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Full Paper

Improved neutrosophic estimation of population mean: Application to temperature data and simulation study

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Abstract: Neutrosophic estimation of unknown population parameters has become one of the most recent areas of focus in sampling theory over the past few years. The growing interest in this approach stems from the limitations of classical statistics when dealing with unreliable, indeterminate and uncertain data. Classical statistics only handles with accurate, crisp and unambiguous information. This study introduces an innovative generalised estimator for estimating the population mean under simple random sampling. The proposed generalised estimator can generate a variety of ratio and product exponential-type families of estimators, which are based on both conventional and non-conventional auxiliary information. Expressions for the bias, mean squared error and minimum mean square error have been derived. The proposed estimators provide an interval within which the population parameter is likely to lie with maximum probability. The superiority of the suggested estimators is demonstrated through both natural and simulated data.

Keywords: auxiliary information, mean squared error, neutrosophic data, percentage relative efficiency

INTRODUCTION

In the field of sampling theory, particularly in the estimation of population parameters, much of the research focuses on exact, determinate, crisp or unambiguous measurements of variables within classical statistics. However, problems arise when there is uncertainty, ambiguity or imprecision in the data as traditional/classical statistical techniques do not yield favourable results. The solution to this issue lies in neutrosophic statistical methods.

Neutrosophic statistics, which is a relatively new field, is gaining increasing attention from researchers nowadays. The field of neutrosophy was introduced by Smarandache [1]. It addresses uncertain, imperfect, unclear, incomplete and ambiguous data. When indeterminacy is zero, neutrosophic statistics resemble classical statistics. A large number of indeterminate data sets are used in daily life and many disciplines have applied neutrosophic theory, including statistical models for neutrosophic treatment, the integration of renewable energy through tools such as solar panels and wind turbines, studies related to COVID-19 and the Omicron variant.

There are so many research variables in real life that it might be expensive to collect data, particularly when those data are unclear. It will therefore be risky and expensive to use conventional techniques for uncertain data to ascertain the parameter's unknown true value.

Some notable contributions in the study of neutrosophic environment are highlighted here. Rao and Aslam [2] inspected a time-truncated repetitive sampling plan through detailed investigation of Italian COVID-19 data for Weibull distribution under indeterminacy. Tahir et al. [3] transformed some already existing estimators used for crisp data into neutrosophic ratio-type estimators for estimating the finite population mean. Kumar et al. [4] proposed a neutrosophic exponential-type estimator by considering neutrosophic study and auxiliary variables to estimate the population mean in the presence of uncertainty. Yadav et al. [5] explored a neutrosophic estimator using auxiliary information, which showed superior accuracy and robustness over classical methods in uncertain environments. Zaman and Sozen [6] proposed neutrosophic ratio-type and multivariate exponential estimators for population mean estimation under uncertainty using Turkey's monthly temperature data. Theoretical and simulation results demonstrated their effectiveness for environmental data analysis. Yadav and Prasad [7] introduced neutrosophic factor-type exponential estimators for population mean estimation, handling ambiguity in the data. Empirical and simulation results showed these estimators outperform traditional methods. Vishwakarma and Singh [8] utilised neutrosophic subsidiary information to suggest a neutrosophic ranked-set sampling method, and a generalised estimator to estimate the population mean.

Raghav [9] recommended a generalised neutrosophic exponential robust ratio-type estimator to estimate the finite population mean. He demonstrated the superiority of his estimator through a real-life data based on stock price and some simulation study. Singh et al. [10] proposed a generalised neutrosophic robust ratio-type estimator for the estimation of population mean in the occurrence of outliers. Singh et al. [11] unveiled some product and ratio exponential estimators under neutrosophic stratified random sampling scheme. They executed an analysis based on real and artificial data to showcase the dominance of their estimators. Irfan et al. [12] and Shahzad et al. [13, 14] also gave significant contribution in the estimation of population mean under simple random sampling without replacement (SRSWOR) scheme. For recent contributions on neutrosophic estimation of unknown population parameters under various sampling schemes in survey sampling, the reader is referred to Aslam [15], Ullah et al. [16], Yadav and Parsad [17], Yadav and Smarandache [18], Singh et al. [19], Kumari et al. [20], Masood et al. [21], Alqudah et al. [22], Shahzad et al. [23], Priya and Kumar [24], Azeem [25], Yadav et al. [26], Kumar et al. [27], Purwar et al. [28], Singh et al. [29], Ravindrabahadur et al. [30], Kumari et al. [31], Kumar and Kumar [32], Yadav et al. [33], Priya and Kumar [34], Basha and Usman [35], Verma et al. [36], Hussain et al. [37], Zaman and Sozen [38], Kumar and Kumar [39] and Tiwari et al. [40].

A substantial body of research exists in sampling theory in which non-conventional measures of auxiliary variables are employed to enhance the estimation of unknown population parameters. Selected contributions are briefly reviewed in the following. Zohaib and Irfan [41] developed robust estimators for the population median under stratified sampling, employing mean squared error (MSE) and relative root mean square error in assessing efficiency. Irfan et al. [42] proposed generalised class of population median estimators under SRSWOR using robust auxiliary measures such as the decile mean, Hodges–Lehmann estimator and tri-mean. Empirical evidence showed superior performance over existing estimators. Irfan et al. [43] generalised difference-cum-exponential estimator for population mean under SRSWOR using conventional and non-conventional auxiliary information. Some contributions in the area of non-conventional measures have been reported by Javed et al. [44, 45] and Irfan et al. [46].

