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Full Paper

On the structure of duals in paranormed sequence spaces of quaternions

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Abstract: This study develops a duality theory for paranormed sequence spaces whose terms take values in the non-commutative algebra of quaternions. Starting from Maddox-type positive exponent sequences, we introduce quaternion-valued analogues of the bounded, convergent, null and p-absolutely summable sequence classes. We then determine the corresponding α-, β- and γ-duals of these spaces. The main novelty of the study is that the duality problem is not treated as a direct scalar-field replacement of the classical real or complex case. Instead, the non-commutativity of quaternion multiplication is explicitly incorporated into the structure of the dual spaces. In particular, the convergence and boundedness conditions appearing in the definitions of the β- and γ-duals require a distinction between left and right multiplication, a phenomenon absent from classical sequence-space theory over commutative scalar fields. The obtained results extend Maddox-type paranormed sequence-space duality to the quaternionic setting and clarify the joint role of the paranorm topology and the order of quaternionic multiplication.

Keywords: paranormed sequence spaces, quaternion space, quaternion numbers, α-dual, β-dual, γ-dual.

INTRODUCTION

Quaternions serve as an extension of complex numbers, offering significant insights through both algebraic and geometrical frameworks. While complex numbers effectively represent points or rotations within the two-dimensional xy-plane, their algebraic structure inherently reflects the commutative nature of $2D$ rotations. That is, the sequence in which rotations are applied does not alter the final orientation, mirroring the complex multiplication rule: $z_1z_2 = z_2z_1$.

However, this commutative property fails to generalise to three-dimensional space. In $3D$, the sequence in which rotations are executed significantly impacts the outcome, demonstrating that spatial rotations are generally non-commutative. For instance, it can be readily shown that a $90°$ rotation about the x-axis followed by a $90°$ rotation about the y-axis yields a different result than performing these rotations in the reverse order.

The transition from real to complex numbers involves discarding the concept of order while preserving associativity and commutativity in arithmetic operations. Recognising the limitations of complex numbers in modelling $3D$ dynamics leads to the realisation that any algebraic system designed to describe such rotations must inherently accommodate a non-commutative product.

Introduced by the Irish mathematician William Rowan Hamilton in 1843, quaternions provide this necessary extension. Foundational discussions on quaternion analysis were done by Hamilton [1], whose work generalises complex numbers to a four-component structure—spanned by the basic elements $1, \mathrm{i}, \mathrm{j}$ and k—creating a robust mathematical framework essential for representing spatial rotations.

Beyond their role in theoretical mathematics, quaternions have substantial utility in applied fields, particularly where computations involving three-dimensional rotations are necessary. Applications include $3D$ computer graphics, robotics, computer vision, magnetic resonance imaging [2] and crystallographic texture analysis [3]. The body of literature on quaternions is vast, ranging from Hamilton's original texts [1] to modern explorations such as a significant study by Ward [4]. Related quaternionic and quaternion-type algebraic structures have also been studied from different viewpoints. For instance, Ercan and Yüce [5] investigated several properties of dual quaternions including extensions of Euler's and De Moivre's formulas and matrix representations. Such works indicate that quaternion-related number systems often require algebraic tools that are different from those used in the real or complex settings. In the present paper, however, our focus is not on dual quaternions, but on sequence spaces whose terms belong to the quaternion algebra itself and on the manner in which non-commutative quaternion multiplication changes the form of Köthe-Toeplitz-type duality.

Analysing sequence spaces over quaternion algebra, as opposed to the real or complex fields, introduces additional theoretical difficulties. The main difficulty stems from the non-commutativity of quaternion multiplication. In classical real or complex sequence spaces, the order of multiplication does not affect the duality definitions since $x_k a_k = a_k x_k$. In the quaternionic setting, however, the products $x_k a_k$ and $a_k x_k$ are generally different. Therefore, the convergence of series and the boundedness of partial sums occurring in the definitions of the β- and γ-duals depend essentially on whether multiplication is taken from the left or from the right. This makes it necessary to distinguish left and right dual structures.

The literature on sequence spaces and their duals has a long development. Classical investigations of Köthe-Toeplitz duals, matrix transformations, multiplier spaces and Maddox-type sequence spaces laid the foundation for the modern theory of α-, β- and γ-duals [6-16]. These works mainly consider sequence spaces over real or complex scalar fields and provide the standard tools for studying convergence, summability, matrix domains and perfection properties. In particular, Maddox-type spaces generated by positive exponent sequences form an important class of paranormed sequence spaces since the exponent sequence controls the underlying topology and produces spaces that are generally not normable in the usual sense.

In more recent studies, paranormed sequence spaces have been developed in several directions including Riesz and weighted mean sequence spaces, generalised difference sequence spaces, Euler and Fibonacci-type constructions, matrix domains, double sequence spaces and related matrix transformations [17-25]. Another active line of research studies matrix-generated paranormed sequence spaces obtained from special matrices such as the Tribonacci, Catalan, Catalan-Motzkin, Lambda-Pascal, Narayana, Schröder and Motzkin matrices [26-34]. In addition to these developments, Kadak et al. [35] investigated classical paranormed sequence spaces and their related duals over the non-Newtonian complex field, showing that duality questions can also be formulated in non-classical scalar frameworks. Singh and Malik [36] studied paranormed Nörlund N^t-difference sequence spaces and determined their α-, β- and γ-duals together with Schauder bases. Topological aspects of generalised paranormed sequence spaces have also been considered in the setting of total paranormed double sequence spaces [37].

Recent developments based on q-calculus provide another important direction in the theory of paranormed sequence spaces. Srivastava et al. [38] introduced the q-analogue of the paranormed Cesàro sequence spaces obtained as matrix domains in Maddox-type spaces, and investigated their Schauder bases, α-, β- and γ--duals, and matrix transformations. Ellidokuzoğlu and Demiriz [39] introduced new paranormed sequence spaces derived by the q-second difference matrix and studied their algebraic, topological and duality properties. These papers typically introduce new domains of triangular, conservative, difference or q-analogue matrices in Maddox-type spaces, examine their topological properties, determine Schauder bases, compute α-, β- and γ-duals, and characterise certain matrix classes. Thus, the recent literature shows that the computation of dual spaces remains a central problem in the theory of paranormed sequence spaces.

