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Full Paper

Tubular Mannheim partner surfaces with Frenet frame in Euclidean space $\mathbb{E}^3$

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Abstract: We define the tubular Mannheim partner surfaces. We calculate the Gaussian and mean curvature and investigate such surfaces for singularity and necessary and sufficient conditions for parameter curves on this surface to be asymptotic, geodesic and line of curvature. In addition, the principal curvatures of the tubular Mannheim partner surface are determined from the Gaussian and mean curvatures. Finally, an explicit example of a Mannheim curve and its partner curve is given; the tubular Mannheim partner surface is constructed and the obtained results are analysed.

Keywords: tubular Mannheim partner surface, tubular surface, Mannheim curve, Mannheim partner curve, Mannheim pair

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INTRODUCTION

Tubular surfaces, also known as pipe surfaces, are a fascinating subject of study in differential geometry. These surfaces can be seen as particular cases of channel surfaces introduced by Gaspar Monge [1] in 1850, who described such surfaces as the envelopes of a one-parameter family of moving spheres with a variable radius, tubular surfaces being the case in which the radius is constant.

Tubular surfaces have become an important area of research in mathematics and several practical areas. In engineering, for example, they are used to design and analyse pipelines, cables and other structural components [2]. In computer graphics tubular surfaces are employed to create three-dimensional models of long, slender elements with precision and efficiency [3] and are also widely used in computer-aided geometric design [4].

The study of tubular surfaces involves exploring various geometric and topological properties including the curvature of the central curve, torsion and how these characteristics influence the overall geometry of the resulting surface. In the context of three-dimensional Euclidean space, a tubular surface can be defined as the set of all points that are at a fixed distance from a central spatial curve, often referred to as the axis of the pipe. This configuration generates shapes that can resemble hoses, cylinders or flexible pipes, depending on the nature of the central curve and the radius of the pipe. Tubular surfaces have been studied from various perspectives. In the light of the Darboux and Bishop frames by Doğan and Yayli [5,6], necessary and sufficient conditions for the non-singularity of pipe surfaces were investigated by Maekawa et al. [7]. Canal surfaces and their principal geometric properties, such as the computation of area and Gaussian curvature, were examined by Xu and Feng [8]. Tubular surfaces in Galilean space were investigated by Dede [9] and Karacan and Tunçer [10] explored tubular surfaces of Weingarten types in Pseudo-Galilean space. Kızıltuğ et al. [11] examined tubular surfaces in the light of the Darboux frame in the Galilean 3-space.

Beyond tubular surfaces, recent studies have also focused on surfaces generated from special curves or containing such curves as notable geometric elements. The central problem in this context is to determine when a given curve (or a curve derived from it) turns out to be a geodesic, an asymptotic curve, a line of curvature, or the like. One of the pioneering contributions in this area was presented by Wang et al. [12], who constructed a family of parametric surfaces sharing a prescribed curve as a common geodesic through the use of marching-scale functions. Bayram and Bilici [13] constructed a family of surfaces that admits the involute of a given curve as an asymptotic curve, establishing necessary and sufficient conditions for this property to hold. Furthermore, they derived geometric conditions related to the involute under which a ruled surface exhibits this characteristic. Bilici and Köseoğlu [14,15] investigated what they termed an tubular involutive and involutive sweeping surfaces. The first is a tubular surface whose central curve is the involute of a regular curve with curvature function $\kappa \neq 0$, and the second is obtained by sweeping one curve along an involute. In these studies besides determining the conditions under which the surface is regular, the Gaussian and mean curvatures of the surface were computed, and necessary and sufficient conditions were established for its coordinate curves to be asymptotic curves, geodesics or lines of curvature. Additional studies involving tubular, involutive and ruled surfaces can be found in the literature [16-24].

Another relevant topic in differential geometry is the study of curves with special properties. Among these curves are the so-called Mannheim curves, introduced in 1878 by Amédée Mannheim [25]. If there exists a corresponding relationship between the space curves α and α^* such that at the corresponding points of the curves the principal normal lines of α coincide with the binormal lines of α^*, then α is called a Mannheim curve and α^* a Mannheim partner curve of α. The pair (α, α^*) is said to be a Mannheim pair. Mannheim curves are characterised by the relation $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a non-zero constant. On Mannheim curves, Liu and Wang [26] provided necessary and sufficient conditions for a curve to be a Mannheim partner curve in Euclidean 3-space and Minkowski 3-space. The Mannheim partner curves in Euclidean 3-space were studied by Orbay and Kasap [27], who obtained relationships between the curvatures and torsions of the curves forming a Mannheim pair. Slant helices that are Mannheim curves were explored by Çay and Yaylı [28]. Mendonça [29] analysed the correspondence between the Bishop frame of a curve in $\mathbb{E}^3$ and its

Mannheim partner curve, obtaining a relationship between their respective Bishop curvatures. Tul and Sarioğlugil [30] and Mendonça [31] defined and studied Mannheim curves in Heisenberg 3-spaces and in the special linear group respectively.