Research Gap

Limited research has been found on sample surveys for estimating the population mean using known neutrosophic auxiliary variables. Currently, there are not enough significant publications on this statistical topic. Since the study and auxiliary variables are neutrosophic in nature, traditional ratio estimation methods cannot be applied. The present study focuses on an innovative neutrosophic family of estimators for improved estimation of the population mean under SRSWOR. This study aims to fill the gap by introducing a generalised neutrosophic exponential-type estimator that can provide a variety of families of estimators. The novelty of this work lies in the fact that no studies have offered such a flexible generalised estimator. These estimators are based on two types of auxiliary information: conventional and non-conventional. This work should provide researchers with a wide range of appropriate estimators to choose for estimating the population mean based on the available auxiliary information.

METHODS

Consider $\Theta = \{\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_N\}$ as a finite population consisting of N units. Let a neutrosophic random sample of size $n_N \in [n_L,\ n_U]$ be chosen from this population under simple SRSWOR. Other useful notations under neutrosophic environment are given in Table 1.

Table 1. Some useful notations under neutrosophic environment

Description	Study variable		Auxiliary variable
i^{th} observation	Population	$Y_N(i) \in [Y_L,\ Y_U]$	$X_N(i) \in [X_L, X_U]$
Average		$\bar{Y}_N(i) \in [\bar{Y}_L, \bar{Y}_U]$	$\bar{X}_N(i) \in [\bar{X}_L, \bar{X}_U]$
i^{th} observation	Sample	$y_N(i) \in [y_L, y_U]$	$x_N(i) \in [x_L, x_U]$
Average		$\bar{y}_N(i) \in [\bar{y}_L, \bar{y}_U]$	$\bar{x}_N(i) \in [\bar{x}_L, \bar{x}_U]$
Variance	$\sigma^2_{yN} \in \left[\sigma^2_{yL}, \sigma^2_{yU}\right]$		$\sigma^2_{xN} \in [\sigma^2_{xL}, \sigma^2_{xU}]$
Coefficient of variation	$C^2_{yN} \in \left[C^2_{yL}, C^2_{yU}\right]$ where $C^2_{yN} = \sigma^2_{yN}(\bar{Y}^2_N)^{-1}$		$C^2_{xN} \in [C^2_{xL}, C^2_{xU}]$ where $C^2_{xN} = \sigma^2_{xN}(\bar{X}^2_N)^{-1}$

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Table 1. (Continued)

Description	Study variable	Auxiliary variable
Coefficients	Correlation: $\rho_{xyN} \in [\rho_{xyL}, \rho_{xyU}]$ where $\rho_{xyN} = \frac{\sigma_{xyN}}{\sigma_{xN}\sigma_{yN}}$ Skewness: $\beta_{1(x)N} \in [\beta_{1(x)L}, \beta_{1(x)U}]$	Kurtosis: $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$
Relative error terms and their expectations	$\xi_{yN} = \frac{(\bar{y}_N - \bar{Y}_N)}{\bar{Y}_N}, \xi_{yN} \in [\xi_{yNL}, \xi_{yNU}]$	$\xi_{xN} = \frac{(\bar{x}_N - \bar{X}_N)}{\bar{X}_N}, \xi_{xN} \in [\xi_{xNL}, \xi_{xNU}]$
	$E(\xi_{yN}) = 0$	$E(\xi_{xN}) = 0$
	$E(\xi_{yN}^2) = \theta_N C_{yN}^2, \ \xi_{yN}^2 \in [\xi_{yNL}^2, \xi_{yNU}^2]$	$E(\xi_{xN}^2) = \theta_N C_{xN}^2, \ \xi_{xN}^2 \in [\xi_{xNL}^2, \xi_{xNU}^2]$
	$E(\xi_{yN}\xi_{xN}) = \theta_N C_{xN} C_{yN} \rho_{xyN} = \theta_N C_{yxN} = \theta_N k_N C_{xN}^2$ where $\theta_N = \frac{1-f_N}{n_N}$; $\theta_N \in [\theta_L, \theta_U]$ $f_N = \frac{n_N}{N_N}$, $k_N = \rho_{xyN}\left(\frac{C_{yN}}{C_{xN}}\right)$ and $n_N \in [n_L, n_U]$	

Reviewing Neutrosophic Estimators

The key contributions to the estimation of the true but unknown population mean in the case of neutrosophic data under a simple random sampling scheme are given in Table 2. This table illustrates the neutrosophic estimators for the population mean provided by Tahir et al. [3]

Proposed Neutrosophic Estimator

This section aims to suggest an innovative neutrosophic family of estimators for enhanced estimation of the population mean under simple random sampling. Singh and Solanki [54] proposed a class of estimators for the unknown population mean under determinate data. Getting motivated by their work, we introduce an innovative generalised estimator for estimating the population mean under indeterminate/neutrosophic data. Eq. (1) shows the suggested generalised estimator.

$$\bar{y}_{Np} = w_{1N}\bar{y}_N\left[\frac{\alpha\bar{X}_N + \gamma}{\alpha\bar{x}_N + \gamma}\right]^g + w_{2N}\bar{y}_N\, exp\left[\frac{\delta\alpha(\bar{X}_N - \bar{x}_N)}{\alpha(\bar{X}_N + \bar{x}_N) + 2\gamma}\right] \quad (1)$$

where $\alpha(\neq 0)$ and γ are real numbers assumed to be known parameters of the auxiliary variable, and w_{1N} and w_{2N} are the weights to be determined to minimise the MSE. Being constants, the (g, δ) assume values (1, -1) for designing the different estimators.

