigations include decompositions involving m-potent and nilpotent elements [27], properties of nil-clean elements and uniquely nil-clean elements [28], and characterisations of strongly nil-clean elements [29]. Decompositions involving central elements and elements of the Jacobson radical were also considered by Ma et al. [30]. These works further illustrate the broad interest in replacing one or more components of the classical clean decomposition while retaining useful structural information.

The interaction between the periodic-element and regular-element approaches has also been investigated in several related settings. Klingler et al. [31] studied semi-clean group rings and obtained characterisations for commutative local coefficient rings and abelian groups. Ramezan-

Nassab [32] further investigated clean and semi-clean group rings over suitable semi-local rings. Khurana et al. [33] introduced special clean elements and related their decomposition to unit-regularity and Bott--Duffin invertibility. Singh and Pandeya [34] generalised semi-clean decompositions by allowing finitely many units, leading to n-semi-clean and σ-semi-clean rings and corresponding notions for ideals. Nil-semi-clean rings were introduced by Ahmad and Al-Mallah [35], who considered decompositions involving semi-idempotent and periodic elements. Although these investigations broaden the theory of semi-clean and related rings, the regular-element component in these decompositions has not been combined with the periodic element component in the manner considered here.

The two principal directions described above suggest a natural question. Starting with the clean decomposition $a = e + u$, one may replace an idempotent e by a periodic element, obtaining the semi-clean setting, or replace the unit u by a regular element, obtaining the r-clean decomposition. It is therefore natural to ask what happens when both replacements are made simultaneously. More precisely, one is led to consider decompositions of the form $a = p + r$, where p is periodic and r is a von Neumann regular element. This combination provides the motivation for the class introduced in this paper.

We introduce this class as follows. A ring R is called semi-r-clean if every element of R can be expressed as the sum of a periodic element and a von Neumann regular element. The definition simultaneously incorporates the two principal generalisations of clean rings discussed above. Since every idempotent is periodic, every r-clean ring is semi-r-clean. Likewise, since every unit is von Neumann regular, every semi-clean ring is semi-r-clean. Thus, the class of semi-r-clean rings contains both the r-clean and semi-clean classes. The examples established later in the paper show that these inclusions are not merely formal: there are semi-r-clean rings which are neither r-clean nor semi-clean.

The purpose of this paper is to develop the basic theory of semi-r-clean rings and to examine its behaviour under several standard ring constructions. We first establish permanence under homomorphic images and finite direct products. We then investigate conditions under which a semi-r-clean ring becomes semi-clean, including the role of zero divisors. Using the concept of a regular ideal introduced by Brown and McCoy [36], we obtain a lifting criterion relating the semi-r-clean property of a ring to that of a suitable quotient, provided periodic elements lift modulo the regular ideal. We also prove that polynomial rings over commutative rings cannot be semi-r-clean. For rings where 2 is an invertible element, a further decomposition involving a unit, a regular element and an involution is obtained. In addition, we examine the behaviour of the property under Pierce decompositions with respect to central idempotents. Finally, we study formal triangular matrix rings and ideal extensions, obtaining both sufficient conditions and partial converses for the semi-r-clean property.

Throughout the paper, R denotes a ring, and we use $R_g(R), U_n(R)$ and $P_d(R)$ to denote, respectively, the sets of von Neumann regular, unit and periodic elements of R.

SEMI-r-CLEAN RINGS

We start by presenting an outline of the proposed semi-r-clean concepts.

Definition 1. An element $x \in R$ is termed semi-r-clean if it can be expressed as the sum of a periodic and a regular (Von Neumann) element. A ring R is semi-r-clean if every element of R admits such a decomposition.

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From the definition, one can observe that every regular element and every periodic element is semi-r-clean. For instance, if m is a nilpotent element, then $m = (m-1)+1$. Hence m is semi-r-clean since $(m-1)$ is a unit (and therefore regular) while 1 is periodic. Similarly, if j is in the Jacobson radical, then $j = (j-1)+1$, but $j-1$ is invertible, so j is semi-r-clean.

Next, we provide an example of an element that is semi-r-clean but neither r-clean nor semi-clean.

