Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

Maejo International Journal of Science and Technology

e-ISSN 2697-4746
Available online at www.mijst.mju.ac.th

Full Paper

Strong unique continuation for a class of degenerate elliptic operators

Ahmed Loulit [1] and Ferit Gurbuz [2, *]

[1] Department of Mathematics, Research Center E. Bernheim Solvay Business School, Free University of Brussels (ULB), av F.D. Roosevelt 21, CP 135/01, B-1050 Brussels, Belgium
[2] Department of Mathematics, Kirklareli University, Kirklareli 39100, Turkey
* Corresponding author, e-mail: feritgurbuz@klu.edu.tr

Received: 26 October 2025 / Accepted: 30 April 2026 / Published: 7 May 2026

Abstract: The unique continuation property for solutions of second-order elliptic equations has been widely studied owing to its fundamental role in partial differential equations and related fields. While extensive results are available for strictly elliptic operators, comparatively little is known in the degenerate setting. In this paper we establish a strong unique continuation property for a class of degenerate elliptic operators involving potentials in the Kato class. The analysis is carried out under minimal regularity assumptions and the results extend several known contributions in the literature. In particular, we show that solutions that vanish in a neighbourhood, or satisfy appropriate vanishing conditions, must be identically zero. The theoretical findings are further supported by an illustrative example, highlighting the behaviour of solutions and confirming the robustness of the unique continuation property even in the presence of degeneracy and singular potentials.

Keywords: unique continuation, Kato class, degenerate elliptic operators, semigroups of operators, partial differential equations

---

INTRODUCTION

In his work "Schrödinger semigroups", Simon [1] conjectured that the operator $L = -\Delta + V$ satisfies the unique continuation property when the potential V belongs locally to the Kato class K^n_{loc}. More precisely, if $\Omega \subset \mathbb{R}^n$ is a connected open set, then any solution u of $Lu = 0$ in Ω that vanishes on a non-empty open subset $\Omega_0 \subset \Omega$ must be identically zero. This conjecture was positively resolved by Fabes et al. [2] in the case where $V \in K^n_{\mathrm{loc}}$, $n \geq 3$, is a radial function.

In the present work we extend the result of Ling [3] for the class of operators

$$Lu = -\operatorname{div}(|u|^{p-2}\nabla u) + V|u|^{p-2}u + f(x, u, \nabla u), p \geq 2, \quad (1)$$

where $u: \Omega \to \mathbb{R}^s$, $s \geq 1$ under the following assumptions:

(i) $V \in K^n_{\mathrm{loc}}$ is a radial function,

(ii) $| f(x,u,\nabla u) | \leq | b(x) || u |^{p-2} | \nabla u |$,

(iii) for every $x_0 \in \Omega$, there exists $r_0 > 0$ and an increasing function $h: (0, r_0) \to \mathbb{R}^+$ such that

$$\int_0^{r_0} \frac{h(r)}{r} dr < \infty$$

and

$$| b(x) | \leq C \frac{h(| x - x_0 |)}{| x - x_0 |}, C > 0.$$

Here, $B_{r_0}(x_0)$ denotes the ball centred at x_0 with radius r_0.

The first work addressing the unique continuation property for such operators is by Ling [3], who proved that if $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ is a solution of

$$-\mathrm{div}(| u |^{p-2} \nabla u) + V | u |^{p-2} u = 0$$

with $V \in L^\infty(\Omega)$, and if

$$\int_{\partial B_r(x_0)} | u |^{2(p-1)} = O(r^N), \forall N > 0,$$

then $u \equiv 0$ in Ω.

Since $L^\infty \subset K^n$, our results generalise those of Ling [3]. We recall that a measurable function V belongs to the Kato class $K^n(\Omega)$, $n \geq 3$, if

$$\lim_{r \to 0} \sup_{x \in \Omega} \int_{|x-y|<r} \frac{| V(y) |}{| x - y |^{n-2}} dy = 0.$$

Moreover, $V \in K^n_{\mathrm{loc}}(\Omega)$ if this condition holds locally. It is well known that $L^p_{\mathrm{loc}} \subset K^n_{\mathrm{loc}}$ for $p > n/2$ while the two classes are generally incomparable when $1 < p \leq n/2$. We also note that Aizenman and Simon [4] showed that the natural functional setting ensuring the continuity of solutions to the equation

$$Lu = \Delta u + Vu = 0$$

is given by the Kato class. This result was later extended by Chiarenza et al. [5] to more general operators of the form

$$Lu = -\mathrm{div}(A(x)\nabla u) + Vu,$$

where $A(x)$ is a uniformly elliptic matrix with bounded measurable coefficients. Several related results have been obtained in the literature [5-13].