Remark 1. It is worth mentioning here that the proposed generalised neutrosophic estimator can provide a variety of estimators. For instance, by placing different values of g and δ in Eq. (1), we obtain

i) Ratio-ratio type exponential estimators $[(g, \delta) = (1, 1)]$:

$$\bar{y}_{NpR} = w_{1N}\bar{y}_N\left[\frac{\alpha\bar{X}_N + \gamma}{\alpha\bar{x}_N + \gamma}\right] + w_{2N}\bar{y}_N\, exp\left[\frac{\alpha(\bar{X}_N - \bar{x}_N)}{\alpha(\bar{X}_N + \bar{x}_N) + 2\gamma}\right]$$

ii) Product-product type exponential estimators $[(g, \delta) = (-1, -1)]$:

$$\bar{y}_{NpP} = w_{1N}\bar{y}_N\left[\frac{\alpha\bar{x}_N + \gamma}{\alpha\bar{X}_N + \gamma}\right] + w_{2N}\bar{y}_N\, exp\left[\frac{-\alpha(\bar{X}_N - \bar{x}_N)}{\alpha(\bar{X}_N + \bar{x}_N) + 2\gamma}\right]$$

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iii) Ratio-product type exponential estimators $[(g, \delta) = (1, -1)]$:

$$\bar{y}_{NpRP} = w_{1N}\bar{y}_N \left[\frac{\alpha\bar{X}_N + \gamma}{\alpha\bar{x}_N + \gamma}\right] + w_{2N}\bar{y}_N\, exp\left[\frac{-\alpha(\bar{X}_N - \bar{x}_N)}{\alpha(\bar{X}_N + \bar{x}_N) + 2\gamma}\right]$$

iv) Product-ratio type exponential estimators $[(g, \delta) = (-1, 1)]$:

$$\bar{y}_{NpPR} = w_{1N}\bar{y}_N \left[\frac{\alpha\bar{x}_N + \gamma}{\alpha\bar{X}_N + \gamma}\right] + w_{2N}\bar{y}_N\, exp\left[\frac{\alpha(\bar{X}_N - \bar{x}_N)}{\alpha(\bar{X}_N + \bar{x}_N) + 2\gamma}\right]$$

Table 2. Existing estimators proposed by Tahir et al. [3]

Estimator	MSE
Robson [47] $\bar{y}_{RN} = \frac{\bar{y}_N}{\bar{x}N}\bar{X}_N$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + C_{xN}^2 - 2C_{xN}\,C_{yN}\,\rho_{xyN}\right]$
Sisodia and Dwivedi [48] $\bar{y}_{SDrN} = \bar{y}_N\,\frac{\bar{X}N + C_{xN}}{\bar{x}N + C_{xN}}$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N}{\bar{X}_N + C_{xN}}\right)C_{xN}C_{yN}\rho_{xyN}\right]$
Singh and Kakran [49] $\bar{y}_{SKrN} = \bar{y}_N\,\frac{\bar{X}_N + \beta_{2(x)N}}{\bar{x}_N + \beta_{2(x)N}}$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N + \beta_{2(x)N}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N}{\bar{X}_N + \beta_{2(x)N}}\right)C_{xN}C_{yN}\rho_{xyN}\right]$
Upadhyaya and Singh [50] $\bar{y}_{USrN} = \bar{y}_N\,\frac{\bar{X}_N\,\beta_{2(x)N} + C_{xN}}{\bar{x}_N\,\beta_{2(x)N} + C_{xN}}$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + \left(\frac{\bar{X}_N\,\beta_{2(x)N}}{\bar{X}_N\,\beta_{2(x)N} + C_{xN}}\right)^2 C_{xN}^2 - 2\left(\frac{\bar{X}_N\,\beta_{2(x)N}}{\bar{X}_N\,\beta_{2(x)N} + C_{xN}}\right)C_{xN}C_{yN}\rho_{xyN}\right]$
Bahl and Tuteja [51] $\bar{y}_{BTrN} = \bar{y}_N\,exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right)$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + \frac{1}{4}C_{xN}^2 - C_{xN}C_{yN}\rho_{xyN}\right]$
Singh et al. [52] $\bar{y}_{RrN} = \bar{y}_N\,exp\left[\frac{(a\bar{X}_N + b) - (a\bar{x}_N + b)}{(a\bar{X}_N + b) + (a\bar{x}_N + b)}\right]$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + \left(\frac{a\bar{X}_N}{2(a\bar{X}_N + b)}\right)^2 C_{xN}^2 - \frac{1}{2}\left(\frac{2a\bar{X}_N}{a\bar{X}_N + b}\right) C_{xN}C_{yN}\rho_{xyN}\right]$
Khan et al. [53] $\bar{y}_{KNN} = \bar{y}_N exp\left[\alpha\left(\frac{\bar{X}_N^{\frac{1}{n}} - \bar{x}_N^{\frac{1}{n}}}{\bar{X}_N^{\frac{1}{n}} + (a-1)\bar{x}_N^{\frac{1}{n}}}\right)\right]$	$\theta_N\bar{Y}_N^2\left[C_{yN}^2 + \frac{\alpha^2 C_{xN}^2}{a^2h^2} - \frac{2\alpha C_{xN}C_{yN}\rho_{xyN}}{ah}\right]$ $MSE((\bar{y}_{KNN})_{min} = \theta_N\bar{Y}_N^2\,C_{yN}^2(1 - \rho_{xyN}^2)$

Remark 2. Choosing different values of α and γ will generate a variety of estimators for the above i-iv classes. These choices can either be real numbers or be made based on the available conventional and non-conventional parameters associated with auxiliary variable X. The choices are given in Table 3.