Example 1. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, 3 \text{ does not divide } n \text{ and } 5 \text{ does not divide } n\right\}$. Consider an element $A \in M_2(R)$, where

$$A = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ 0 & 0 \end{bmatrix}.$$

The matrix A can be decomposed as

$$A = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & \frac{7}{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = E + F.$$

A straightforward calculation demonstrates that the matrix E is regular but not a unit, and the matrix F is periodic but not idempotent. Hence the matrix A is neither r-clean nor semi-clean. Note that every r-clean ring is semi-r-clean since idempotents are periodic. However, as Example 1 shows, the converse is not true.

It is worth noting that a subring of a semi-r-clean ring need not itself be semi-r-clean. The following example illustrates this fact.

Example 2. Consider the ring

$$R = \mathbb{Z}_{(5)} \cap \mathbb{Z}_{(3)} = \left\{\frac{x}{y} : x, y \in \mathbb{Z}, 5 \text{ does not divide } y \text{ and } 3 \text{ does not divide } y\right\}.$$

One can easily verify that R is a semi-r-clean ring. However, note that $\mathbb{Z} \subseteq R$ but $\mathbb{Z}$ is not semi-r-clean. We next present several results concerning the semi-r-clean property under different ring constructions.

Proposition 1. If R is a semi-r-clean ring, then every homomorphic image of R is also semi-r-clean.

Proof. Let $\Phi : R \to R'$ be a ring homomorphism and consider $y' \in \Phi(R)$. Then $\exists y \in R$ such that $y' = \Phi(y)$. As R is semi-r-clean, we write

$$y = p + g,$$

where $p \in P_d(R)$ and $g \in R_g(R)$. Hence

$$y' = \Phi(y) = \Phi(p+g) = \Phi(p) + \Phi(g).$$

Note that $(\Phi(p))^m = \Phi(p^m) = \Phi(p^n) = (\Phi(p))^n$ for some $m, n \in N$, showing that $\Phi(p) \in P_d(\Phi(R))$. Similarly, since $g \in R_g(R)$, it follows that $\Phi(g) \in R_g(\Phi(R))$. Therefore, $y' = \Phi(p) + \Phi(g)$ admits the semi-r-clean decomposition in $\Phi(R)$. Thus, $\Phi(R)$ is semi-r-clean.

The stability of semi-r-clean rings under homomorphic images naturally leads to the question of whether the property is also preserved under direct products. The following proposition addresses this.

Proposition 2. Let $R = \prod_{\beta \in J} R_\beta$ be a finite direct product of rings. Then

(i) If R is semi-r-clean, then each component ring R_β is semi-r-clean.

(ii) Conversely, if each R_β is semi-r-clean, then so is R.

Proof. (i) This part implies from Proposition 1.
(ii) Since J is finite, $R = \prod_{\beta\in J} R_\beta$ is periodic if and only if each R_β is periodic. Let $y = (y_\beta)_{\beta\in J} \in R$. Since each R_β is semi-r-clean, we can write $y_\beta = p_\beta + g_\beta$ with $p_\beta \in P_d(R_\beta)$ and $g_\beta \in R_g(R_\beta)$. Thus, for each β, there exists $t_\beta \in R_\beta$ such that $g_\beta t_\beta g_\beta = g_\beta$, which shows that g_β is regular. Now define $p = (p_\beta)_{\beta\in J}$ and $g = (g_\beta)_{\beta\in J}$. Then $p \in P_d(R)$ and $g \in R_g(R)$ because both periodicity and regularity hold component-wise. Hence $y = p + g$ and therefore R is semi-r-clean.

Although Example 1 demonstrates that semi-r-clean elements are not necessarily semi-clean, we can establish conditions under which every semi-r-clean element becomes semi-clean. The following result provides such conditions by imposing a structural constraint on the underlying ring.

Proposition 3. Let R be a semi-r-clean ring without zero divisors. Then each non-zero element of R is semi-clean.