In what follows, we assume that $n \geq 3$. Due to the local nature of the problem, we restrict our analysis to a fixed ball centred at the origin, namely the unit ball B_1. More generally, we denote by B_r the open ball centred at the origin with radius $0 < r < 1$. We assume that the potential $V \in K^n(B_1)$ is radial, i.e. $V(x) = V(| x |)$ for all $x \in B_1$. According to Proposition 4.10 of Aizenman and Simon [4], the condition $V \in K^n(B_1)$ is equivalent to

$$\varepsilon(r_0) = \int_0^{r_0} r\, g(r) dr < \infty, \ \forall r_0 \in (0,1),$$

where $g(r) = | V(r) |$.

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

STATEMENT OF RESULTS

In this section we consider solutions of the operator

$$Lu = -\mathrm{div}(|\, u \,|^{p-2}\, \nabla u) + V \,|\, u \,|^{p-2}\, u + f(x,u,\nabla u), p \geq 2$$

under the assumptions (i), (ii) and (iii).

Theorem 1. Let $u \in W^{2,2}_{\mathrm{loc}}(B_1)$ be a continuous solution of the above equation in B_1. Then there exists $r_0 > 0$, depending only on the local properties of V and b, such that if there exist constants $A, \alpha > 0$ satisfying

$$\int_{B_r} |\, u \,|^{2(p-1)}\, dx = O\left(\exp\left[-\frac{A}{r^{\alpha\epsilon(r_0)}}\right]\right) \text{ as } r \to 0.$$

Then $u \equiv 0$ in B_{r_0}.

To prove this theorem, we follow the approach of Fabes et al. [2], which is based on geometric and variational techniques [14,15]. The main idea is to derive quantitative estimates for the solutions of the operator L, which provide detailed information about the behaviour of their zeros. More precisely, we obtain the following result.

Theorem 2. Let $u \in W^{2,2}_{\mathrm{loc}}(B_1)$ be a continuous solution of equation (1) in B_1. Then there exist $r_0 > 0$ depending on the local norm of V, a constant $c = c(n) > 0$, a constant $B > 0$ depending on u, r, and the local properties of V and b, and a constant $\beta > 0$ depending on n and the local properties of V and b, such that if $u \equiv 0$ in B_r for all $r \in (0, r_0/2)$, then

$$\int_{B_{2r}} |\, u \,|^{2(p-1)}\, dx \leq c\exp\left[\frac{B}{r^{\beta\epsilon(r_0)}}\right] \int_{B_r} |\, u \,|^{2(p-1)}\, dx.$$

Theorem 3. Let $u \in W^{2,2}_{\mathrm{loc}}(B_1)$ be a solution of (1) in B_1, with $V \in L^{\infty}(B_1)$ and $f \equiv 0$. Then there exists $r_0 > 0$, depending on the L^{∞}-norm of V, such that if

$$\int_{B_r} |\, u \,|^{2(p-1)}\, dx = O(r^N), \forall N > 0,$$

as $r \to 0$, then $u \equiv 0$ in B_{r_0}.

The proof of Theorem 3 is based, among other arguments, on quantitative estimates of the solutions of the operator L, which provide information about the structure of their zero sets. More precisely, we have the following result.

Theorem 4. Let $u \in W^{2,2}_{\mathrm{loc}}(B_1)$ be a solution of (1) in B_1, with $V \in L^{\infty}(B_1)$. Then there exists $r_0 > 0$, sufficiently small and depending on the L^{∞}-norm of V, such that

$$\int_{B_{2R}} |\, u \,|^{2(p-1)}\, dx \leq c \int_{B_R} |\, u \,|^{2(p-1)}\, dx, \forall R \in (0, r_0/2),$$

where the constant c does not depend on R.