Table 3. Conventional and non-conventional measures

Conventional measure	Non-conventional measure
Standard deviation: $S_{xN} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{X}_N)^2}{(N-1)}}$	Median: M_{DN}
Coefficient of variation: $C_{xN} = \frac{S_{xN}}{\bar{X}_N}$	Quartile deviation: $QD_N = \frac{Q_{3N} - Q_{1N}}{2}$
Coefficient of skewness: $\beta_{1(xN)}$	Mid-range: $MR_N = \frac{x_{(1)} + x_{(N)}}{2}$
Coefficient of kurtosis: $\beta_{2(xN)}$	Inter-quartile range: $IQR_N = Q_{3N} - Q_{1N}$
Coefficient of correlation: $\rho_{yxN} = \frac{S_{yxN}}{S_{yN} S_{xN}}$	Quartile average: $QA_N = \frac{Q_{3N} + Q_{1N}}{2}$
	Tri-mean: $TM_N = \frac{Q_{1N} + 2Q_{2N} + Q_{3N}}{4}$
	Hodges-Lehmann: $HL_N = \text{Median}\left(\frac{x_j + x_k}{2}\right)$ $1 \le j \le k \le N$

Remark 3. Although we can select any of the measures given in Table 3 as all of them produce efficient results, only a few measures, i.e. $(\alpha = 1, \gamma = 1)$, $(\alpha = 1, \gamma = S_{xN})$, $(\alpha = C_{xN}, \gamma = S_{xN})$, $(\alpha = C_{xN}, \gamma = 1)$, $(\alpha = TM_N, \gamma = HL_N)$, $(\alpha = QD_N, \gamma = HL_N)$, have been used in order to keep the manuscript length manageable. The corresponding results for these measures are presented in Tables 5-9.

Bias, MSE and minimum MSE of $\bar{y}_{Np}$

Expressing the '$\bar{y}_{Np}$' in terms of $\xi's$, we have

$$\bar{y}_{Np} = w_{1N}\bar{Y}_N(1 + \xi_{yN})(1 + \nu\xi_{xN})^{-g} + w_{2N}\bar{Y}_N(1 + \xi_{yN})exp\left\{-\frac{\delta\nu\xi_{xN}}{2}\left(1 + \frac{\nu\xi_{xN}}{2}\right)^{-1}\right\} \quad (2)$$

where $\nu = \frac{\alpha\bar{X}_N}{(\alpha\bar{X}_N + \gamma)}$.

By expanding the right-hand side of Eq. (2), ignoring the terms of 's with powers greater than two and subtracting $\bar{Y}$ from both sides, we obtain

$$(\bar{y}_{Np} - \bar{Y}) = \bar{Y}_N\left[w_{1N}\left\{1 + \xi_{yN} - \nu g(\xi_{xN} + \xi_{yN}\xi_{xN}) + \frac{g(g+1)}{2}\nu^2\xi_{xN}^2\right\}\right.$$
$$\left. + w_{2N}\left\{1 + \xi_{yN} - \frac{\delta\nu}{2}(\xi_{xN} + \xi_{yN}\xi_{xN}) + \frac{\delta(\delta+2)}{8}\nu^2\xi_{xN}^2\right\} - 1\right] \quad (3)$$

To obtain the bias of the proposed neutrosophic estimator $\bar{y}_{Np}$ up to the first order of approximation, we take the expectation on both sides of Eq. (3):

$$Bias(\bar{y}_{Np}) \cong \bar{Y}_N\left[w_{1N}\left\{1 + \left(\frac{\theta_N\nu g}{2}\right)C_{xN}^2(\nu(g+1) - 2k)\right\}\right.$$
$$\left. + w_{2N}\left\{1 + \left(\frac{\theta_N\delta\nu}{8}\right)C_{xN}^2(\nu(\delta+2) - 4k)\right\} - 1\right]. \quad (4)$$

Taking square on both sides of Eq. (3), we have

$$(\bar{y}_{Np} - \bar{Y})^2 \cong \bar{Y}_N^2[1 + w_{1N}^2\{1 + 2\xi_{yN} - 2\nu g\xi_{xN} + \xi_{yN}^2 - 4\nu g\xi_{yN}\xi_{xN} + \nu^2 g(2g+1)\xi_{xN}^2\}$$
$$+ w_{2N}^2\left\{1 + 2\xi_{yN} - \delta\nu\xi_{xN} + \xi_{yN}^2 - 2\delta\nu\xi_{yN}\xi_{xN} + \frac{\delta(\delta+2)}{4}\nu^2\xi_{xN}^2 + \frac{\delta^2\nu^2\xi_{xN}^2}{4}\right\}$$

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$$+2w_{1N}w_{2N}\left\{1+2\xi_{yN}-\frac{v(2g+\delta)}{2}\xi_{xN}-v(2g+\delta)\xi_{yN}\xi_{xN}+\xi_{yN}^2+\frac{1}{8}[(2g+\delta)^2+2(2g+\delta)]v^2\xi_{xN}^2\right\}$$
$$-2w_{1N}\left\{1+\xi_{yN}-vg(\xi_{xN}+\xi_{yN}\xi_{xN})+\frac{g(g+1)}{2}v^2\xi_{xN}^2\right\}$$
$$-2w_{2N}\left\{1+\xi_{yN}-\frac{\delta v}{2}(\xi_{xN}+\xi_{yN}\xi_{xN})+\frac{\delta(\delta+2)}{8}v^2\xi_{xN}^2\right\}\Bigg] \quad (5)$$