Inspired by the study carried out by Bilici and Köseoğlu [14], in this paper we study tubular surfaces that have the partner curve of a Mannheim curve as the central curve, and we will call such surfaces tubular Mannheim partner surfaces. Next, we present some geometric preliminaries on Mannheim curves and regular surfaces. Then we present the tubular Mannheim partner surface S, calculate the Gaussian, the mean curvatures, the principal curvatures and show results of dealing with the regularity of the surface, and show a relationship similar to that found by Bilici and Köseoğlu [14] in Theorem 3, between the Gaussian and mean curvatures of S. In addition, we give the conditions for the coordinate curves to be asymptotic curves, geodesics and lines of curvature. Finally, we present an explicit example of a Mannheim curve, its partner curve, the Frenet apparatus of both curves and the tubular Mannheim partner surface S^*, and we analyse this surface in light of the results obtained in this paper.

PRELIMINARIES

Consider the Euclidean space $\mathbb{E}^3$ with inner product

$$\langle \cdot,\cdot \rangle = dx^2 + dy^2 + dz^2,$$

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Let $\alpha: I \longrightarrow \mathbb{E}^3$ be a differentiable curve in the Euclidean space defined on an open interval I, parametrised by arc-length and let $\{T_\alpha = \alpha', N_\alpha, B_\alpha\}$ be the Frenet frame satisfying [32]

$$\begin{cases} T'_\alpha = \kappa_\alpha N_\alpha, \\ N'_\alpha = -\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha, \\ B'_\alpha = -\tau_\alpha N_\alpha, \end{cases} \tag{1}$$

where κ_α and τ_α are differentiable functions on I called the curvature and the torsion of α respectively, T_α is the tangent vector, N_α is the principal normal vector and B_α is the binormal vector of α.

Definition 1. If there exists a corresponding relationship between the space curves $\alpha(s)$ and $\alpha^*(s^*)$, with $s^* = s^*(s)$ being a differentiable and injective function, such that at the corresponding points of the curves, the principal normal lines of α coincide with the binormal lines of α^*, then α is called a Mannheim curve and α^* a Mannheim partner curve of α. The pair (α, α^*) is said to be a Mannheim pair. We denote by $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_{\alpha^*}, N_{\alpha^*}, B_{\alpha^*}\}$ their respective Frenet frames.

Remark 1. If (α, α^*) is a Mannheim pair, then there exists a non-zero constant λ such that the relation $\alpha^* = \alpha + \lambda N_\alpha$ holds [27].

Lemma 1, obtained through straightforward calculations based on Theorems 3 and 5 of Orbay and Kasap [27], establishes a relationship between the Frenet apparatus of the Mannheim pair, which is extensively used in the subsequent calculations.

Lemma 1. Let $\{T_\alpha, N_\alpha, B_\alpha, \kappa, \tau\}$, $\{T_{\alpha^*}, N_{\alpha^*}, B_{\alpha^*}, \kappa^*, \tau^*\}$ be the Frenet apparatus of the Mannheim pair (α, α^*) respectively, and $\Psi(s, s^*) = \angle\big(T_\alpha(s), T_{\alpha^*}(s^*)\big)$. Then

$$T_\alpha = cos\,\Psi\ T_{\alpha^*} + sin\,\Psi\ N_{\alpha^*}, \tag{2}$$

$$B_\alpha = -\,sin\,\Psi\ T_{\alpha^*} + cos\,\Psi\ N_{\alpha^*}, \tag{3}$$

$$T_{\alpha^*} = cos\,\Psi\ T_\alpha - sin\,\Psi\ B_\alpha, \tag{4}$$

$$N_{\alpha^*} = \sin\Psi\ T_\alpha + \cos\Psi\ B_\alpha, \tag{5}$$

$$\kappa^* = -\frac{d\Psi}{ds^*}, \qquad \tau^* = -\frac{1}{\lambda}\tan\Psi, \tag{6}$$

$$\tau^* = \frac{\kappa}{\lambda\tau}, \qquad \cos\Psi = \frac{ds^*}{ds}, \tag{7}$$

$$\kappa = \frac{1}{\lambda}\sin^2\Psi, \qquad \tau = -\frac{1}{\lambda}\sin\Psi\ \cos\Psi, \tag{8}$$

$$\tau^* = -\kappa\tan\Psi + \tau. \tag{9}$$

Remark 2. Since the angle Ψ measures the rotation between the vectors T_α and T_{α^*} and it is a property of the geometric point of the curve, not of the parametrisation, when solving the system given by equations (2) and (3) (which is invertible), we obtain Ψ expressed in terms of the known vectors. These vectors are functions of s on the left-hand side of system and of s^* on the right-hand side of system. The consistency of the system forces Ψ to depend on both parameters, but thanks to the relation $s^* = s^*(s)$, we can write $\Psi(s,s^*) = \Psi\big(s,s^*(s)\big) = \theta(s)$ and also $\Psi(s,s^*) = \Psi(s(s^*),s^*) = \theta(s^*)$.