Remark. Observe that the continuity assumption on the solution u is no longer required in Theorem 3.

PROOF OF RESULTS

Let $u \in W^{2,2}_{\mathrm{loc}}(B_1)$ be a solution of

$$Lu = -\mathrm{div}(|\, u \,|^{p-2}\, \nabla u) + V \,|\, u \,|^{p-2}\, u + f(x,u,\nabla u), p \geq 2.$$

For a ball B_r centred at the origin with radius $0 < r < \frac{1}{2}$, we define

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

$$H(r) = \int_{\partial B_r} w^2 u^2 \, d\sigma,$$

$$D(r) = (p-1) \int_{B_r} w^2 \mid \nabla u \mid^2 dx,$$

$$I(r) = (p-1) \int_{B_r} w^2 \mid \nabla u \mid^2 dx + \int_{B_r} V u^2 w^2 dx + \int_{B_r} wuf(x, u, \nabla u) dx,$$

where $w = \mid u \mid^{p-2}$.

Proposition 1. Let $u \in W^{2,2}_{\text{loc}}(B_1)$ be a continuous solution of equation (1). Then there exists $r_0 > 0$, which is sufficiently small and depends on the local properties of V and b such that either

$$H(r) \neq 0 \quad \forall r \in (0, r_0)$$

or

$$u \equiv 0 \text{ in } B_{r_0}.$$

The proof of this proposition relies on the following well-known inequality:

$$\int_{B_r} \frac{u^2}{|x|^2} dx \leq c_n \left[\frac{1}{r} \int_{\partial B_r} u^2 \, d\sigma + \int_{B_r} \mid \nabla u \mid^2 dx \right], \tag{2}$$

which is valid for all $u \in C^\infty(\mathbb{R}^n)$ and $r > 0$, as well as on the following lemma.

Lemma 1 [2]**.** Let $V \in K^n$. Then there exists a constant $c_n > 0$, depending only on n, such that for all $u \in C^\infty(\mathbb{R}^n)$ and all $r > 0$,

$$\int_{B_r} \mid V \mid u^2 dx \leq c_n \eta(r) \left[\frac{1}{r} \int_{\partial B_r} u^2 \, d\sigma + \int_{B_r} \mid \nabla u \mid^2 dx \right], \tag{3}$$

where

$$\eta(r) = \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \frac{\mid V(y) \mid}{\mid x - y \mid^{n-2}} dy.$$

Proof of Proposition 1.

Assume that $H(r_0) = 0$ for some sufficiently small $r_0 > 0$. Since u is continuous, it follows that

$$u = 0 \text{ on } \partial B_{r_0}.$$

By the divergence theorem, we have

$$\int_{\partial B_{r_0}} w^2 \, u \nabla u \cdot \nu \, d\sigma = (p-1) \int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx + \int_{B_{r_0}} V w^2 u^2 dx + \int_{B_{r_0}} wuf(x, u, \nabla u) dx, \tag{4}$$

where $\nu = \frac{x}{|x|}$.

Since $u = 0$ on ∂B_{r_0}, the left-hand side of (4) vanishes. Hence

$$(p-1) \int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx + \int_{B_{r_0}} V w^2 u^2 dx + \int_{B_{r_0}} wuf(x, u, \nabla u) dx = 0.$$

Using the assumptions on V and b, together with the inequalities (2) and (3), we obtain the estimates

$$\mid \int_{B_{r_0}} wuf(x, u, \nabla u) dx \mid \leq C_1(n, p) h(r_0) \int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx$$

and

$$\int_{B_{r_0}} V w^2 u^2 dx \leq C_2(n, p) \eta(r_0) \int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx.$$

Therefore,

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

$$\int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx \le C_3(n,p)\delta(r_0) \int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx,$$

where

$$\delta(r) = h(r) + \eta(r).$$

Since $\lim_{r\to 0} \delta(r) = 0$, it follows that

$$\int_{B_{r_0}} w^2 \mid \nabla u \mid^2 dx = 0$$

for sufficiently small r_0.