Proof. Let $x \in R$ with $x \neq 0$. As R is semi-r-clean, we write $x = g + p$, where $g \in R_g(R)$ and $p \in P_d(R)$. As g is regular, there exists an element $t \in R$ such that $gtg = g$. As R has no zero divisors, this implies that if $g \neq 0$, then $g \in U_n(R)$; that is, g is a unit. Alternatively, if $g = 0$, then $x = p$. But as $x \neq 0$, we have $p \neq 0$; this implies $p \in U_n(R)$. Thus, in either case x is the sum of a unit (or zero) and a periodic element. Hence x is semi-clean.

Proposition 4. Let R be a ring. Then R is semi-r-clean if and only if each $x \in R$ admits a decomposition of the form $x = g - p$, where $g \in R_g(R)$ and $p \in P_d(R)$.

Proof. Suppose that R is semi-r-clean. Then for any $x \in R$, $\exists\ g \in R_g(R)$ and $p \in P_d(R)$ such that $-x = g + p$. This implies $x = -g - p$. As g is regular, $\exists\ t \in R$ such that $gtg = g$. Thus,

$$(-g)(-t)(-g) = (-g),$$

so $-g \in R_g(R)$. Hence x can be expressed as $x = (-g) - p$, where $-g \in R_g(R)$ and $p \in P_d(R)$ as desired.

Conversely, assume that for each $x \in R$, there is a decomposition $x = g - p$, where $g \in R_g(R)$ and $p \in P_d(R)$. Then $-x = -g + p$. Since $-g \in R_g(R)$, we can write $-x = (-g) + p$, where $-g \in R_g(R)$ and $p \in P_d(R)$. Thus, $-x$ has the semi-r-clean decomposition. Hence R is semi-r-clean.

Theorem 1. Let J be a regular ideal of the ring R and suppose that the periodic elements of R can be lifted modulo J. Then R is semi-r-clean if and only if R/J is semi-r-clean.

Proof. Since R/J is semi-r-clean, for every $b \in R$ the image $b + J \in R/J$ is semi-r-clean. Hence there exists an element $p + J \in P(R/J)$ such that $(b - p) + J \in R_g(R/J)$. This implies that

$$(b - p) - (b - p)t(b - p) \in J$$

for some $t \in R$, as $b - p + J$ is regular in R/J. Since J is a regular ideal, by Lemma 1 [36] $(b - p) \in R_g(R)$.

Furthermore, as periodic elements in R/J lift to periodic elements in R, hence the element $p \in P_d(R)$. Thus, b is semi-r-clean. The converse follows directly from Proposition 1.

Proposition 5. For any commutative ring R, the polynomial ring $R[x]$ fails to be semi-r-clean.

Proof. Assume that an element $x \in R[x]$ can be expressed as $x = g + p$, where $g \in R_g(R[x])$ and $p \in R[x]$ is periodic, i.e. $p^m = p^n$ for some $m > n$. Since the base ring R is commutative, it follows that periodic elements in the polynomial ring $R[x]$ must lie in R. Hence $p \in R$ and consequently $g - p \in R_g(R[x])$. But by Lemma 2.11 [20], this leads to the conclusion that $1 \in R[x]$ is nilpotent, which is clearly impossible. This contradiction confirms that $R[x]$ cannot be semi-r-clean.

Note that by Proposition 1, $R \cong \frac{R[[x]]}{(x)}$. Hence if $R[[x]]$ is semi-r-clean, then so is R.

Theorem 2. In any semi-r-clean ring R with 2 invertible elements, each element admits a decomposition as the sum of a unit, a regular and an element whose square is 1.

Proof. Let $m \in R$ and as R is semi-r-clean and $\frac{1}{2} \in R$, there exists a decomposition

$$\frac{m+1}{2} = g + p,$$

where $g \in R_g(R)$ and $p \in P_d(R)$. It is known that every periodic element can be expressed as the sum of a unit and an idempotent (Lemma 5.1 [15]), so we may write

$$\frac{m+1}{2} = g + u + d,$$

with $u \in U_n(R)$ and $d \in I_d(R)$. Multiplying both sides by 2 gives

$$m + 1 = 2g + 2u + 2d, \text{ so } m = 2g + 2u + (2d - 1).$$

We now verify the properties of each summand. Since $g \in R_g(R)$, $\exists\ t \in R$ such that $gtg = g$, and hence

$$(2g)\left(\frac{t}{2}\right)(2g) = 2(gtg) = 2g,$$

showing that $2g \in R_g(R)$. Also, $2u \in U_n(R)$ as the product of an invertible element and a unit. Finally,

$$(2d-1)^2 = 4d^2 - 4d + 1 = 4d - 4d + 1 = 1$$

since $d^2 = d$. Thus, $2d - 1$ is an involution. Therefore, m is the sum of a regular, a unit and an element whose square is 1 as claimed.