Köthe-Toeplitz duality has also been extended beyond the most classical scalar setting. For example, Dutta [40] investigated Köthe-Toeplitz duals of certain n-normed valued difference sequence spaces. Such studies indicate that sequence-space duality can be adapted to richer ambient structures. Nevertheless, these generalisations still do not address the specific algebraic complication produced by quaternionic multiplication. In particular, they do not require a systematic distinction between products of the forms $x_k a_k$ and $a_k x_k$.

On the other hand, quaternionic functional analysis has developed as an independent research direction. Quaternionic vector spaces, one-sided and two-sided quaternionic Hilbert spaces, quaternionic operator theory and slice hyperholomorphic functional calculus have been studied [41-43]. These studies show that left and right scalar structures are not merely notational choices in quaternionic analysis; rather, they are part of the mathematical structure. However, the existing quaternionic functional-analytic literature does not provide a detailed treatment of Maddox-type paranormed quaternion sequence spaces and their α-, β- and γ-duals.

Consequently, there remains a precise gap between two well-developed lines of research. The first line is the theory of paranormed sequence spaces and their duals, which is almost entirely developed over commutative scalar fields. The second line is quaternionic functional analysis, where one-sided and two-sided scalar structures are essential, but Maddox-type sequence spaces and their Köthe-Toeplitz-type duals have not been systematically studied. The present paper is situated exactly between these lines.

The contribution of the present paper is therefore not merely a formal extension of known real or complex results to quaternion-valued sequences. The essential point is that the quaternionic setting changes the nature of the duality problem itself. The novelty of the paper is threefold. First, we introduce quaternion-valued paranormed sequence spaces associated with a bounded positive exponent sequence $p = (p_k)$. Second, we determine the corresponding α-, β- and γ-duals for the basic quaternion-valued classes of bounded, convergent, null and p-absolutely summable sequences. Third, we clarify how the order of quaternionic multiplication naturally leads to left and right β- and γ-dual structures, a phenomenon absent from the classical real and complex theories.

PRELIMINARIES

Let ω represent the space comprising all sequences of real numbers. Any vector subspace of ω is termed a sequence space. The standard notations ℓ_∞, c and c_0 denote the spaces of all bounded sequences, convergent sequences and null sequences (i.e. converging to zero) respectively.

A paranormed space is defined as a linear topological space X over the real field $\mathbb{R}$ equipped with a function $g: X \to \mathbb{R}$ (the paranorm). This function must satisfy the following properties: g is subadditive, $g(\theta) = 0$ (where θ is the zero vector in X), and g is symmetric ($g(u) = g(-u)$ for all $u \in X$). Furthermore, scalar multiplication must be continuous. Specifically, if (α_n) is a sequence of scalars such that $\alpha_n \to \alpha$, and (u_n) is a sequence in X such that $g(u_n - u) \to 0$, then it must follow that $g(\alpha u_n - \alpha u) \to 0$.

Throughout this paper we assume that $p = (p_k)$ is a bounded sequence of strictly positive real numbers (i.e. $0 < inf_k p_k \leq sup_k p_k < \infty$). Let $\sup(p_k) = H$ and $M = \max\{1, H\}$. The linear spaces $c(p), c_0(p), \ell_\infty(p)$ and $\ell(p)$ were introduced by Maddox [6] (see also Nakano [7]) as follows:

$$c(p) = \left\{\mathfrak{f} = (\mathfrak{f}_k) \in \omega: \lim_{k\to\infty} |\mathfrak{f}_k - \mathfrak{f}_0|^{p_k} = 0, \text{ for some } \mathfrak{f}_0 \in \mathbb{C}\right\},$$
$$c_0(p) = \left\{\mathfrak{f} = (\mathfrak{f}_k) \in \omega: \lim_{k\to\infty} |\mathfrak{f}_k|^{p_k} = 0\right\},$$
$$\ell_\infty(p) = \left\{\mathfrak{f} = (\mathfrak{f}_k) \in \omega: \sup_{k\in\mathbb{N}} |\mathfrak{f}_k|^{p_k} < \infty\right\}.$$

These spaces are complete and paranormed by

$$g_1(\mathfrak{f}) = \sup_{k\in\mathbb{N}} |\mathfrak{f}_k|^{p_k/M}, \text{ provided that } \inf_{k\in\mathbb{N}} p_k > 0.$$

Additionally, the space

$$\ell(p) = \{\mathfrak{f} = (\mathfrak{f}_k) \in \omega: \textstyle\sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\}$$

is complete and paranormed by

$$g_2(\mathfrak{f}) = (\textstyle\sum_k |\mathfrak{f}_k|^{p_k})^{1/M}.$$

The classical spaces c(p), c0(p), ℓ∞(p) and ℓ(p) and their matrix domains constitute the standard Maddox-type framework used throughout this paper. As discussed in the Introduction, the novelty here is not the redefinition of these scalar spaces, but their systematic use in the quaternionic setting, where the non-commutativity of multiplication affects the structure of the β- and γ-duals.