Hence u is constant in B_{r_0}. Since $u = 0$ on ∂B_{r_0}, we conclude that

$$u \equiv 0 \text{ in } B_{r_0}.$$

We now introduce the frequency function

$$N(r) = \frac{rI(r)}{H(r)}$$

and define

$$\Omega_{r_0} = \{r \in (0,r_0): N(r) > \max(1, N(r_0))\}.$$

Corollary 1 [16]**.** The function $r \mapsto N(r)$ is absolutely continuous on $(0,r_0)$ and therefore differentiable almost everywhere. Consequently, Ω_{r_0} is an open subset of $\mathbb{R}$ and can be written as

$$\Omega_{r_0} = \bigcup_{j=1}^{\infty} (a_j, b_j), \qquad (a_j, b_j) \subset \Omega_{r_0}.$$

Lemma 2. Let u and r_0 be as in Proposition 1. Then there exists a constant $c > 0$, depending only on the local norm of V such that

$$D(r) \le c\, I(r), \qquad \forall r \in \Omega_{r_0}.$$

Proof. We have

$$I(r) = D(r) + \int_{B_r} V\, w^2 u^2 dx + \int_{B_r} wuf(x,u,\nabla u) dx,$$

which implies

$$D(r) \le I(r) + \int_{B_r} \mid V \mid w^2 u^2 dx + \int_{B_r} \mid wuf(x,u,\nabla u) \mid dx.$$

As in the proof of Proposition 1, we obtain

$$D(r) \le I(r) + C(n,p)\delta(r) \left[\frac{H(r)}{r} + D(r)\right].$$

On the other hand, since $r \in \Omega_{r_0}$, we have $N(r) > 1$. Hence

$$\frac{H(r)}{r} < I(r).$$

Therefore,

$$D(r) \le I(r) + C(n,p)\delta(r)(I(r) + D(r)).$$

Choosing $r < r_0$ sufficiently small, we obtain

$$D(r) \le c\, I(r).$$

Lemma 3. We have

$$H'(r) = \frac{n-1}{r} H(r) + 2(p-1)I(r).$$

Proof. Let $\nu = \frac{x}{|x|}$. By the divergence theorem,

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

$$H(r) = \int_{\partial B_r} w^2 u^2 d\sigma = \int_{B_r} \nabla(w^2 u^2) \cdot \nu \, dx + (n-1) \int_{B_r} \frac{w^2 u^2}{|x|} \, dx.$$

Expanding the gradient term yields

$$H(r) = 2(p-1) \int_{B_r} w^2 u \nabla u \cdot \nu \, dx + (n-1) \int_{B_r} \frac{w^2 u^2}{|x|} \, dx.$$

Differentiating with respect to r, we obtain

$$H'(r) = 2(p-1) \int_{\partial B_r} w^2 u \nabla u \cdot \nu \, d\sigma + \frac{n-1}{r} \int_{\partial B_r} w^2 u^2 \, d\sigma.$$

Using again the divergence theorem,

$$\int_{\partial B_r} w^2 u \nabla u \cdot \nu \, d\sigma = I(r),$$

hence

$$H'(r) = 2(p-1) I(r) + \frac{n-1}{r} H(r).$$

Lemma 4. We have

$$\begin{aligned} I'(r) = {} & \frac{n-2}{r} I(r) + 2(p-1) \int_{\partial B_r} w^2 (\nabla u \cdot \nu)^2 d\sigma \\ & + \int_{\partial B_r} V w^2 u^2 d\sigma + \int_{\partial B_r} wuf(x,u,\nabla u) d\sigma \\ & - \frac{n-2}{r} \int_{B_r} V w^2 u^2 dx - \frac{2(p-1)}{r} \int_{B_r} (\nabla u \cdot x) V w^2 u dx \\ & - \frac{n-2}{r} \int_{B_r} wuf(x,u,\nabla u) dx - \frac{2(p-1)}{r} \int_{B_r} (\nabla u \cdot x) wf(x,u,\nabla u) dx. \end{aligned}$$