Regular surfaces $S \subset \mathbb{E}^3$ and their fundamental properties are central concepts in differential geometry. Locally, a regular surface S can be described through a parametrisation $\sigma(s,\vartheta)$, where (s,ϑ) are parameters that vary in an open region of the plane R^2.

To study the intrinsic and extrinsic properties of regular surfaces such as area, angle and length on a regular surface, we use the first fundamental form $I_p(w) = \langle w,w\rangle$, whose coefficients at $\sigma(s,\vartheta)$ are given by [33]

$$E = \langle\sigma_s,\sigma_s\rangle,\ F = \langle\sigma_s,\sigma_\vartheta\rangle \ \text{ and } \ G = \langle\sigma_\vartheta,\sigma_\vartheta\rangle.$$

To study the deviation of the regular surface from the tangent plane we use the second fundamental form $II_p(w) = -\langle d\eta_p(w),w\rangle$, where $\eta = \frac{\sigma_s\times\sigma_\vartheta}{\|\sigma_s\times\sigma_\vartheta\|}$ is the normal vector field of the surface S at $\sigma(s,\vartheta)$. The coefficients of the second fundamental form are given by [33]

$$e = \langle\eta,\sigma_{ss}\rangle, \quad f = \langle\eta,\sigma_{s\vartheta}\rangle \ \text{ and } \ g = \langle\eta,\sigma_{\vartheta\vartheta}\rangle.$$

Important geometrical concepts that describe how a surface curves around a specific point are the Gaussian and mean curvatures [32]. Geometrically, the mean curvature measures the average tendency of the surface to curve in one direction or another, whereas Gaussian curvature describes how the surface curves in all possible directions around the point in question. The Gaussian curvature and the mean curvature of the surface S can be calculated by

$$K = \frac{eg-f^2}{EG-F^2} \quad \text{and} \quad H = \frac{Eg-2Ff+eG}{2(EG-F^2)}, \tag{10}$$

respectively.

A canal surface is a surface formed as the envelope of a family of spheres whose centres lie on a space curve $\alpha(s)$, its directrix or central curve. If the radius of the generated spheres are constant, the canal surface is called a pipe surface or tubular surface, and a parametrisation for this surface is given by

$$\sigma(s,\vartheta) = \alpha(s) + r\big(\cos\vartheta\ N_\alpha(s) + \sin\vartheta\ B_\alpha(s)\big), s \in I \text{ and } \vartheta \in (0,2\pi), \tag{11}$$

where $r \in R, r > 0$ is the radius of tubular surface.

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PROPERTIES OF TUBULAR MANNHEIM PARTNER SURFACES

In this section the geometric properties of the tubular Mannheim partner surfaces are examined. We assume that the angular function θ is non-constant and with isolated singular points. The regularity conditions are established, and the points at which the Gaussian and mean curvatures vanish are identified. Furthermore, the conditions for a parameter curve to be a geodesic, an asymptotic curve and a line of curvature are investigated.

Using Remark 1 and equations (4) and (5), we can write

$$\begin{aligned} \sigma^*(s^*,\vartheta) &= \alpha^*(s^*) + r(\cos\vartheta\ N_{\alpha^*} + \sin\vartheta\ B_{\alpha^*}) \\ &= \alpha(s) + r\cos\vartheta\sin\theta\ T_\alpha + (\lambda - r\sin\vartheta)\, N_\alpha + r\cos\vartheta\cos\theta\ B_\alpha, \end{aligned} \tag{12}$$

with $s \in I$ and $\vartheta \in (0,2\pi)$.

If we take the partial derivatives of (12) with respect to s and ϑ we get the following tangent vectors:

$$\begin{aligned} \sigma^*{}_{s^*} &= \left(\cos\theta + r\theta'\cos\vartheta + \frac{r}{\lambda}\sin\vartheta\sin\theta\tan\theta\right) T_\alpha + \frac{r}{\lambda}\cos\vartheta\tan\theta\ N_\alpha \\ &\quad + \left(-\sin\theta - r\theta'\tan\theta\cos\vartheta + \frac{r}{\lambda}\sin\vartheta\sin\theta\right) B_\alpha, \end{aligned} \tag{13}$$

$$\sigma^*{}_\vartheta = -r\sin\vartheta\sin\theta\, T_\alpha - r\cos\vartheta\ N_\alpha - r\sin\vartheta\cos\theta\, B_\alpha, \tag{14}$$

where the " $'$ " denotes the derivative with respect to s. From (13) and (14), the coefficients of the first fundamental form are

$$E^* = (1 + r\theta'\sec\theta\cos\vartheta)^2 + \left(\frac{r}{\lambda}\tan\theta\right)^2, \tag{15}$$

$$F^* = -\frac{r^2}{\lambda}\tan\theta, \tag{16}$$

$$G^* = r^2. \tag{17}$$

By taking the vector product of $\sigma^*{}_{s^*}$ and $\sigma^*{}_\vartheta$, we get the normal vector field of the tubular Mannheim partner surface S as

$$\eta = -\cos\vartheta\sin\theta\, T_\alpha + \sin\vartheta\ N_\alpha - \cos\vartheta\cos\theta\, B_\alpha. \tag{18}$$

Thus, we can give the following result.