We now review fundamental notations associated with quaternion spaces. The space of quaternions, Q, is a four-dimensional real algebra. The null element is denoted by 0_Q, and the multiplicative identity by 1_Q. The space Q is spanned by the basis $\{1, i, j, k\}$, where i, j and k are imaginary units satisfying the relations:

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i \text{ and } ki = -ik = j.$$

A quaternion ρ is expressed as $\rho = y_0 + y_1 i + y_2 j + y_3 k$, where $y_0, y_1, y_2 y_3 \in \mathbb{R}$. Given $\rho \in Q$, we define:

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(i) The conjugate of ρ: $\bar{\rho} = y_0 - y_1 i - y_2 j - y_3 k$,

(ii) The norm of ρ: $\|\rho\| = \sqrt{\rho\bar{\rho}} = \sqrt{y_0^2 + y_1^2 + y_2^2 + y_3^2} \in \mathbb{R}$,

(iii) The real part of ρ: $\mathrm{Re}(\rho) = \frac{1}{2}(\rho + \bar{\rho}) = y_0 \in \mathbb{R}$,

(iv) The imaginary part of ρ: $\mathrm{Im}(\rho) = \frac{1}{2}(\rho - \bar{\rho}) = y_1 i + y_2 j + y_3 k$.

A quaternion $\rho \in \mathbf{Q}$ is termed real if $\rho = \mathrm{Re}(\rho)$ (if and only if $\rho = \bar{\rho}$) and imaginary if $\rho = \mathrm{Im}(\rho)$ (if and only if $\bar{\rho} = -\rho$).

MAIN RESULTS

In this section using the bounded sequence $\mathrm{p} = (\mathrm{p_k})$ of strictly positive real numbers, we introduce the corresponding classes of sequences over the quaternion numbers (QN):

$$\ell_\infty(\mathrm{p})(\mathrm{QN}) = \left\{\varsigma = (\varsigma_\mathrm{k}) \in \omega(\mathrm{QN}): \sup_\mathrm{k} \|\varsigma_\mathrm{k}\|_{\mathrm{QN}}^{\mathrm{p_k}} < \infty\right\},$$

$$\mathrm{c}(\mathrm{p})(\mathrm{QN}) = \left\{\varsigma = (\varsigma_\mathrm{k}) \in \omega(\mathrm{QN}): \lim_{\mathrm{k}\to\infty} \|\varsigma_\mathrm{k} - \mathrm{t_0}\|_{\mathrm{QN}}^{\mathrm{p_k}} = 0, \text{ for some } \mathrm{t_0} \in \mathrm{QN}\right\},$$

$$\mathrm{c_0}(\mathrm{p})(\mathrm{QN}) = \left\{\varsigma = (\varsigma_\mathrm{k}) \in \omega(\mathrm{QN}): \lim_{\mathrm{k}\to\infty} \|\varsigma_\mathrm{k}\|_{\mathrm{QN}}^{\mathrm{p_k}} = 0\right\},$$

$$\ell(\mathrm{p})(\mathrm{QN}) = \{\varsigma = (\varsigma_\mathrm{k}) \in \omega(\mathrm{QN}): \textstyle\sum_{\mathrm{k}=1}^{\infty} \|\varsigma_\mathrm{k}\|_{\mathrm{QN}}^{\mathrm{p_k}} < \infty\}.$$

We denote by $\omega(\mathrm{QN})$ the space of all quaternion sequences. The spaces defined above represent the classes of bounded, convergent, null and absolutely p-summable quaternion sequences respectively.

If $\mathrm{p_k} = \mathrm{p}$ (a constant sequence), these spaces reduce to the standard spaces $\ell_\infty(\mathrm{QN}), \mathrm{c}(\mathrm{QN}), \mathrm{c_0}(\mathrm{QN}), \ell_\mathrm{p}(\mathrm{QN})$ respectively. It is readily verified that the paranormed versions coincide with their standard counterparts if and only if $0 < \inf \mathrm{p_k} \leq \sup \mathrm{p_k} < \infty$. Since $\mathrm{p} = (\mathrm{p_k})$ is bounded as assumed in this study, these sequence spaces are linear under pointwise operations.

Definition 1. A quaternion sequence space $\mathrm{X}(\mathrm{QN})$ is said to be solid (or normal) if $(\eta_\mathrm{k}) \in \mathrm{X}(\mathrm{QN})$ whenever $(\tau_\mathrm{k}) \in \mathrm{X}(\mathrm{QN})$ and $\|\eta_\mathrm{k}\|_{\mathrm{QN}} \leq \|\tau_\mathrm{k}\|_{\mathrm{QN}}$ for all $\mathrm{k} \in \mathbb{N}$.

Equivalently, $\mathrm{X}(\mathrm{QN})$ is solid if $(\tau_\mathrm{k}) \in \mathrm{X}(\mathrm{QN})$ implies $(\beta_\mathrm{k}\tau_\mathrm{k}) \in \mathrm{X}(\mathrm{QN})$ for every quaternion sequence $\beta = (\beta_k) \in \omega(\mathrm{QN})$ such that $|\beta_\mathrm{k}| \leq 1$ for all $\mathrm{k} \in \mathbb{N}$.

Remark 1. It is straightforward to show that the spaces $\ell_\infty(\mathrm{p})(\mathrm{QN}), \mathrm{c_0}(\mathrm{p})(\mathrm{QN})$ and $\ell(\mathrm{p})(\mathrm{QN})$ are solid.

The definition of dual spaces in the QN context requires careful handling because quaternion multiplication is non-commutative. When analysing the convergence (β-dual) or boundedness (γ-dual) of series involving products, the result is sensitive to the order of the factors. Consequently, we must distinguish between left and right duals.