Proof. From the definition,

$$I(r) = (p-1) \int_{B_r} w^2 |\nabla u|^2 \, dx + \int_{B_r} V w^2 u^2 dx + \int_{B_r} wuf(x,u,\nabla u) dx,$$

differentiating, we obtain

$$I'(r) = (p-1) \int_{\partial B_r} w^2 |\nabla u|^2 \, d\sigma + \int_{\partial B_r} V w^2 u^2 d\sigma + \int_{\partial B_r} wuf(x,u,\nabla u) d\sigma.$$

Using the identity

$$\int_{\partial B_r} w^2 |\nabla u|^2 \, d\sigma = \frac{1}{r} \int_{B_r} \nabla(w^2 |\nabla u|^2) \cdot x \, dx + \frac{n}{r} \int_{B_r} w^2 |\nabla u|^2 \, dx,$$

and applying the divergence theorem together with standard computations, we obtain

$$\begin{aligned} \int_{\partial B_r} w^2 |\nabla u|^2 \, d\sigma = {} & \frac{n-2}{r} \int_{B_r} w^2 |\nabla u|^2 \, dx + 2 \int_{\partial B_r} w^2 (\nabla u \cdot \nu)^2 d\sigma \\ & - \frac{2}{r} \int_{B_r} w (\nabla u \cdot x) \operatorname{div}(w \nabla u) dx. \end{aligned}$$

Using the identity

$$\operatorname{div}(w \nabla u) = Vwu + f(x,u,\nabla u),$$

and substituting into the above expression, we obtain the desired formula for $I'(r)$.

Lemma 5. Let $N(r)$ be defined by

$$N(r) = \frac{rI(r)}{H(r)}.$$

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

Then there exist constants $c > 0$, depending only on the local properties of V and b, and $L > 0$, depending on u and the local properties of V and b, such that

$$N(r) \leq \frac{L}{r^{c\varepsilon(r_0)}}, \qquad \forall r \in (0, r_0).$$

Proof. We first observe that

$$\int_{\partial B_r} V\, u^2 w^2 d\sigma \leq c\, g(r)\, H(r)$$

and, for $r \in \Omega_{r_0}$,

$$H(r) \leq rI(r).$$

Hence

$$\int_{\partial B_r} V\, u^2 w^2 d\sigma \leq c\, r g(r)\, I(r).$$

On the other hand,

$$\int_{B_r} |\, V \,| u^2 w^2 dx + \int_{B_r} |\, wuf(x, u, \nabla u) \,|\, dx \leq c\, \delta(r) \left(\frac{H(r)}{r} + D(r)\right).$$

Using Lemma 2, we deduce

$$\frac{n-2}{r}\left(\int_{B_r} |V|\, u^2 w^2 dx + \int_{B_r} |wuf(x, u, \nabla u)|\, dx\right) \leq C \frac{\delta(r)}{r} I(r).$$

Define

$$\tilde{I}(r) = D(r) + \int_{B_r} |\, V \,| u^2 w^2 dx + \int_{B_r} |\, wuf(x, u, \nabla u) \,|\, dx.$$

Then

$$I(r) \leq \tilde{I}(r) \leq CI(r), \qquad \forall r \in \Omega_{r_0}.$$

Moreover,

$$I(r) = \int_{\partial B_r} w^2\, u \nabla u \cdot \nu d\sigma.$$

We estimate the term involving V:

$$\frac{2(p-1)}{r} \left| \int_{B_r} (\nabla u \cdot x) V u\, w^2 dx \right| \leq \frac{2(p-1)}{r} \int_0^r \rho g(\rho) I(\rho) d\rho \leq \frac{2(p-1)}{r} \tilde{I}(r) \int_0^r \rho g(\rho) d\rho.$$

Thus,

$$\frac{2(p-1)}{r} \left| \int_{B_r} (\nabla u \cdot x) V u\, w^2 dx \right| \leq c \frac{\varepsilon(r)}{r} I(r).$$

We now estimate the terms involving $f(x, u, \nabla u)$. For this, we use the inequality

$$\int_{\partial B_r} w^2\, |\, \nabla u \,|^2\, d\sigma \leq C \int_{\partial B_r} w^2\, (\nabla u \cdot \nu)^2 d\sigma.$$

Furthermore, by the Cauchy-Schwarz inequality,

$$I(r) \leq C \left(\int_{\partial B_r} w^2\, (\nabla u \cdot \nu)^2 d\sigma\right)^{1/2} \left(\int_{\partial B_r} w^2\, u^2 d\sigma\right)^{1/2}.$$