Proposition 6. If c is a central idempotent element of R such that cRc and $(1-c)R(1-c)$ are both semi-r-clean, then R is a semi-r-clean ring.

Proof. By Pierce decomposition of a ring, we have

$$R = (1-c)R(1-c) + (1-c)Rc + cR(1-c) + cRc.$$

Since c is a central idempotent element, we have

$$R = (1-c)R(1-c) + cRc.$$

Let $z = a + b$, where $a \in \bar{c}R\bar{c}$ and $b \in cRc$, where $(1-c)R(1-c) = \bar{c}R\bar{c}$. By assumption, both a and b are semi-r-clean elements. Hence there exist g_1, g_2, p_1 and p_2 such that

$$a = g_1 + p_1 \text{ and } b = g_2 + p_2,$$

where $g_1 \in R_g(\bar{c}R\bar{c})$, $g_2 \in R_g(cRc)$, $p_1 \in \bar{c}R\bar{c}$ and $p_2 \in cRc$.

Thus, there exist $r_1 \in \bar{c}R\bar{c}$ and $r_2 \in cRc$ such that

$$g_1 = g_1 r_1 g_1 \text{ and } g_2 = g_2 r_2 g_2.$$

Moreover, there exist natural numbers $m_1, m_2, n_1, n_2 \in N$ with $n_1 < m_1$ and $n_2 < m_2$ such that ${p_1}^{n_1} = {p_1}^{m_1}$ and ${p_2}^{n_2} = {p_2}^{m_2}$. Hence $z = a + b = (g_1 + p_1) + (g_2 + p_2) = (g_1 + g_2) + (p_1 + p_2)$.

We now prove that $(g_1 + g_2)$ is regular and $(p_1 + p_2)$ is periodic.

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Step 1. $(g_1 + g_2)$ is regular.
Since $u \in cRc$ and $v \in \bar{c}R\bar{c}$, we have $uv = vu = 0$. Therefore,

$$(g_1 + g_2)(r_1 + r_2)(g_1 + g_2) = g_1 r_1 g_1 + g_2 r_2 g_2 = g_1 + g_2 .$$

Thus, $(g_1 + g_2)$ is regular.
Step 2. $(p_1 + p_2)$ is periodic.
Observe that

$$(p_1 + p_2)^n = {p_1}^n + {p_2}^n.$$

Without loss of generality, suppose that $n_2 \geq n_1$. Then

$${p_1}^{n_1} = {p_1}^{m_1} \Rightarrow {p_1}^{m_1 - n_1} {p_1}^{n_1} = {p_1}^{2(m_1 - n_1)} {p_1}^{m_1} = \cdots = {p_1}^{k(m_1 - n_1) + n_1} ,$$

and similarly

$${p_2}^{n_2} = {p_2}^{m_2} \Rightarrow {p_2}^{m_2 - n_2} {p_2}^{n_2} = {p_2}^{2(m_2 - n_2)} {p_2}^{m_2} = \cdots = {p_2}^{k(m_2 - n_2) + n_2}.$$

Hence

$$p_1^{n_2} = p_1^{n_1 + (n_2 - n_1)} = p_1^{k(m_1 - n_1) + n_1 + (n_2 - n_1)} = p_1^{k(m_1 - n_1) + n_2}$$

By a similar argument,

$$p_2^{n_2} = p_2^{k(m_2 - n_2) + n_2} .$$

If we now set $n_3 = n_2$ and $m_3 = (m_2 - n_2)(m_1 - n_1) + n_2$, then

$$p_1^{n_3} = p_1^{m_3} \text{ and } p_2^{n_3} = p_2^{m_3}.$$

Thus, $(p_1 + p_2)^{n_3} = p_1^{n_3} + p_2^{n_3} = p_1^{m_3} + p_2^{m_3}$. Hence $(p_1 + p_2)$ is periodic.