Definition 2. Let X ($\mathbf{QN}$) be a quaternion sequence space. We define the following dual spaces:

(I) The α-dual (or Köthe-Toeplitz dual) is defined as

$$[X(\mathbf{QN})]^\alpha = \{\tau = (\tau_k) \in \omega(\mathbf{QN}): \textstyle\sum_{k=1}^{\infty} \|\varsigma_k\tau_k\|_{\mathbf{QN}} < \infty \text{ for all } (\varsigma_k) \in X(\mathbf{QN})\}.$$

(II) The **right β-dual** (right generalised Köthe-Toeplitz dual) and **left β-dual** are defined as

$$[X(\mathbf{QN})]^{\beta R} = \{\tau = (\tau_k) \in \omega(\mathbf{QN}): \textstyle\sum_{k=1}^{\infty} \varsigma_k\tau_k \text{ converges for all } (\varsigma_k) \in X(\mathbf{QN})\},$$

$$[X(\mathbf{QN})]^{\beta L} = \{\tau = (\tau_k) \in \omega(\mathbf{QN}): \textstyle\sum_{k=1}^{\infty} \tau_k\varsigma_k \text{ converges for all } (\varsigma_k) \in X(\mathbf{QN})\}.$$

(III) The **right γ-dual** and **left γ-dual** are defined as

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$$[X(\mathbf{QN})]^{\gamma R} = \left\{\tau = (\tau_k) \in \omega(\mathbf{QN}) : \sup_n \|\textstyle\sum_{k=1}^{n} \varsigma_k \tau_k\|_{\mathbf{QN}} < \infty \text{ for all } (\varsigma_k) \in X(\mathbf{QN})\right\},$$

$$[X(\mathbf{QN})]^{\gamma L} = \left\{\tau = (\tau_k) \in \omega(\mathbf{QN}) : \sup_n \|\textstyle\sum_{k=1}^{n} \tau_k \varsigma_k\|_{\mathbf{QN}} < \infty \text{ for all } (\varsigma_k) \in X(\mathbf{QN})\right\}.$$

Remark 2. (I) Due to the non-commutativity of QN, the left and right β-duals (and γ-duals) are generally distinct, i.e. $[\mathrm{X(QN)}]^{\beta R} \neq [\mathrm{X(QN)}]^{\beta L}$.

(II) The α-dual does not require this distinction. Since the quaternion norm is multiplicative ($\|ab\| = \|a\| \cdot \|b\|$), we have $\|\varsigma_k\tau_k\| = \|\tau_k\varsigma_k\|$. Therefore, the convergence of the series in the definition of the α-dual is independent of the multiplication order, and the left and right α -duals coincide.

(III) In the subsequent analysis we focus on the α-dual and the right β- and γ-duals. For notational simplicity, we denote $[\mathrm{X(QN)}]^{\beta R}$ and $[X(\mathbf{QN})]^{\gamma R}$ as $[\mathrm{X(QN)}]^{\beta}$ and $[X(\mathbf{QN})]^{\gamma}$ respectively, which refer to right multiplication.

In the second-dual results we use the alternating convention $[\mathrm{X(QN)}]^{\beta\beta} := \left([\mathrm{X(QN)}]^{\beta_R}\right)^{\beta_L}$. The following lemma, whose proof follows directly from the definitions, will be used frequently.

Lemma 1. Let $X(\mathbf{QN})$ be a quaternion sequence space. Then the following properties hold:

(i) $\emptyset \subset [X(\mathbf{QN})]^{\alpha} \subset [X(\mathbf{QN})]^{\beta} \subset [X(\mathbf{QN})]^{\gamma}$.

(ii) $[X(\mathbf{QN})]^{\xi}$ is a linear space of ω(QN), for $\xi \in \{\alpha, \beta, \gamma\}$.

(iii) If $X(\mathbf{QN}) \subset Y(\mathbf{QN})$, then $[Y(\mathbf{QN})]^{\xi} \subset [X(\mathbf{QN})]^{\xi}$ for $\xi \in \{\alpha, \beta, \gamma\}$.

(iv) $X(\mathbf{QN}) \subset [X(\mathbf{QN})]^{\xi\xi}$, for $\xi \in \{\alpha, \beta, \gamma\}$.

(v) $\left[\cup\ X_j(\mathbf{QN})\right]^{\xi} = \cap\ \left[X_j(\mathbf{QN})\right]^{\xi}$, for $\xi \in \{\alpha, \beta, \gamma\}$.

(vi) If $X(\mathbf{QN})$ is normal, then $[X(\mathbf{QN})]^{\alpha} = [X(\mathbf{QN})]^{\beta} = [X(\mathbf{QN})]^{\gamma}$.

Definition 3. For a sequence space X(QN), if $\mathrm{X(QN)} = [\mathrm{X(QN)}]^{\xi\xi}$, then X(QN) is called a ξ-space, where $\xi \in \{\alpha, \beta, \gamma\}$. In particular, a β-space is called a perfect sequence space or a Köthe-Toeplitz reflexive space.

We now determine the α-, β- and γ-duals of the introduced sequence classes and investigate their perfection properties.

Theorem 1. If $\mathrm{p_k} > 0$ for all $\mathrm{k} \in \mathbb{N}$, then $[\ell_\infty(\mathrm{p})(\mathrm{QN})]^{\alpha} = \mathrm{T}_\infty(\mathrm{p})(\mathrm{QN})$, where

$$\mathrm{T}_\infty(\mathrm{p})(\mathrm{QN}) = \bigcap_{\mathrm{U}>1} \left\{\tau = (\tau_k) \in \omega(\mathrm{QN}) : \sum_{k=1}^{\infty} \|\tau_k\|_{\mathrm{QN}} \mathrm{U}^{1/\mathrm{p_k}} < \infty\right\}.$$

Proof. ($\subseteq$) Let $\varsigma = (\varsigma_k) \in \ell_\infty(\mathrm{p})(\mathrm{QN})$. Consider $\tau = (\tau_k) \in \mathrm{T}_\infty(\mathrm{p})(\mathrm{QN})$ and $\lambda = \sup_k \|\varsigma_k\|_{\mathrm{QN}}^{\mathrm{p_k}}$. Choose $\mathrm{U} > \max\{1, \lambda\}$. Then

$$\sum_{k=1}^{\infty} \|\varsigma_k\tau_k\|_{\mathrm{QN}} \leq \sum_{k=1}^{\infty} \|\tau_k\|_{\mathrm{QN}} \mathrm{U}^{1/\mathrm{p_k}} < \infty.$$

Thus,

$$\mathrm{T}_\infty(\mathrm{p})(\mathrm{QN}) \subseteq [\ell_\infty(\mathrm{p})(\mathrm{QN})]^{\alpha}.$$