Using $H(r) \leq rI(r)$, we obtain

$$I(r) \leq Cr \int_{\partial B_r} w^2\, (\nabla u \cdot \nu)^2 d\sigma.$$

Case 1: Assume that

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

$$\int_{\partial B_r} w^2\,(\nabla u\cdot\nu)^2 \int_{\partial B_r} w^2\,u^2 \le 2\left(\int_{\partial B_r} w^2\,u(\nabla u\cdot\nu)\right)^2.$$

Then

$$\left|\int_{\partial B_r} wuf(x,u,\nabla u)d\sigma\right| \le C\frac{h(r)}{r}I(r).$$

Using Lemma 4, we obtain

$$\frac{I'(r)}{I(r)} \ge \frac{n-2}{r} + \frac{2(p-1)}{I(r)}\int_{\partial B_r} w^2\,(\nabla u\cdot\nu)^2 - C\left(rg(r)+\frac{\varepsilon(r)}{r}\right).$$

Combining this with Lemma 3, we deduce

$$\frac{N'(r)}{N(r)} \ge -C\left(rg(r)+\frac{\varepsilon(r)}{r}\right).$$

Case 2: Assume that

$$\int_{\partial B_r} w^2\,(\nabla u\cdot\nu)^2 \int_{\partial B_r} w^2\,u^2 \ge 2\left(\int_{\partial B_r} w^2\,u(\nabla u\cdot\nu)\right)^2.$$

Then

$$\left|\int_{\partial B_r} wuf(x,u,\nabla u)d\sigma\right| \le Ch(r)\left(\frac{I(r)}{r} + \int_{\partial B_r} w^2\,(\nabla u\cdot\nu)^2\right).$$

Proceeding as before, we obtain

$$\frac{N'(r)}{N(r)} \ge -C\left(rg(r)+\frac{\varepsilon(r)}{r}\right).$$

Conclusion: Let $(a_j, b_j) \subset \Omega_{r_0}$. Integrating over (r, b_j), we obtain

$$\ln\frac{N(b_j)}{N(r)} \ge -c\left(\int_0^{r_0} \rho\, g(\rho)\, d\rho + \varepsilon(r_0)\ln\frac{b_j}{r}\right).$$

Thus,

$$N(r) \le N(b_j)\, e^{c\varepsilon(r_0)}\left(\frac{b_j}{r}\right)^{c\varepsilon(r_0)}.$$

Since $N(b_j) \le \max(1, N(r_0))$, we conclude

$$N(r) \le \frac{L}{r^{c\varepsilon(r_0)}}$$

for some constant $L > 0$.

Proof of Theorem 2. From Lemma 3 and the definition of $N(r)$, we have

$$\frac{d}{dr}\left(\ln\frac{H(r)}{r^{n-1}}\right) = \frac{H'(r)}{H(r)} - \frac{n-1}{r} = 2(p-1)\frac{I(r)}{H(r)} = 2(p-1)\frac{N(r)}{r}.$$

Integrating the above identity between R and $2R$, with $0 < 2R < r_0$, we obtain

$$\ln\left(2^{1-n}\frac{H(2R)}{H(R)}\right) \le 2(p-1)\int_R^{2R}\frac{N(r)}{r}dr.$$

Using Lemma 5, we deduce

$$\ln\left(2^{1-n}\frac{H(2R)}{H(R)}\right) \le 2(p-1)\int_R^{2R}\frac{L}{r^{1+c\varepsilon(r_0)}}dr \le \frac{2(p-1)L}{c\varepsilon(r_0)}R^{-c\varepsilon(r_0)}.$$

Therefore,

$$H(2R) \le 2^{n-1}\exp\left(\frac{B}{R^{\beta}}\right)H(R), \forall R \in (0, r_0/2),$$

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

where
$$B = \frac{2(p-1)L}{c\varepsilon(r_0)}, \beta = c\varepsilon(r_0).$$
Since
$$H(R) = \int_{\partial B_R} w^2 u^2 d\sigma = \int_{\partial B_R} | u |^{2(p-1)} d\sigma,$$
we obtain, by integration,
$$\int_{B_{2R}} | u |^{2(p-1)} dx \le C\exp\left(\frac{B}{R^\beta}\right) \int_{B_R} | u |^{2(p-1)} dx, \forall R \in (0, r_0/2).$$