In order to obtain the MSE of $\bar{y}_{Np}$, taking the expectation on both side of Eq. (5), we get

$$MSE(\bar{y}_{Np})=\bar{Y}_N^2[1+w_{1N}^2A_1+w_{2N}^2A_2+2w_{1N}w_{2N}A_3-2w_{1N}A_4-2w_{2N}A_5] \quad (6)$$

where

$$A_1=\left[1+\theta_N\{C_{yN}^2+(v^2(2g^2+g)-4vgk)C_{xN}^2\}\right]$$
$$A_2=\left[1+\theta_N\left\{C_{yN}^2+\left(\frac{v^2(\delta^2+\delta)}{2}-2\delta vk\right)C_{xN}^2\right\}\right]$$
$$A_3=\left[1+\theta_N\left\{C_{yN}^2\left(\frac{[(2g+\delta)^2+2(2g+\delta)]v^2}{8}-v(2g+\delta)k\right)C_{xN}^2\right\}\right]$$
$$A_4=\left[1+\theta_N\left\{\frac{v^2(g^2+g)}{2}-vgk\right\}C_{xN}^2\right]$$
$$A_5=\left[1+\theta_N\left\{\frac{(\delta^2+2\delta)}{8}v^2-\left(\frac{\delta v}{2}\right)k\right\}C_{xN}^2\right]$$

Partially differentiating $MSE(\bar{y}_{Np})$ with respect to w_{1N} and w_{2N} and equating them to zero, we obtain the following optimal weights of w_{1N} and w_{2N}:

$$w_{1N(opt)}=\frac{(A_2A_4-A_3A_5)}{(A_1A_2-A_3^2)} \quad ; \quad w_{2N(opt)}=\frac{(A_1A_5-A_3A_4)}{(A_1A_2-A_3^2)}$$

Putting the above optimal weights in Eq. (6), the expression for minimum MSE of the proposed neutrosophic estimator becomes

$$MSE_{min}(\bar{y}_{Np})\cong\bar{Y}_N^2\left[1-\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right] \quad (7)$$

DISCUSSION

A flow chart in Figure 1 presents the appropriate selection of neutrosophic versus classical methods for the estimation of the true but unknown parameters of the population. To assess the dominance of the suggested estimators, we conducted two types of studies: one based on observational/real-life data and the other based on simulated data. To study the relative efficiency of the proposed estimators, we calculated the per cent relative efficiencies (PREs) of all the estimators. A PRE may be defined as the ratio of MSE of the existing estimator to that of the proposed estimator. The MSE measures the divergence of the estimator's values from the true parameter value. PREs are calculated with respect to $\bar{y}_{RN}$ through the following formulae:

$$PRE(estimator)=\frac{MSE(\bar{y}_{RN})}{MSE\ (estimator)}\times 100$$

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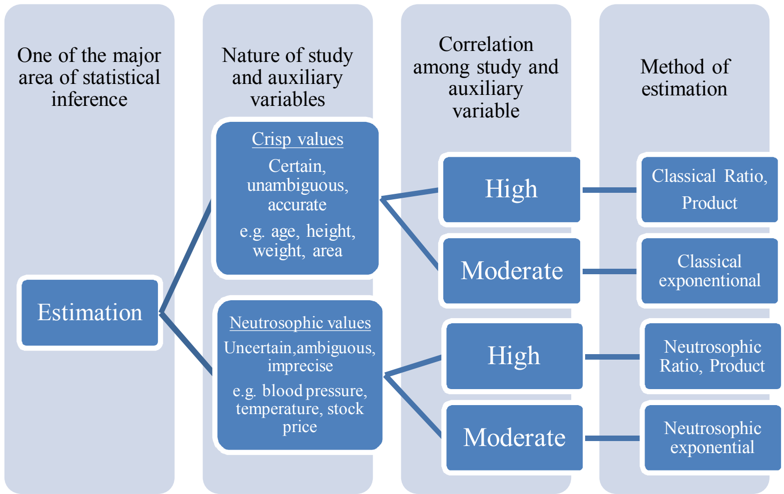

Figure 1. Decision flow chart for selecting classical and neutrosophic approaches

Observational Study

A real-life study based on monthly temperature data for Lahore in Punjab, Pakistan from 2014 to 2023, is taken from the weather website [55] for numerical illustration. These data are chosen because of the neutrosophic nature and fluctuating values of temperature as the temperature varies within an interval with vague values. The summary of statistics of temperature data is given in Table 4.

Table 4. Summary of statistics of temperature data

Month	$[\bar{Y}_{NL}, \bar{Y}_{NU}]$	$[C_{yNL}, C_{yNU}]$	$[\rho_{xyNL}, \rho_{xyNU}]$	$[\bar{X}_{NL}, \bar{X}_{NU}]$
January	[44, 63]	[0.035, 0.0402]	[-0.4835, 0.3606]	[5.5, 5.5]
February	[50, 73]	[0.039, 0.0488]	[0.3409, 0.4273]	$[C_{xNL}, C_{xNU}]$
March	[59, 81]	[0.0541, 0.0515]	[0.4242, 0.5764]	[0.5505, 0.5505]
April	[69, 94]	[0.0403, 0.0390]	[0.1878, 0.1774]	$[\beta_{(2)NL}, \beta_{(2)NU}]$
May	[77, 101]	[0.0310, 0.0292]	[-0.5289, -0.0729]	[-1.2, -1.2]
June	[80, 102]	[0.0224, 0.0286]	[-0.4314, -0.0767]	
July	[80, 95]	[0.0131, 0.0191]	[-0.5511, -0.0817]	
August	[80, 95]	[0.0163, 0.0143]	[-0.0509, 0.3885]	
September	[77, 94]	[0.0242, 0.0223]	[-0.1625, 0.4033]	
October	[68, 90]	[0.0314, 0.0253]	[-0.2865, -0.1222]	
November	[55, 78]	[0.0321, 0.0260]	[-0.2277, 0.4267]	
December	[45, 68]	[0.030, 0.0407]	[-0.1747, 0.5627]	

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Here,

- Y = Temperature and $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$ is the interval for all lower and upper bounds of the month-wise total average temperature.
- X = Coding of time from 1 to 10 (number of years) and $\bar{x}_N \in [\bar{x}_L, \bar{x}_U]$ is the interval for the lower and upper bounds of the average coding from 1 to 10.
- Population available: N=35 (years), sample taken: $n_N = [10, 10]$ (years).