Lemma 2. The tubular Mannheim partner surface S^* has singular points if and only if

$$1 + r\theta'\sec\theta\cos\vartheta = 0. \tag{19}$$

Proof. The tubular Mannheim partner surface S^* has singular points if and only if $E^*G^* - F^{*2} = 0$. By equations (15), (16) and (17), we get

$$E^*G^* - F^{*2} = r^2(1 + r\theta'\sec\theta\cos\vartheta)^2.$$

Since $r > 0$, we have $E^*G^* - F^{*2} = 0 \Leftrightarrow 1 + r\theta'\sec\theta\cos\vartheta = 0.$ □

Theorem 1 follows directly from Lemma 2:

Theorem 1. The tubular Mannheim partner surface S^* is a regular surface if and only if

$$1 + r\theta'\sec\theta\cos\vartheta \neq 0.$$

Proof. The tubular Mannheim partner surface S^* is a regular surface if and only if $E^*G^* - F^{*2} \neq 0$. The proof follows directly from Lemma 2. □

The second-order partial derivatives of the parametrisation σ^* of S^* are

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$$\sigma^*{}_{s^*s^*} = \sec\theta\Big(\Big[\theta'\sin\theta\Big(\frac{r}{\lambda}\sin\vartheta\,(1+\sec^2\theta)-1\Big)+r\theta''\cos\vartheta-\frac{r}{\lambda^2}\sin^2\theta\tan\theta\cos\vartheta\Big]T_\alpha$$
$$+\Big[\frac{r}{\lambda^2}\sin\theta\tan\theta\sin\vartheta+\frac{r}{\lambda}\theta'\sec^2\theta\cos\vartheta\Big]N_\alpha$$
$$-\Big[r\tan\theta\cos\vartheta\Big(\frac{1}{\lambda^2}\sin\,\theta\cos\theta+\theta''\Big)$$
$$-\theta'\cos\theta\Big(\frac{r}{\lambda}\sin\vartheta-1\Big)+r(\theta')^2\sec^2\,\theta\cos\vartheta\Big]B_\alpha\Big), \tag{20}$$

$$\sigma^*{}_{s^*\vartheta} = \Big[-r\,\theta'\sin\vartheta-\frac{r}{\lambda}\sin\theta\tan\theta\cos\vartheta\Big]T_\alpha-\frac{r}{\lambda}\tan\theta\,\sin\vartheta\,N_\alpha$$
$$+\Big[r\theta'\tan\theta\sin\vartheta+\frac{r}{\lambda}\sin\theta\cos\vartheta\Big]B_\alpha, \tag{21}$$

$$\sigma^*{}_{\vartheta\vartheta} = -r\cos\vartheta\sin\theta\,T_\alpha+r\sin\vartheta\,N_\alpha-r\cos\vartheta\cos\theta\,B_\alpha. \tag{22}$$

From (20), (21) and (22), the coefficients of the second fundamental form are

$$e^* = \theta'\sec\theta\cos\vartheta\,(1+r\theta'\sec\theta\cos\vartheta)+\frac{r}{\lambda^2}\tan^2\theta, \tag{23}$$
$$f^* = -\frac{r}{\lambda}\tan\theta, \tag{24}$$
$$g^* = r. \tag{25}$$

At points where the surface is regular, we can give the following Theorem.

Theorem 2. The Gaussian and mean curvatures of the tubular Mannheim partner surface S^* at $\sigma^*(s^*,\vartheta)$ are, respectively,

$$K^* = \frac{\theta'\sec\theta\cos\vartheta}{r(1+r\theta'\sec\theta\cos\vartheta)}, \tag{26}$$
$$H^* = \frac{1+2r\theta'\sec\theta\cos\vartheta}{2r(1+r\theta'\sec\theta\cos\vartheta)}. \tag{27}$$

Proof. Replacing $(15), (16), (17), (23), (24)$ and (25) in (10), the Gaussian and mean curvatures of the tubular Mannheim partner surface are obtained through a direct calculation. □

Corollary 1. The Gaussian and mean curvatures of the tubular Mannheim partner surface S^* satisfy the relation

$$H^* = \frac{1}{2}\Big(K^*r+\frac{1}{r}\Big). \tag{28}$$

In particular, S^* is a linear Weingarten surface (K^* and H^* satisfy $aK^*+bH^*=c$, where a, b and c are constants).