As a direct consequence of the above theorem and using the method of induction, we can state the following corollary.

Corollary 1. Let $1 = c_1 + c_2 + \cdots + c_n$, where c_j are central orthogonal idempotents and $c_j R c_j$ is semi-r-clean for all j. Then R is semi-r-clean.

Ashrafi and Nasibi [20, 21] find some conditions on rings so that formal lower triangular matrix rings are r-clean. In the subsequent proposition we establish specific criteria for rings to ensure that the lower triangular matrix ring is semi-r-clean.

Proposition 7. Let X and Y be rings while $M = {}_X M_Y$ is a bimodule. If the formal triangular matrix ring $T = \begin{bmatrix} X & O \\ M & Y \end{bmatrix}$ is semi-r-clean, then both X and Y are semi-r-clean.

Proof. Suppose that $T = \begin{bmatrix} X & O \\ M & Y \end{bmatrix}$ is semi-r-clean. Then by definition, for any $x \in X, y \in Y$ and $m \in M$, we can write

$$\begin{bmatrix} x & 0 \\ m & y \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ g_2 & g_3 \end{bmatrix} + \begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix},$$

where $\begin{bmatrix} g_1 & 0 \\ g_2 & g_3 \end{bmatrix} \in R_g(T)$ and $\begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix} \in P_d(T)$.
Thus, $x = g_1 + p_1$ and $= g_3 + p_3$.
As $\begin{bmatrix} g_1 & 0 \\ g_2 & g_3 \end{bmatrix} \in R_g(T)$, there exists $\begin{bmatrix} s_1 & 0 \\ s_2 & s_3 \end{bmatrix} \in T$ such that

$$\begin{bmatrix} g_1 & 0 \\ g_2 & g_3 \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ g_2 & g_3 \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ s_2 & s_3 \end{bmatrix} \begin{bmatrix} g_1 & 0 \\ g_2 & g_3 \end{bmatrix} = \begin{bmatrix} g_1 s_1 g_1 & 0 \\ g_3 s_3 g_2 + g_2 s_1 g_1 + g_3 s_2 g_1 & g_3 s_3 g_3 \end{bmatrix}.$$

So $g_1 \in R_g(X)$ and $g_3 \in R_g(Y)$.

Next, since $\begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix} \in P_d(T)$, then there exist positive integers m, n such that

$$\begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix}^m = \begin{bmatrix} p_1 & 0 \\ p_2 & p_3 \end{bmatrix}^n.$$

A trivial calculation shows that $p_1 \in P_d(X)$ and $p_3 \in P_d(Y)$. Therefore, $x \in X$ and $y \in Y$ admit the semi-r-clean decomposition. Hence both X and Y are semi-r-clean rings.

Proposition 8. If the ring of all lower triangular matrices of order n over R is semi-r-clean, then R is a semi-r-clean ring.

The next result discusses the partial converse of the previous result.

Proposition 9. Let X and Y be rings and let $M = {}_X R_Y$ be a bimodule. Assume that one of the rings X or Y is clean and the other one is semi-r-clean. Then the formal triangular matrix ring $T = \begin{bmatrix} X & O \\ M & Y \end{bmatrix}$ is a semi-r-clean ring.

Proof. Without loss of generality, assume that X is a semi-r-clean ring and Y is clean. Let $A = \begin{bmatrix} x & 0 \\ m & y \end{bmatrix} \in T$. Since X is a semi-$r$-clean ring and Y is clean, we may write

$$x = p_1 + g \text{ and } y = p_2 + u,$$

where $p_1, p_2 \in P_d(R), g \in R_g(R)$ and $u \in U_n(R)$.

Consider the decomposition $A = P + Q$, where $P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ and $Q = \begin{bmatrix} g & 0 \\ m & u \end{bmatrix}$. We first show that Q is regular. As $g \in R_g(R)$, $\exists t \in R$ such that $gtg = g$. Then

$$\begin{bmatrix} g & 0 \\ m & u \end{bmatrix} \begin{bmatrix} t & 0 \\ -u^{-1}mt & u^{-1} \end{bmatrix} \begin{bmatrix} g & 0 \\ m & u \end{bmatrix} = \begin{bmatrix} g & 0 \\ m & u \end{bmatrix}.$$

Hence $Q \in R_g(T)$.