($\supseteq$) Conversely, assume $\tau = (\tau_k) \in [\ell_\infty(\mathrm{p})(\mathrm{QN})]^{\alpha}$. Suppose, for contradiction, that $\tau = (\tau_k) \notin \mathrm{T}_\infty(\mathrm{p})(\mathrm{QN})$. Then there exists a positive integer $\mathrm{U_0} > 1$ such that $\sum_{k=1}^{\infty} \|\tau_k\|_{\mathrm{QN}} \mathrm{U}_0^{1/\mathrm{p_k}} = \infty$. Define a sequence $\varsigma = (\varsigma_k)$ by $\varsigma_k = \mathrm{U}_0^{1/\mathrm{p_k}}$ for all $\mathrm{k} \in \mathbb{N}$. Clearly, $\varsigma \in \ell_\infty(\mathrm{p})(\mathrm{QN})$. However,

$\sum_{k=1}^{\infty} \|\varsigma_k\tau_k\|_{QN} = \infty$. This contradicts the assumption that $\tau \in [\ell_\infty(p)(QN)]^\alpha$. Therefore, $\tau \in T_\infty(p)(QN)$ and hence

$$[\ell_\infty(p)(QN)]^\alpha \subseteq T_\infty(p)(QN).$$

Hence $[\ell_\infty(p)(QN)]^\alpha = T_\infty(p)(QN)$.

Remark 3. Since $\ell_\infty(p)(QN)$ is solid, applying Lemma 1 (vi), we obtain

$$[\ell_\infty(p)(QN)]^\alpha = [\ell_\infty(p)(QN)]^\beta = [\ell_\infty(p)(QN)]^\gamma = T_\infty(p)(QN).$$

Theorem 2. If $p_k > 0$ for all $k \in \mathbb{N}$, then $[c_0(p)(QN)]^\alpha = T_0(p)(QN)$, where

$$T_0(p)(QN) = \bigcup_{U>1} \left\{\tau = (\tau_k) \in \omega(QN): \sum_{k=1}^{\infty} \|\tau_k\|_{QN} U^{-1/p_k} < \infty\right\}.$$

Proof. ($\subseteq$) Let $\tau = (\tau_k) \in T_0(p)(QN)$. Then there exists $U > 1$ such that $\sum_{k=1}^{\infty} \|\tau_k\|_{QN} U^{-1/p_k} < \infty$. Let $\varsigma = (\varsigma_k) \in c_0(p)(QN)$. Then there exists $K_0 \in \mathbb{N}$ such that $\|\varsigma_k\|_{QN}^{p_k} < \frac{1}{U}$ for all $k \geq K_0$. Hence

$$\sum_{k=1}^{K_0-1} \|\varsigma_k\tau_k\|_{QN} + \sum_{k=K_0}^{\infty} \|\varsigma_k\tau_k\|_{QN} \leq \sum_{k=1}^{K_0-1} \|\varsigma_k\tau_k\|_{QN} + \sum_{k=K_0}^{\infty} \|\tau_k\|_{QN} U^{-1/p_k} < \infty.$$

Thus, $T_0(p)(QN) \subseteq [c_0(p)(QN)]^\alpha$.

($\supseteq$) Conversely, suppose $\tau = (\tau_k) \in [c_0(p)(QN)]^\alpha$, but $\tau = (\tau_k) \notin T_0(p)(QN)$. By an argument analogous to that used in Theorem 1, we can construct a sequence $\varsigma \in c_0(p)(QN)$ such that $\sum_{k=1}^{\infty} \|\varsigma_k\eta_k\|_{QN}$ diverges. This leads to a contradiction. Therefore,

$$[c_0(p)(QN)]^\alpha \subseteq T_0(p)(QN).$$

Consequently,

$$[c_0(p)(QN)]^\alpha = T_0(p)(QN).$$

Remark 4. Since $c_0(p)(QN)$ is solid, by Lemma 1 (vi), we have

$$[c_0(p)(QN)]^\alpha = [c_0(p)(QN)]^\beta = [c_0(p)(QN)]^\gamma = T_0(p)(QN).$$

Theorem 3. If $p_k > 0$ for all $k \in \mathbb{N}$, then

(i) $[c(p)(\mathbf{QN})]^\alpha = T_0(p)(\mathbf{QN}) \cap \ell_1(\mathbf{QN})$,

(ii) $[c(p)(\mathbf{QN})]^\beta = T_0(p)(\mathbf{QN}) \cap cs(\mathbf{QN})$,

(iii) $[c(p)(\mathbf{QN})]^\gamma = T_0(p)(\mathbf{QN}) \cap bs(\mathbf{QN})$,

where $T_0(p)(\mathbf{QN})$ is defined in Theorem 2, and

$$\ell_1(\mathbf{QN}) = \left\{\tau = (\tau_k) \in \omega(\mathbf{QN}): \sum_{k=1}^{\infty} \|\tau_k\|_{\mathbf{QN}} < \infty\right\},$$

$$cs(\mathbf{QN}) = \left\{\tau = (\tau_k) \in \omega(\mathbf{QN}): \text{the series} \sum_{k=1}^{n} \tau_k \text{ converges}\right\},$$

$$bs(\mathbf{QN}) = \left\{\tau = (\tau_k) \in \omega(\mathbf{QN}): \sup_n \left\|\sum_{k=1}^{n} \tau_k\right\|_{\mathbf{QN}} < \infty\right\}.$$

Proof. We prove part (i); the proofs of parts (ii) and (iii) are similar.