Corollary 2. The principal curvatures of the tubular Mannheim partner surface S^* are given by

$$\kappa^*{}_1 = K^*r \qquad \text{and} \qquad \kappa^*{}_2 = \frac{1}{r}. \tag{29}$$

Proof. Since the principal curvatures are given by

$$\kappa^*{}_1 = H^*+\sqrt{H^{*2}-K^*} \qquad \text{and} \qquad \kappa^*{}_2 = H^*-\sqrt{H^{*2}-K^*},$$

using (28) we obtain $\kappa^*{}_1 = K^*r$ and from $H^* = \frac{\kappa^*{}_1+\kappa^*{}_2}{2}$ it follows that $\kappa^*{}_2 = \frac{1}{r}$. □

Corollary 3. The set of points on the tubular Mannheim partner surface S^* where the Gaussian curvature vanishes is given by s-parameter curves $\vartheta = \frac{\pi}{2}$ and $\vartheta = \frac{3\pi}{2}$.

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Corollary 4. The set of points on the tubular Mannheim partner surface S^* where the mean curvature vanishes are $\sigma^*(s^*, \vartheta)$, where (s, ϑ) satisfies the equation

$$1 + 2r\theta'(s)\sec\theta(s)\cos\vartheta = 0. \tag{30}$$

Theorem 3. Let S^* be tubular Mannheim partner surface in $\mathbb{E}^3$.

(i) The *s*-parameter curve of S^* is an asymptotic curve if and only if $\vartheta = \vartheta_0$ satisfies the equation

$$\theta'(s)\sec\theta(s)\cos\vartheta\,(1 + r\theta'(s)\sec\theta(s)\cos\vartheta) + \frac{r}{\lambda^2}\tan^2\theta(s) = 0. \tag{31}$$

(ii) None of the ϑ-parameter curves of S^* can be an asymptotic curve.

Proof. (i) The *s*-parameter curve of S^* is an asymptotic curve if and only if, at $\sigma^*(s^*, \vartheta_0)$,

$$\langle \eta, {\sigma^*}_{s^* s^*} \rangle = 0 \quad \Leftrightarrow \quad e^* = 0.$$

The proof follows from (23).

(ii) The ϑ-parameter curve of S^* is an asymptotic curve if and only if, at $\sigma^*({s^*}_0, \vartheta)$,

$$\langle \eta, {\sigma^*}_{\vartheta\vartheta} \rangle = 0 \quad \Leftrightarrow \quad g^* = 0.$$

Since $g^* = r > 0$, we obtain (ii). □

Theorem 4. Let S^* be a tubular Mannheim partner surface in $\mathbb{E}^3$.

(i) None of the s-parameter curves of S^* are geodesics.

(ii) Any ϑ-parameter curve of S^* is a geodesic.

Proof. (i) A necessary and sufficient condition for an *s*-parameter curve of S^* to be a geodesic is that, at $\sigma^*(s^*, \vartheta_0)$,

$$\eta \times {\sigma^*}_{s^* s^*} = 0 \quad \Leftrightarrow \quad ({\sigma^*}_{s^*} \times {\sigma^*}_{\vartheta}) \times {\sigma^*}_{s^* s^*} = 0,$$

which implies

$$E^*_{s^*} = 0 \tag{32}$$

and

$$E^*_{\vartheta} = 2F^*_{s^*}. \tag{33}$$

The equations (32) and (33) are equivalent, respectively, to

$$(1 + r\theta'\sec\theta\cos\vartheta)r\cos\vartheta\;(\theta'\sec\theta)' + \frac{r^2}{\lambda^2}\theta'\sec\theta\tan\theta = 0, \tag{34}$$

$$\sin\vartheta\,(1 + r\theta'\sec\theta\cos\vartheta) = \frac{r}{\lambda}\sec^2\theta. \tag{35}$$

Differentiating (35) with respect to *s* and substituting in (34), we obtain

$$1 + r\theta'\sec\theta\cos\vartheta = -\frac{r}{2\lambda}\sin\vartheta. \tag{36}$$

Using (36) in (35), we get θ = constant. Since θ is not constant, the proof is completed.

(ii) A necessary and sufficient condition for a ϑ-parameter curve of S^* to be a geodesic is that, at $\sigma^*({s^*}_0, \vartheta)$,

$$\eta \times {\sigma^*}_{\vartheta\vartheta} = 0.$$

From equations (18) and (22), we have

$$\eta \times {\sigma^*}_{\vartheta\vartheta} = \eta \times (r\,\eta) = 0.$$

The proof follows. □

Theorem 5. The parameter curves of the tubular Mannheim partner surface S^* in $\mathbb{E}^3$ cannot be lines of curvature.