The observations made from Tables 5-8 are as follows:

i) It can be clearly seen from the indeterminacy interval results given in Tables 5-8 that $\bar{y}_{PN}$ has higher PRE (i.e. $PRE_N \in [PRE_L, PRE_U]$) interval values (bolded values) compared to the existing estimators, i.e. $\bar{y}_{SDrN}, \bar{y}_{SKrN}, \bar{y}_{USrN}, \bar{y}_{BTrN}, \bar{y}_{RrN}$ and $\bar{y}_{KNN}$.

ii) All the proposed classes of estimators mentioned in Remark 1 exhibit higher PREs compared to all other estimators discussed in Table 2.

iii) All the proposed estimators perform efficiently regardless of whether conventional or non-conventional auxiliary information is available.

Simulation Study

In this section a Monte Carlo simulation study of the proposed innovative neutrosophic generalised estimator is conducted using R language to evaluate its performance based on simulated neutrosophic data. The following steps are performed:

- Neutrosophic survey/study variable Y_N is generated through neutrosophic normal distribution NN($[\bar{Y}_{NL}, \bar{Y}_{NU}]$,$[\sigma^2_{yNL}, \sigma^2_{yNU}]$) with parameters ([76.0, 84.9],[166.41, 295.84]).
- Neutrosophic auxiliary variable X_N is generated through a neutrosophic normal distribution NN($[\bar{X}_{NL}, \bar{X}_{NU}]$,$[\sigma^2_{xNL}, \sigma^2_{xNU}]$) with parameters ([171.2, 180.4]), [33.64,44.89]).
- To generate multivariate normal distributions, we use $Cov(Y_N, X_N) = \rho_{xyN}\sqrt{\delta^2{}_{YN}\,\delta^2{}_{xN}}$, where $[\rho_{xyNL}, \ \rho_{xyNU}]$= [0.0285, 0.0104].
- For sample sizes $n \in$ [20, 20] neutrosophic estimates are obtained with 1000 iterations.
- Neutrosophic PREs are obtained for the proposed neutrosophic estimator $\bar{y}_{PN}$ and the existing neutrosophic estimators $\bar{y}_{SDrN}, \bar{y}_{SKrN}, \bar{y}_{USrN}, \bar{y}_{BTrN}, \bar{y}_{RrN}$ and $\bar{y}_{KNN}$. The results are reported in Table 9.

Table 9 shows the indeterminacy interval results of the PRE_N values for the proposed and all competing estimators discussed throughout this study. The bolded values in the last column represent the highest PRE_N values, highlighting the superiority of the proposed classes (ratio-ratio type, product-product type, ratio-product type and product-ratio type) of neutrosophic estimators. Furthermore, the enhanced efficiency of $\bar{y}_{PN}$ holds for both conventional and non-conventional auxiliary information.

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Table 5. PREs of existing and proposed estimators relative to $\bar{y}_{RNL}, \bar{y}_{RNU}$: Some members of ratio-ratio-type exponential class of estimators $\bar{y}_{NpR}$

$[\bar{y}_{SDrNL}, \bar{y}_{SDrNU}]$	$[\bar{y}_{SKrNL}, \bar{y}_{SKrNU}]$	$[\bar{y}_{USrNL}, \bar{y}_{USrNU}]$	$[\bar{y}_{BTrNL}, \bar{y}_{BTrNU}]$	$[\bar{y}_{RrNL}, \bar{y}_{RrNU}]$	$[\bar{y}_{KNNL}, \bar{y}_{KNNU}]$	$[\bar{y}_{PNL}, \bar{y}_{PNU}]$
α =1, γ=1						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[30683.714, 34331.224]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[14503.037, 25252.6933]**
[121.819, 122.289]	[60.222, 59.727]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[13628.547, 14523.952]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[21031.048, 22710.735]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[40813.517, 93839.162]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[77376.251, 104303.823]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[722346.831, 866316.277]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[242146.319, 576130.089]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[88413.891, 114119.864]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.639]	[31737.351, 53041.664]	**[40595.414, 87364.071]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[41272.971, 74425.687]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[22582.965, 56108.953]**
$\alpha = 1, \gamma = S_{xN}$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[27689.393, 28769.160]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[13862.777, 23327.298]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[13113.250, 14033.040]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.903]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[19470.998, 20901.701]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[34502.733, 62184.367]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[55176.586, 70981.768]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[189989.644, 299090.562]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[173842.332, 223441.803]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[66293.459, 74675.769]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[34226.487, 61328.777]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[36067.595, 54947.208]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[20405.453, 47313.867]**

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Table 6. PREs of existing and proposed estimators relative to $\bar{y}_{RNL}, \bar{y}_{RNU}$: Some members of product-product-type exponential class of estimators $\bar{y}_{NpP}$