($\subseteq$) Let $\tau = (\tau_k) \in T_0(p)(QN) \cap \ell_1(QN)$. Then there exists $U > 1$ such that

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$$\sum_{k=1}^{\infty} \|\tau_k\|_{QN} U^{-1/p_k} < \infty \text{ and } \sum_{k=1}^{\infty} \|\tau_k\|_{QN} < \infty. \tag{1}$$

Let $\varsigma = (\varsigma_k) \in c(p)(QN)$. For sufficiently large k, we have

$$\|\varsigma_k - t\|_{QN}^{p_k} < \frac{1}{U} \Rightarrow \|\varsigma_k - t\|_{QN} < U^{-1/p_k}. \tag{2}$$

Using (1) and (2), we obtain

$$\sum_{k=1}^{\infty} \|\varsigma_k \tau_k\|_{QN} \leq \sum_{k=1}^{\infty} \|\varsigma_k - t\|_{QN} \|\tau_k\|_{QN} + \|t\|_{QN} \sum_{k=1}^{\infty} \|\tau_k\|_{QN}$$
$$\leq \sum_{k=1}^{\infty} \|\tau_k\|_{QN} U^{-\frac{1}{p_k}} + \|t\|_{QN} \sum_{k=1}^{\infty} \|\tau_k\|_{QN} < \infty.$$

Thus,

$$T_0(p)(QN) \cap \ell_1(QN) \subseteq [c(p)(QN)]^{\alpha}.$$

($\supseteq$) Conversely, suppose $\tau = (\tau_k) \in [c(p)(QN)]^{\alpha}$. Since the constant sequence $e = (1,1,1,\dots)$ belongs to $c(p)(QN)$, we must have

$$\sum_{k=1}^{\infty} \|\tau_k e_k\|_{QN} < \infty,$$

which implies $\tau \in \ell_1(QN)$. Furthermore, since

$$c_0(p)(QN) \subseteq c(p)(QN),$$

by Lemma 1 (iii),

$$[c(p)(QN)]^{\alpha} \subseteq [c_0(p)(QN)]^{\alpha}.$$

By Theorem 2, $[c(p)(QN)]^{\alpha} \subseteq T_0(p)(QN)$. Therefore,

$$[c(p)(QN)]^{\alpha} \subseteq T_0(p)(QN) \cap \ell_1(QN).$$

Combining the inclusions, we conclude

$$[c(p)(QN)]^{\alpha} = T_0(p)(QN) \cap \ell_1(QN).$$

Theorem 4. If $0 < p_k \leq 1$ for all $k \in \mathbb{N}$, then $[\ell(p)(QN)]^{\alpha} = \ell_{\infty}(p)(QN)$.

Proof. ($\subseteq$) Let $\varsigma = (\varsigma_k) \in \ell(p)(QN)$ and $\tau = (\tau_k) \in \ell_{\infty}(p)(QN)$. Set $M = \max_k \left\{1, \sup_k \|\tau_k\|_{QN}^{p_k}\right\}$. Then $\|\tau_k\|_{QN}^{p_k} \leq M$ for all $k \in \mathbb{N}$. Since $\varsigma \in \ell(p)(QN)$, we have $\|\varsigma_k\|_{QN}^{p_k} \to 0$. Therefore, $\|\varsigma_k \tau_k\|_{QN} \leq 1$ for all sufficiently large k and hence

$$\|\varsigma_k \tau_k\|_{QN} \leq \|\varsigma_k \tau_k\|_{QN}^{p_k}, \qquad k > K_0.$$

Using the multiplicativity of the quaternion norm, we obtain

$$\sum_{k=1}^{\infty} \|\varsigma_k \tau_k\|_{QN} = \sum_{k=1}^{K_0} \|\varsigma_k \tau_k\|_{QN} + \sum_{k=K_0+1}^{\infty} \|\varsigma_k \tau_k\|_{QN}$$
$$\leq \sum_{k=1}^{K_0} \|\varsigma_k \tau_k\|_{QN} + \sum_{k=K_0+1}^{\infty} \|\varsigma_k \tau_k\|_{QN}^{p_k}$$
$$\leq \sum_{k=1}^{K_0} \|\varsigma_k \tau_k\|_{QN} + \sum_{k=K_0+1}^{\infty} \|\varsigma_k\|_{QN}^{p_k} \|\tau_k\|_{QN}^{p_k}$$

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$$\leq \sum_{k=1}^{K_0} \|\varsigma_k \tau_k\|_{QN} + M \sum_{k=K_0+1}^{\infty} \|\varsigma_k\|_{QN}^{p_k} < \infty.$$

Hence

$$\ell_\infty(p)(QN) \subseteq [\ell(p)(QN)]^\alpha.$$

($\supseteq$) Conversely, assume $\tau = (\tau_k) \in [\ell(p)(QN)]^\alpha$, but $\tau = (\tau_k) \notin \ell_\infty(p)(QN)$. Then we can find a strictly increasing sequence of indices (k_n) such that $\|\tau_{k_n}\|_{QN}^{p_{k_n}} \geq n^2$.

Define a sequence $\varsigma = (\varsigma_k)$ as follows:

$$\varsigma_k = \begin{cases} \tau_k^{-1}, & \text{if } k = k_n \\ \theta, & \text{if } k \neq k_n. \end{cases}$$

Since $\|\tau_{k_n}\|_{QN}^{p_{k_n}} \geq n^2$, each τ_{k_n} is non-zero and hence $\tau_{k_n}^{-1}$ exists in QN. Then

$$\sum_{k=1}^{\infty} \|\varsigma_k\|_{QN}^{p_k} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

This implies that $\varsigma = (\varsigma_k) \in \ell(p)(QN)$. However,

$$\sum_{k=1}^{\infty} \|\varsigma_k \tau_k\|_{QN} = \sum_{k=1}^{\infty} 1 = \infty.$$

This contradicts the assumption that $\tau \in [\ell(p)(QN)]^\alpha$. Therefore,

$$[\ell(p)(QN)]^\alpha = \ell_\infty(p)(QN).$$

Theorem 5. If $p_k > 1$ for all $k \in \mathbb{N}$, then $[\ell(p)(QN)]^\alpha = T(p)(QN)$, where

$$T(p)(QN) = \bigcup_{U>1} \left\{ \tau = (\tau_k) \in \omega(QN): \sum_{k=1}^{\infty} \|\tau_k\|_{QN}^{q_k} U^{-q_k/p_k} < \infty \right\}$$

and $\frac{1}{p_k} + \frac{1}{q_k} = 1$, for all $k \in \mathbb{N}$.