Proof. A necessary and sufficient condition for a parameter curve of S^* to be a line of curvature is that

$$F^* = f^* = 0.$$

From equations (16) and (24), at an arbitrary point $\sigma^*(s^*, \vartheta)$, we have

$$F^* \neq 0 \quad \text{and} \quad f^* \neq 0.$$

The proof follows. □

EXAMPLE

The parametrisation of slant helices that are Mannheim curves, together with an explicit example, was intrinsically presented by Çay and Yaylı [28]. Since a slant helix is a Mannheim curve if and only if its Mannheim partner is a general helix [34], we present in this section an example of a Mannheim pair of this type. The tubular surface is constructed around the partner curve and the results established are applied to make explicit some important curves on this surface. The tubular Mannheim partner surface and the corresponding curves are illustrated using the Mathematica software with $r = \frac{1}{5}$.

Example 1. Let $\alpha(s)$ be a Slant-Mannheim curve (Figure 1) given by the parametric equation:

$$\alpha(s) = \frac{1}{2}\Big(ln(sech\, s) - tanh^2 s, arcsin(sech\, s) + sech\, s\, tanh\, s\,, \sqrt{3}\, ln(sech\, s) + \frac{\sqrt{3}}{3}\Big),$$

with Frenet apparatus

$$T_\alpha = \frac{1}{2}\big(2\, tanh^3 s - 3\, tanh\, s\,, 2\, sech^3 s, -\sqrt{3}\, tanh\, s\big),$$
$$N_\alpha = \frac{1}{2}\big(\sqrt{3}\,(2\, tanh^2 s - 1), -2\sqrt{3}\, sech\, s\, tanh\, s, -1\big),$$
$$B_\alpha = \frac{1}{2}\big(2\, sech^3 s - 3\, sech\, s\,, -2\, tanh^3 s, \sqrt{3}\, sech\, s\big),$$
$$\kappa = \sqrt{3}\, sech^2 s, \qquad \tau = -\sqrt{3}\, tanh\, s\, sech\, s,$$

where $s < 0$. Its Mannheim partner curve (Figure 1) is the general helix given by

$$\alpha^*(s^*) = \frac{1}{2}\,\big(s^* - e^{2s^*}, e^{s^*}\sqrt{1 - e^{2s^*}} + arcsin\big(e^{s^*}\big), \sqrt{3}\, s^*\big),$$

with Frenet apparatus

$$T_{\alpha^*} = \frac{1}{2}\big(1 - 2\, e^{2s^*}, 2\, e^{s^*}\sqrt{1 - e^{2s^*}}\,, \sqrt{3}\big),$$
$$N_{\alpha^*} = \frac{1}{2}\big(-4e^{s^*}\sqrt{1 - e^{2s^*}}, 2 - 4\, e^{2s^*}\,, 0\big),$$
$$B_{\alpha^*} = \frac{1}{2}\big(-\sqrt{3}\,\big(1 - 2\, e^{2s^*}\big), -2\sqrt{3}\, e^{s^*}\sqrt{1 - e^{2s^*}}\,, 1\big),$$
$$\kappa^* = \frac{e^{s^*}}{\sqrt{1 - e^{2s^*}}}, \qquad \tau^* = \frac{\sqrt{3}\, e^{s^*}}{\sqrt{1 - e^{2s^*}}}.$$

Here, $s^* < 0$, with $s^*(s) = ln\big(sech(s)\big)$.

Using equation (2), we get the theta function of the pair (α, α^*) given by

$$\theta(s) = arctan(csch\, s)\,, \qquad \forall s < 0. \tag{37}$$

The distance between α and α^* is $\lambda = \frac{\sqrt{3}}{3}$.

Using equation (12), a parametrisation of the tubular surface S with central curve α and that of the tubular Mannheim partner surface S^* with central curve α^* (Figures 2 and 3 respectively) are

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$$\sigma(s,\vartheta) = \frac{1}{2}\begin{bmatrix} ln(sech\,s) - tanh^2 s + r\sqrt{3}\,(2\,tanh^2 s - 1)cos\,\vartheta + r\,sech\,s(2\,sech^2 s - 3)\,sin\,\vartheta \\ arcsin(sech\,s) + sech\,s\,tanh\,s\big(1 - 2\sqrt{3}\,r\,cos\,\vartheta\big) - 2\,r\,tanh^3 s\ sin\,\vartheta \\ \sqrt{3}ln(sech\,s) - r\,cos\,\vartheta + \sqrt{3}\,r\,sech\,s\,sin\,\vartheta \end{bmatrix}$$

and

$$\sigma^*(s^*,\vartheta) = \frac{1}{2}\begin{bmatrix} ln(sech\,s) - sech^2 s + 4r\,sech\,s\,tanh\,s\,cos\,\vartheta + \sqrt{3}r(2\,sech^2 s - 1)\,sin\,\vartheta \\ arcsin(sech\,s) - 2r(2sech^2 s - 1)cos\,\vartheta + \big(2\sqrt{3}\,r\,sin\,\vartheta - 1\big)sech\,s\,tanh\,s \\ \sqrt{3}ln(sech\,s) + r sin\,\vartheta \end{bmatrix},$$

where $r > 0$ is constant, $s < 0$ and $\vartheta \in (0,2\pi)$.