$[\bar{y}_{SDrNL}, \bar{y}_{SDrNU}]$	$[\bar{y}_{SKrNL}, \bar{y}_{SKrNU}]$	$[\bar{y}_{USrNL}, \bar{y}_{USrNU}]$	$[\bar{y}_{BTrNL}, \bar{y}_{BTrNU}]$	$[\bar{y}_{RrNL}, \bar{y}_{RrNU}]$	$[\bar{y}_{KNNL}, \bar{y}_{KNNU}]$	$[\bar{y}_{PNL}, \bar{y}_{PNU}]$
$\alpha = C_{xN}, \gamma = S_{xN}$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[26736.021, 27216.659]**
[121.584, 121.668]	[60.477, 60.399]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[13786.985, 23108.160]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[13089.518, 14092.759]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[19128.353, 20509.555]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[32791.589, 54052.505]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[49210.552, 64115.775]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[135653.987, 206928.549]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[147452.288, 168332.024]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[62886.815, 66566.964]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[32438.308, 55049.631]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[35323.975, 50080.334]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[19693.552, 46441.929]**
$\alpha = C_{xN}, \gamma = 1$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[28821.243, 30076.898]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[14483.263, 25204.514]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[13746.184, 14939.825]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[20405.064, 21992.953]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[36500.824, 63885.006]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[57417.965, 80438.582]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[228127.354, 619799.364]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[296657.312, 368094.581]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[80491.894, 84016.372]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[36012.217, 66042.801]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[40395.248, 59102.172]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[20892.193, 56056.703]**

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Table 7. PREs of existing and proposed estimators relative to $\bar{\boldsymbol{y}}_{\boldsymbol{RNL}}, \bar{\boldsymbol{y}}_{\boldsymbol{RNU}}$: Some members of ratio-product-type exponential class of estimators $\bar{y}_{NpRP}$

$[\bar{y}_{SDrNL}, \bar{y}_{SDrNU}]$	$[\bar{y}_{SKrNL}, \bar{y}_{SKrNU}]$	$[\bar{y}_{USrNL}, \bar{y}_{USrNU}]$	$[\bar{y}_{BTrNL}, \bar{y}_{BTrNU}]$	$[\bar{y}_{RrNL}, \bar{y}_{RrNU}]$	$[\bar{y}_{KNNL}, \bar{y}_{KNNU}]$	$[\bar{y}_{PNL}, \bar{y}_{PNU}]$
$\alpha = C_{xN}, \gamma = S_{xN}$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[26690.213, 27471.138]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[13898.672, 23334.430]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[13203.774, 14234.709]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[19260.332, 20654.333]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[32929.052, 53770.564]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[49090.451, 64363.117]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[134065.787, 207524.368]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[149844.462, 168999.023]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[63664.373, 66717.203]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[32553.203, 55062.383]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[35707.238, 50154.392]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[19756.697, 47100.407]**
$\alpha = C_{xN}, \gamma = 1$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[28511.039, 30920.981]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[14803.007, 25948.813]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[14082.076, 15396.543]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[20775.572, 22406.662]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[36828.314, 62226.899]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[56528.695, 81067.929]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[211591.940, 619632.579]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[326343.626, 371581.746]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[84177.829, 84120.117]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[36252.109, 65634.228]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[41815.059, 59027.764]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[21011.700, 58986.874]**

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Table 8. PREs of existing and proposed estimators relative to $\bar{\boldsymbol{y}}_{\boldsymbol{RNL}}, \bar{\boldsymbol{y}}_{\boldsymbol{RNU}}$: Some members of product-ratio-type exponential class of estimators $\bar{y}_{NpPR}$

$[\bar{y}_{SDrNL}, \bar{y}_{SDrNU}]$	$[\bar{y}_{SKrNL}, \bar{y}_{SKrNU}]$	$[\bar{y}_{USrNL}, \bar{y}_{USrNU}]$	$[\bar{y}_{BTrNL}, \bar{y}_{BTrNU}]$	$[\bar{y}_{RrNL}, \bar{y}_{RrNU}]$	$[\bar{y}_{KNNL}, \bar{y}_{KNNU}]$	$[\bar{y}_{PNL}, \bar{y}_{PNU}]$
$\boldsymbol{\alpha = TM_N, \gamma = HL_N}$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[30958.459, 33067.542]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[15255,738, 27350.274]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[14452.525, 15744.693]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[21808.543, 23608.089]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[40539.013, 74759.156]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[66504.738, 101209.738]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[471791.214, 1229102.314]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[1232973.643, 3707035.870]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[102977.899, 106449.685]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[39894.932, 78932.573]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[45761.667, 69462.939]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[22230.787, 66618.822]**
$\boldsymbol{\alpha = QD_N,, \gamma = HL_N}$						
[120.105, 121.498]	[60.586, 62.133]	[83.734, 84.544]	[370.198, 414.708]	[502.302, 585.753]	[26331.556, 26620.960]	**[28219.701, 29253.938]**
[121.584, 121.668]	[60.399, 60.477]	[83.636, 83.680]	[416.196, 420.359]	[587.434, 596.474]	[13640.984, 22660.920]	**[14355.084, 24617.926]**
[121.819, 122.289]	[59.727, 60.222]	[83.280, 83.545]	[424.241, 441.533]	[602.835, 636.980]	[12948.103, 13897.813]	**[13615.555, 14707.036]**
[121.202, 121.219]	[60.884, 60.901]	[83.893, 83.902]	[404.082, 404.782]	[564.928, 566.360]	[18867.571, 20205.764]	**[20127.141, 21651.405]**
[120.314, 120.841]	[61.307, 61.904]	[84.114, 84.423]	[377.017, 393.110]	[515.174, 544.809]	[32062.290, 52250.690]	**[35439.992, 60342.410]**
[120.438, 120.903]	[61.246, 61.763]	[84.080, 84.350]	[380.761, 395.602]	[522.042, 549.893]	[47684.029, 61230.080]	**[54605.917, 74551.796]**
[120.546, 120.958]	[61.189, 61.647]	[84.049, 84.289]	[384.518, 397.787]	[529.312, 554.317]	[123995.996, 178560.140]	**[184658.435, 368433.574]**
[120.970, 121.282]	[60.833, 61.175]	[83.862, 84.042]	[398.147, 408.513]	[554.971, 574.755]	[131358.084, 149066.487]	**[220753.602, 262577.171]**
[120.821, 121.414]	[60.685, 61.336]	[83.785, 84.127]	[392.995, 412.663]	[544.993, 582.474]	[59720.623, 63516.094]	**[73882.770, 77651.764]**
[120.658, 120.772]	[61.384, 61.517]	[84.154, 84.222]	[387.685, 390.890]	[534.960, 540.693]	[31737.351, 53041.664]	**[34994.478, 62234.296]**
[120.710, 121.563]	[60.519, 61.458]	[83.698, 84.191]	[389.275, 417.312]	[537.888, 591.070]	[34287.892, 48401.909]	**[38757.380, 56078.107]**
[120.583, 121.746]	[60.326, 61.587]	[83.595, 84.262]	[384.266, 424.025]	[527.820, 604.391]	[19459.591, 44542.563]	**[20639.589, 52531.388]**