Proof. ($\subseteq$) Let $\varsigma = (\varsigma_k) \in \ell(p)(QN)$ and $\tau = (\tau_k) \in T(p)(QN)$. By the generalised Young inequality, for some $U > 1$ associated with τ, we have

$$\sum_{k=1}^{\infty} \|\varsigma_k \tau_k\|_{QN} \leq \sum_{k=1}^{\infty} U \|\varsigma_k\|_{QN}^{p_k} + \sum_{k=1}^{\infty} \|\tau_k\|_{QN}^{q_k} U^{-q_k/p_k} < \infty.$$

Consequently,

$$T(p)(QN) \subseteq [\ell(p)(QN)]^\alpha.$$

($\supseteq$) For the reverse inclusion, suppose that $\tau = (\tau_k) \in [\ell(p)(QN)]^\alpha$, but $\tau \notin T(p)(QN)$. We can then choose a strictly increasing sequence of integers $0 = n_0 < n_1 < n_2 < \cdots$ such that

$$Y_m = \sum_{I(m)} \|\tau_k\|_{QN}^{q_k} (m+1)^{-q_k/p_k} > 1, \tag{3}$$

where $m \in \mathbb{N}$ and $I(m) = \{k: n_{m-1} + 1 \leq k \leq n_m\}$. Next, we define a sequence $\varsigma = (\varsigma_k)$ by

$$\varsigma_k = \|\tau_k\|_{QN}^{q_k - 1} (m+1)^{-q_k} Y_m^{-1}$$

for $k \in I(m)$. Using the identity $\frac{1}{p_k} + \frac{1}{q_k} = 1$, we examine the convergence of ς. A direct calculation gives the following estimate:

$$\sum_{k \in I(m)} \|\varsigma_k\|_{QN}^{p_k} \leq (m+1)^{-2}.$$

Thus,

$$\sum_m \sum_{k \in I(m)} \|\varsigma_k\|_{QN}^{p_k} \leq \sum_m \frac{1}{(m+1)^2} < \infty \text{ and hence } \varsigma \in \ell(p)(QN).$$

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However, using (3), we obtain

$$\sum_{k\in I(m)} \|\varsigma_k\tau_k\|_{QN} = \frac{1}{m+1}.$$

This implies

$$\sum_{m}\sum_{k\in I(m)} \|\varsigma_k\tau_k\|_{QN} = \sum_{m} \frac{1}{m+1},$$

which diverges. This contradiction shows that $\tau \in T(p)(QN)$. Therefore,

$$[\ell(p)(QN)]^{\alpha} = T(p)(QN).$$

Remark 5. Since $\ell(p)(QN)$ is solid, Lemma 1 (vi) gives

$$[\ell(p)(QN)]^{\xi} = [\ell(p)(QN)]^{\beta} = [\ell(p)(QN)]^{\gamma}.$$

Consequently, for $\xi \in \{\alpha,\beta,\gamma\}$,

$$\ell_\infty(p)(QN), \text{ if } 0 < p_k \le 1 \text{ for all } k \in \mathbb{N}$$
$$T(p)(QN), \text{ if } p_k > 1 \text{ for all } k \in \mathbb{N}.$$

For Theorems 6 and 7, $p = (p_k)$ is allowed to be an arbitrary sequence of strictly positive real numbers.

Theorem 6. If $p_k > 0$ for all $k \in \mathbb{N}$, then $[\ell_\infty(p)(QN)]^{\beta\beta} = \check{T}_\infty(p)(QN)$, where

$$\check{T}_\infty(p)(QN) = \bigcap_{U>1}\left\{\tau = (\tau_k) \in \omega(QN): \sup_k \|\tau_k\|_{QN} U^{1/p_k} < \infty\right\}.$$

Proof. By Remark 3,

$$[\ell_\infty(p)(QN)]^{\beta} = T_\infty(p)(QN) = \cup_{U>1} E_U ,$$

where

$$E_U = \left\{\eta = (\eta_k) \in \omega(QN): \sum_{k=1}^{\infty} \|\eta_k\|_{QN} U^{-1/p_k} < \infty\right\}.$$

By Lemma 1 (v),

$$[\ell_\infty(p)(QN)]^{\beta\beta} = \cap_{U>1} E_U^{\beta} .$$

For each $U > 1$, the β-dual of E_U is characterised by

$$E_U^{\beta} = \left\{\tau = (\tau_k) \in \omega(QN): \sup_k \|\tau_k\|_{QN} U^{1/p_k} < \infty\right\}.$$

Therefore,

$$[\ell_\infty(p)(QN)]^{\beta\beta} = \check{T}_\infty(p)(QN).$$

Theorem 7. $\ell_\infty(p)(QN)$ is perfect if and only if $p \in \ell_\infty$.

Proof. (Sufficiency) Suppose that $p = (p_k) \in \ell_\infty$ and let $H =: \sup_k p_k < \infty$. Let $\tau = (\tau_k) \in [\ell_\infty(p)(QN)]^{\beta\beta}$. By Theorem 6, taking $U = 2$, there exists a constant $C > 0$ such that

$$\sup_k \|\tau_k\|_{QN} 2^{1/p_k} \le C.$$

Hence

$$\|\tau_k\|_{QN}^{p_k} \le \frac{C^{p_k}}{2},$$

Since $0 < p_k \le H$ for all $k \in \mathbb{N}$, we have

$$C^{p_k} \le \max_H\{1, C^H\}.$$

Consequently,

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$$\|\tau_k\|_{QN}^{p_k} \le \frac{C^{p_k}}{2} \le \max_H\{1, C^H\}/2$$

for all $k \in \mathbb{N}$. Therefore, $\tau \in \ell_\infty(p)(QN)$. Together with Lemma 1 (iv), this yields

$$[\ell_\infty(p)(QN)]^{\beta\beta} = \ell_\infty(p)(QN).$$

Thus, $\ell_\infty(p)(QN)$ is perfect.