Calculating the coefficients of the first and second fundamental forms of S^*, we obtain

$$E^* = [(rcos\vartheta + sinh\,s)^2 + 3r^2]\,csch^2\,s, \quad F^* = -\sqrt{3}\,r^2\,csch\,s, \quad G^* = r^2,$$
$$e^* = [cos\vartheta\,(rcos\vartheta + sinh\,s) + 3r]\,csch^2\,s, \quad f^* = -\sqrt{3}\,r\,csch\,s, \quad g^* = r.$$

The Gaussian and mean curvatures are given by

$$K^* = \frac{cos\vartheta}{r\,(rcos\vartheta + sinh\,s)} \quad \text{and} \quad H^* = \frac{2rcos\vartheta + sinh\,s}{2r\,(rcos\vartheta + sinh\,s)}.$$

In light of the results found previously, we obtain the following:

i) Using (19), we show that the singular points of S^* are the points of the curve (Figure 4),

$$\mathrm{s}(\vartheta) = \ln\big(\sqrt{1 + \mathrm{r}^2\cos^2\vartheta} - \mathrm{r}\cos\vartheta\big), \text{where } (0,\pi/2) \cup (3\pi/2,2\pi) \text{ is its domain.}$$

ii) From Corollary 3, we show that the flat points on the surface are the s-parameter curves given by (Figure 5)

$$\sigma^*\left(s^*,\frac{\pi}{2}\right) = \frac{1}{2}\begin{bmatrix} ln(sech\,s) + (2\sqrt{3}\,r - 1)sech^2\,s - \sqrt{3}\,r \\ arcsin(sech\,s) + \big(2\sqrt{3}r - 1\big)sech\,s\,tanh\,s \\ \sqrt{3}ln(sech\,s) + r \end{bmatrix},$$

$$\sigma^*\left(s^*,\frac{3\pi}{2}\right) = \frac{1}{2}\begin{bmatrix} ln(sech\,s) - \big(2\sqrt{3}\,r + 1\big)sech^2 s + \sqrt{3}\,r \\ arcsin(sech\,s) - \big(2\sqrt{3}\,r + 1\big)sech\,s\,tanh\,s \\ \sqrt{3}ln(sech\,s) - r \end{bmatrix}.$$

iii) Using (37) in (30), we find that the points that nullify the mean curvature lie on the curve given by the equations (Figure 6):

$$s(\vartheta) = ln(\sqrt{1 + 4r^2\,cos^2\,\vartheta} - 2r\,cos\,\vartheta), \quad \text{where} \quad \vartheta \in (0,\pi/2) \cup (3\pi/2,2\pi),$$

$$\sigma^*(s^*,\vartheta) = \frac{1}{2}\begin{bmatrix} -\frac{1}{2}ln(1 + 4r^2\,cos^2\,\vartheta) + \dfrac{(1 - 4r^2\,cos^2\,\vartheta)\big(2 + \sqrt{3}\,r\,sin\,\vartheta\big) - 3}{1 + 4r^2\,cos^2\,\vartheta} \\ arcsin\left(\dfrac{1}{\sqrt{1 + 4r^2\,cos^2\,\vartheta}}\right) + \dfrac{8r^3\,cos^3\,\vartheta - 2\sqrt{3}r^2\,sin(2\vartheta)}{1 + 4r^2\,cos^2\,\vartheta} \\ -\frac{\sqrt{3}}{2}ln(1 + 4r^2\,cos^2\,\vartheta) + r\,sin\,\vartheta \end{bmatrix}.$$

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iv) Substituting (37) in (31), we show that an s-parameter curve is an asymptotic if and only if there exists $\vartheta = \vartheta_0$ constant satisfying the equation

$$\cos\vartheta\ (1 + r\,\mathrm{csch}\, s \cos\vartheta) + 3\,r\,\mathrm{csch}\, s = 0, \qquad \forall\, s < 0. \tag{38}$$

Note that (38) is a quadratic equation in $\cos\vartheta$ that has no solution for $\vartheta = \vartheta_0$ and $\forall\, s < 0$. Therefore, none of the s-parameter curves of S^* is an asymptotic curve.

v) Consider $s = s_0 = \ln(\sqrt{2} - 1)$ in the surface parametrisation. Thus, we obtain a geodesic given by the ϑ-parameter curve with equation (Figure 7)

$$\sigma^*({s^*}_0, \vartheta) = \frac{1}{2}\begin{bmatrix} -\frac{1}{2}\ln 2 - \frac{1}{2} - 2\,r\cos\vartheta \\ \frac{\pi}{4} + \frac{1}{2} - \sqrt{3}\,r\sin\vartheta \\ -\frac{\sqrt{3}}{2}\ln 2 + r\sin\vartheta \end{bmatrix}, \quad \vartheta \in (0, 2\pi).$$