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Table 9. PREs of existing and proposed estimators relative to $[\bar{y}_{RNL}, \bar{y}_{RNU}]$ under simulation study

Ratio-ratio-type exponential estimator $\bar{y}_{NpR}$						
$[\bar{y}_{SDrNL}, \bar{y}_{SDrNU}]$	$[\bar{y}_{SKrNL}, \bar{y}_{SKrNU}]$	$[\bar{y}_{USrNL}, \bar{y}_{USrNU}]$	$[\bar{y}_{BTrNL}, \bar{y}_{BTrNU}]$	$[\bar{y}_{RrNL}, \bar{y}_{RrNU}]$	$[\bar{y}_{KNNL}, \bar{y}_{KNNU}]$	$[\bar{y}_{PNL}, \bar{y}_{PNU}]$
$\alpha = 1, \gamma = 1$						
[100.224, 100.226]	[101.039, 100.351	[100.916, 101.901]	[102.315, 102.449]	[102.863, 102.938]	[102.993, 102.995]	**[103.651, 103.997]**
$\alpha = 1, \gamma = S_{xN}$						
[100.225, 100.227]	[101.036, 100.355]	[100.911, 101.922]	[102.339, 102.442]	[102.893, 102.928]	[102.984, 103.026]	**[103.122, 103.218]**
$\alpha = C_{xN}, \gamma = S_{xN}$						
[102.392, 102.427]	[102.713, 102.935]	[102.934, 103.085]	[102.384, 102.407]	[102.874, 103.080]	[102.939, 103.085]	**[103.079, 103.278]**
Product-product-type exponential estimators $\bar{y}_{NpP}$						
$\alpha = c_x, \gamma = S_{xN}$						
[102.296, 102.387]	[102.613, 102.888]	[102.887, 102.969]	[102.296, 102.370]	[102.827, 102.964]	[102.893, 102.969]	**[103.109, 103.256]**
$\alpha = c_x, \gamma = 1$						
[102.304, 102.353]	[102.619, 102.847]	[102.847, 102.976]	[102.302, 102.339]	[102.786, 102.971]	[102.852, 102.977]	**[103.069, 103.262]**
$\alpha = TM_N, \gamma = HL_N$						
[100.001, 100.001]	[100.002, 100.008]	[100.0067, 100.021]	[102.349, 102.438]	[102.357, 102.447]	[102.980, 103.039]	**[103.194, 103.321]**
Ratio-product-type exponential estimator $\bar{y}_{NpRP}$						
$\alpha = c_x, \gamma = S_{xN}$						
[102.310, 102.477]	[102.629, 102.994]	[102.988, 102.993]	[102.310, 102.453]	[102.933, 102.983	[102.988, 102.998]	**[103.212, 103.270]**
$\alpha = c_x, \gamma = 1$						
[102.313, 102.500]	[102.635, 103.028]	[102.994, 103.027]	[102.315, 102.479]	[102.967, 102.989	[102.995, 103.033]	**[103.246, 103.276]**
$\alpha = QD_N, \gamma = HL_N$						
[100.033, 100.082]	[100.054, 100.431]	[100.476, 100.374]	[102.275, 102.450]	[102.450, 102.766]	[102.942, 102.995]	**[103.216, 103.228]**
Product -ratio- -type exponential estimator $\bar{y}_{NpPR}$						
$\alpha = TM_N, \gamma = HL_N$						
[100.001, 100.001]	[100.002, 100.008]	[100.007, 100.021]	[102.293, 102.399]	[102.301, 102.407]	[102.929, 102.966]	**[103.066, 103.163]**
$\alpha = QD_N, \gamma = HL_N$						
[100.033, 100.079]	[100.054, 100.420]	[100.360, 100.476]	[102.283, 102.385]	[102.459, 102.692]	[102.912, 102.952]	**[103.051, 103.146]**

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CONCLUSIONS

This study demonstrates the effectiveness of neutrosophic estimation techniques in addressing the challenges posed by unreliable and uncertain data in population mean estimation. By introducing a generalised estimator that incorporates both conventional and non-conventional auxiliary information, we have shown that the proposed estimators offer greater flexibility and efficiency compared to classical methods. The derived expressions for bias, MSE and minimum MSE further validate the performance of these estimators. The results from both natural and simulated data emphasise their superiority, which may be further explored for their application in various contexts where ambiguous data are prevalent.

Future research may be performed to enhance neutrosophic estimation by combining it with advanced sampling designs such as stratified random sampling, systematic sampling and two-phase sampling. The proposed work may also be extended to scenarios involving non-response and/or measurement errors, as well as situations with missing observations.

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