Proof of Theorem 1. Let r_0 be as in Theorem 2 and let $r \in (0, r_0/2)$. Assume that
$$u \equiv 0 \text{ in } B_r.$$
Using the estimate from Theorem 2 iteratively, with $r_1 = \frac{r_0}{2}$, we obtain
$$\int_{B_{r_1}} | u |^{2(p-1)} dx \le C^k \exp\left[\frac{B}{r_1^\beta}(1 + 2^\beta + \cdots + 2^{k\beta})\right] \int_{B_{r_1 2^{-k}}} | u |^{2(p-1)} dx.$$
We rewrite this as
$$\int_{B_{r_1}} | u |^{2(p-1)} dx \le C^k \exp\left[\frac{B}{(r_1 2^{-k})^\beta}(1 + 2^{-\beta} + \cdots + 2^{-k\beta})\right] | B_{r_1 2^{-k}} |^\gamma$$
$$\times \frac{1}{| B_{r_1 2^{-k}} |^\gamma} \int_{B_{r_1 2^{-k}}} | u |^{2(p-1)} dx,$$
where $\gamma > 0$ is to be chosen.
Choose γ such that $2^{n\gamma} = C$. Then
$$\int_{B_{r_1}} | u |^{2(p-1)} dx \le | B_{r_1} |^\gamma \exp\left(\frac{B'}{(r_1 2^{-k})^\beta}\right) \frac{1}{| B_{r_1 2^{-k}} |} \int_{B_{r_1 2^{-k}}} | u |^{2(p-1)} dx.$$
Assume that
$$\int_{B_r} | u |^{2(p-1)} dx = O\left(\exp\left[-\frac{A}{r^{\alpha\epsilon(r_0)}}\right]\right),$$
with $A = B'$ and $\alpha > \beta$.
Letting $k \to \infty$, we conclude that
$$u \equiv 0 \text{ in } B_{r_1}.$$

To prove Theorem 3, we will need an additional result. Note that in Theorem 3 we do not assume that the solutions of
$$Lu = -\text{div}(| u |^{p-2} \nabla u) + V | u |^{p-2} u + f(x, u, \nabla u)$$
are continuous. The following result allows us to remove the continuity assumption.

Proposition 2. Let $u \in W^{2,2}_{\text{loc}}(B_1)$ be a solution of
$$Lu = -\text{div}(| u |^{p-2} \nabla u) + V | u |^{p-2} u + f(x, u, \nabla u), p \ge 2.$$
Then there exists $0 < r_0 < \frac{1}{2}$, depending only on n, p and $\| V \|_{L^\infty}$ such that
$$\int_{B_r} u^2 w^2 \le r \int_{\partial B_r} u^2 w^2, \qquad \forall r \in (0, r_0).$$

Proof. We adapt the argument of Garofalo and Lin [14, p.261]. By the divergence theorem,

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

$$\int_{B_r} \mathrm{div}(w^2 u \nabla u)(r^2 - |x|^2) = -\int_{B_r} w^2 u \nabla u \cdot \nabla(r^2 - |x|^2) = 2\int_{B_r} w^2 u \nabla u \cdot x.$$

It is straightforward to verify that

$$\nabla(w^2 u^2) = 2(p-1) w^2 u \nabla u,$$

and hence

$$(p-1)\int_{B_r} \mathrm{div}(w^2 u \nabla u)(r^2 - |x|^2) = \int_{B_r} \nabla(w^2 u^2) \cdot x.$$

Applying again the divergence theorem, we obtain

$$\int_{B_r} \nabla(w^2 u^2) \cdot x = \int_{\partial B_r} w^2 u^2 \frac{x \cdot x}{|x|} - n\int_{B_r} w^2 u^2 = r\int_{\partial B_r} w^2 u^2 - n\int_{B_r} w^2 u^2.$$