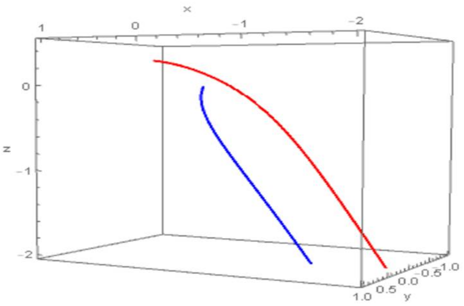

Figure 1. Mannheim pair (α, α^*)

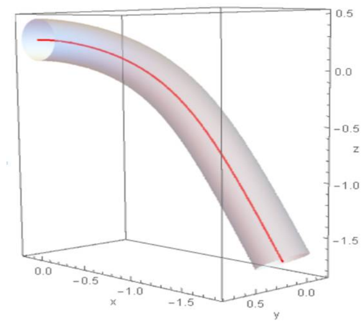

Figure 2. Tubular surface S

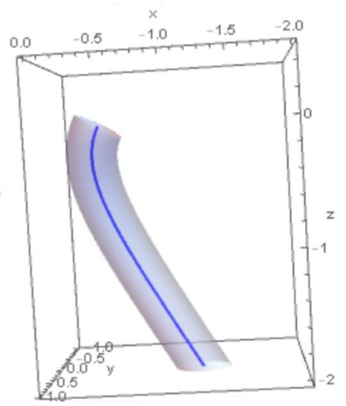

Figure 3. Tubular Mannheim partner surface S^*

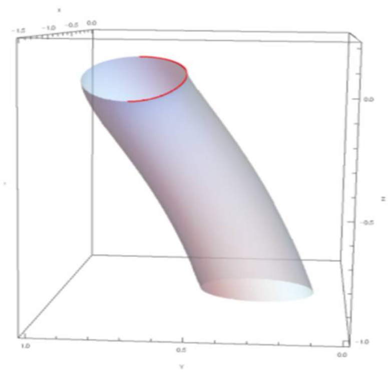

Figure 4. Singular points

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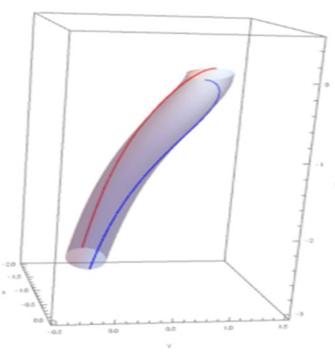

Figure 5. Flat points $\vartheta = \frac{\pi}{2}$ and $\vartheta = \frac{3\pi}{2}$

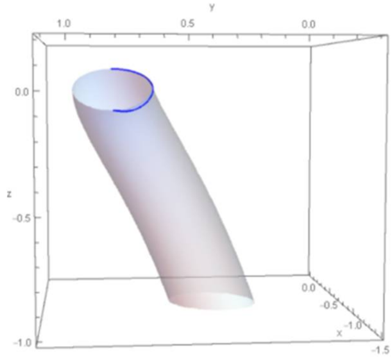

Figure 6. Points where mean curvature vanishes

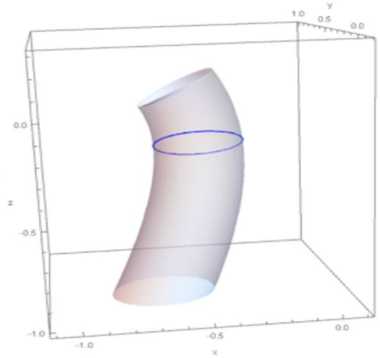

Figure 7. Geodesic at $s_0 = ln(\sqrt{2} - 1)$

CONCLUSIONS

Motivated by the study presented by Bilici and Köseoğlu [14], we investigate tubular surfaces whose central curve is a Mannheim partner curve, referred to as tubular Mannheim partner surfaces. Using the properties and relationships between a Mannheim curve and its partner described by Orbay and Kasap [27], we calculate the coefficients of the first and second fundamental forms in terms of the arc-length parameter of a Mannheim curve and the radial parameter of the tube. We analyse the regularity of the surface and compute its Gaussian, mean and principal curvatures. Moreover, we demonstrate that these surfaces are Weingarten surfaces, as shown in Corollary 1. We also investigate the conditions under which the coordinate curves of this surface can be asymptotic lines, geodesics or lines of curvature. Finally, we present an explicit example of a Mannheim pair with their respective Frenet apparatus and the tubular surface constructed around the partner curve, analysing this surface in light of the results obtained.

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