Next, we show that P is periodic. Since $p_1, p_2 \in P_d(R)$, there exist positive integers m_1, m_2, n_1, n_2 such that

$$p_1{}^{m_1} = p_1{}^{n_1} \text{ and } p_2{}^{m_2} = p_2{}^{n_2}.$$

Let $d = lcm(m_1 - n_1, m_2 - n_2)$ and set $M = max(n_1, n_2) + d$ and $N = max(n_1, n_2)$. Then

$$p_1^M = p_1^N \text{ and } p_2^M = p_2^N.$$

Thus, $p^M = \begin{bmatrix} p_1^M & 0 \\ 0 & p_2^M \end{bmatrix} = \begin{bmatrix} p_1^N & 0 \\ 0 & p_2^N \end{bmatrix} = p^N$. Hence $P \in P_d(T)$.

Therefore, every element $A \in T$ can be expressed as the sum of a regular and a periodic element. Hence T is semi-r-clean. This completes the proof.

Let R be a ring and let X be an $R - R$ bimodule which also forms a ring satisfying the following compatibility conditions:

$$(lm)n = l(mn), (ln)m = l(nm), \text{ and } (ml)n = m(ln), \text{ for all } n \in R \text{ and } l, m \in X.$$

The ideal extension $J(R\,; X)$ is defined as

$$J(R\,; X) = \{(r, l) : r \in R, l \in X\},$$

which becomes a ring with the usual addition and multiplication given by

$$(r, l)(s, l') = (rs, rl' + ls + ll')$$

for all $r, s \in R$ and $l, l' \in X$.

In the following, we discuss some results on ideal extensions in the context of semi-r-clean rings.

Theorem 3. The following statements hold:
(i) If $J(R\,;X)$ is semi-r-clean, then R is semi-r-clean.
(ii) If R is semi-clean and for every $m \in X$, there exists $n \in X$ such that $m + n + mn = 0$, then $J(R\,;X)$ is semi-clean and hence semi-r-clean.
(iii) If R is semi-r-clean and for each $m \in X$ and $r, g \in R$, $msg + msm + gms = m$, then $J(R\,;X)$ is semi-r-clean.

Proof. (i) Note that $J(R\,;X)/(0 \oplus X) \cong R$. Hence by Proposition 1, R is semi-r-clean.
(ii) As R is semi-clean, for every $v \in R$, we can write

$$v = u + p,$$

where $p \in P_d(R)$ and $u \in U_n(R)$. Thus, for $(v, m) \in J(R\,;X)$, we have

$$(v, m) = (u, m) + (p, 0).$$

Clearly, $(p, 0) \in P_d(J(R\,;X))$. Consider the element $(u, m) \in J(R\,;X)$. Since $u \in U_n(R)$, this implies that $(u, 0)$ is a unit in $J(R\,;X)$. Moreover, by assumption, for every $m \in X$, $\exists\, n \in X$ such that

$$m + n + mn = 0.$$

This condition ensures that the element$(1, m) \in J(R\,;X)$ is a unit for any $m \in X$. Consequently, $(u, m) \in U_n(J(R\,;X))$. Therefore, every element $(v, m) \in J(R\,;X)$ can be represented as the sum of a unit and a periodic element. Hence $J(R\,;X)$ is semi-clean and thus semi-r-clean.
(iii) Since R is semi-r-clean, $b = g + p$, where $g \in R_g(R)$ and $p \in P_d(R)$. Therefore, $(b, m) \in J(R\,;X)$, and we can write $(b, m) = (g, m) + (p, 0)$. Since g is regular, $g = gsg$ for some $s \in R$. So $(g, m)(s, 0)(g, m) = (gsg, msg + msm + gms) = (g, m)$ for any $m \in X$ and $g, s \in R$. But $msg + msm + gms = m$. Therefore, $(g, m) \in R_g(J(R\,;X))$ and hence $J(R\,;X)$ is semi-r-clean.

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