(Necessity) Conversely, suppose that $\ell_\infty(p)(QN)$ is perfect but $p \notin \ell_\infty$. Then there exists a strictly increasing sequence (k_n) such that $p_{k_n} > n$. Define a sequence $\varsigma = (\varsigma_k)$ by $\varsigma_{k_n} = 1 + i + j + k$ and $\varsigma_k = 0$ otherwise. For every $U > 1$, we have

$$\sup_k \|\varsigma_k\|_{QN} U^{1/p_k} \le 2U < \infty.$$

Hence by Theorem 6, $\varsigma \in [\ell_\infty(p)(QN)]^{\beta\beta}$. However, $\|\varsigma_k\|_{QN}^{p_k} = \|1 + i + j + k\|_{QN}^{p_k} = 2^{p_{k_n}} > 2^n$, so $\varsigma \notin \ell_\infty(p)(QN)$. This contradicts the assumption that $\ell_\infty(p)(QN)$ is perfect. Therefore, $p = (p_k)$ must be bounded.

Theorem 8. If $p_k > 0$ for all $k \in \mathbb{N}$, then $[c_0(p)(QN)]^{\beta\beta} = \check{T}_0(p)(QN)$, where

$$\check{T}_0(p)(QN) = \bigcap_{U>1} \left\{\tau = (\tau_k) \in \omega(QN): \sup_k \|\tau_k\|_{QN} U^{1/p_k} < \infty\right\}.$$

Proof. By Remark 4,

$$[c_0(p)(QN)]^\beta = T_0(p)(QN) = \bigcup_{U>1} E_U,$$

where

$$E_U = \left\{\tau = (\tau_k) \in \omega(QN): \sum_k \|\tau_k\|_{QN} U^{-1/p_k} < \infty\right\}.$$

Applying Lemma 1 (v), the second dual is

$$[c_0(p)(QN)]^{\beta\beta} = [T_0(p)(QN)]^\beta = \left[\bigcup_{U>1} E_U\right]^\beta = \bigcap_{U>1} E_U^\beta.$$

For each $U > 1$, the β-dual of E_U is

$$E_U^\beta = \left\{\tau = (\tau_k) \in \omega(QN): \sup_k \|\tau_k\|_{QN} U^{1/p_k} < \infty\right\}.$$

Therefore, $[c_0(p)(QN)]^{\beta\beta} = \check{T}_0(p)(QN)$.

In the following theorem, we drop the condition $\inf_k p_k > 0$ and assume only that $p_k > 0$ for all $k \in \mathbb{N}$.

Theorem 9. $c_0(p)(QN)$ is perfect if and only if $p \in c_0$.

Proof. (Sufficiency) Let $p = (p_k) \in c_0$. We aim to show $[c_0(p)(QN)]^{\beta\beta} \subseteq c_0(p)(QN)$. Let $\tau = (\tau_k) \in [c_0(p)(QN)]^{\beta\beta}$. By Theorem 8,

$$\sup_k \|\tau_k\|_{QN} U^{1/p_k} < \infty$$

for all $U > 1$. Suppose, to the contrary, that $\tau = (\tau_k) \notin c_0(p)(QN)$. Then there exist a constant $s > 0$ and a strictly increasing sequence (k_n) such that $\|\tau_k\|_{QN}^{p_k} \ge s$, for $k = k_n$, $n \in \mathbb{N}$. Choose $U > \max_s\{1, s^{-1}\}$. By Theorem 8, there exists a constant CU > 0 such that

$$\|\tau_k\|_{QN} U^{1/p_k} \le CU$$

for all $k \in \mathbb{N}$. Therefore, we obtain

$$(Us)^{1/p_{k_n}} \leq CU$$

for all $n \in \mathbb{N}$. Since $p_{k_n} \to 0$ and $Us > 1$, the left-hand side tends to infinity, which is a contradiction. Hence $\tau \in c_0(p)(QN)$, and therefore $c_0(p)(QN)$ is perfect.

(Necessity) Conversely, let $c_0(p)(QN)$ be perfect, but $p \notin c_0$. Then there exists an increasing sequence (k_m) and a constant $s' > 0$ such that $p_{k_m} \geq s'$ for all $m \in \mathbb{N}$. Define a sequence $\varsigma = (\varsigma_k)$ by $\varsigma_{k_m} = 1 + i + j + k$ and $\varsigma_k = 0$ otherwise. For every $U > 1$,

$$\sup_k \|\varsigma_k\|_{QN} U^{1/p_k} \leq 2U^{1/s'} < \infty.$$

Hence by Theorem 8, $\varsigma \in [c_0(p)(QN)]^{\beta\beta}$. Since $\|\varsigma_{k_m}\|_{QN}^{p_{k_m}} = 2^{p_{k_m}} \geq 2s' > 0$, we have $\varsigma \notin c_0(p)(QN)$. This contradicts the assumption that $c_0(p)(QN)$ is perfect. Thus, $p \in c_0$.

CONCLUSIONS

The results obtained here show that several classical duality phenomena can be extended to quaternion-valued sequence spaces, but only after the order of multiplication is carefully incorporated into the definitions and proofs. Thus, the present paper does not merely restate known real or complex results in a different notation; it also identifies the precise way in which quaternionic non-commutativity modifies the duality structure.

The approach developed in this work may serve as a basis for further investigations of matrix transformations, summability methods, compactness questions and operator-theoretic problems in quaternion-valued sequence spaces. It may also be extended to more general matrix-generated quaternionic sequence spaces and to sequence spaces over other non-commutative or higher dimensional algebraic structures.

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