Thus,

$$(p-1)\int_{B_r} \mathrm{div}(w^2 u \nabla u)(r^2 - |x|^2) = r\int_{\partial B_r} w^2 u^2 - n\int_{B_r} w^2 u^2.$$

On the other hand, since u is a solution, we have

$$\mathrm{div}(w \nabla u) = Vwu,$$

which implies

$$\mathrm{div}(w^2 u \nabla u) = (p-1) w^2 |\nabla u|^2 + V w^2 u^2.$$

Combining the above identities yields

$$r\int_{\partial B_r} w^2 u^2 - n\int_{B_r} w^2 u^2 = (p-1)^2 \int_{B_r} w^2 |\nabla u|^2 (r^2 - |x|^2) + (p-1)\int_{B_r} V w^2 u^2 (r^2 - |x|^2).$$

Therefore,

$$r\int_{\partial B_r} w^2 u^2 \geq \int_{B_r} w^2 u^2 [n + (p-1) V (r^2 - |x|^2)].$$

Let $\| V \|_{L^\infty} = \| V \|_{L^\infty(B_{1/2})}$ and choose $r_0 \in (0, \frac{1}{2})$ such that

$$r_0^2 \leq \frac{n-1}{(p-1)\| V \|_{L^\infty}}.$$

Then for all $r \in (0, r_0)$ and $|x| < r$,

$$n + (p-1) V (r^2 - |x|^2) \geq n - (p-1)\| V \|_{L^\infty} r^2 \geq n - (p-1)\| V \|_{L^\infty} r_0^2 \geq 1.$$

This completes the proof.

Corollary 2. There exists $r_0 > 0$, sufficiently small and depending only on $\| V \|_{L^\infty}$, such that either

$$H(r) \neq 0, \qquad \forall r \in (0, r_0),$$

or

$$u \equiv 0 \text{ in } B_{r_0}.$$

Proof. Assume that $H(r_0) = 0$ for some sufficiently small $r_0 > 0$. Then by Proposition 2,

$$\int_{B_{r_0}} w^2 u^2 \leq r_0 H(r_0) = 0,$$

which implies $u \equiv 0$ in B_{r_0}.

Proof of Theorem 4. Let $I(r), N(r)$ and Ω_{r_0} be defined as before. By Corollary 2,

$$\Omega_{r_0} = \bigcup_{j=1}^{\infty} (a_j, b_j)$$

and

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

$$\frac{N'(r)}{N(r)} = A(r) + B(r).$$

Using the identity

$$I(r) = \int_{\partial B_r} w^2\, u \nabla u \cdot \nu$$

and the Cauchy–Schwarz inequality, we obtain

$$I(r) \le \left(\int_{\partial B_r} w^2\, u^2 \right)^{1/2} \left(\int_{\partial B_r} w^2\, (\nabla u \cdot \nu)^2 \right)^{1/2}.$$

Hence

$$A(r) \ge 0,$$

and therefore

$$\frac{N'(r)}{N(r)} \ge B(r).$$

A careful estimation of the terms involving V shows that there exists a constant

$$K = K(n, p, \| V \|_{L^\infty}, r_0) > 0$$

such that

$$B(r) \ge -K.$$

Thus,

$$\frac{N'(r)}{N(r)} \ge -K,$$

which implies that $N(r)e^{Kr}$ is increasing on each interval (a, b). Consequently, $N(r)$ is bounded on $(0, r_0)$.

From this, we deduce

$$\frac{H'(r)}{H(r)} = \frac{n-1}{r} + \frac{2(p-1)N(r)}{r} \le \frac{c}{r},$$

for some constant $c > 0$. Integrating over $(R, 2R) \subset (0, r_0)$, we obtain

$$H(2R) \le \tilde{c} H(R),$$

and therefore

$$\int_{B_{2R}} | u |^{2(p-1)} \le \tilde{c} \int_{B_R} | u |^{2(p-1)}.$$

Proof of Theorem 3. By iterating the previous inequality, we obtain

$$\int_{B_{2R}} | u |^{2(p-1)} \le \tilde{c}^k \int_{B_{R2^{-k}}} | u |^{2(p-1)}.$$

Assume that

$$\int_{B_r} | u |^{2(p-1)} = O(r^N), \forall N > 0, \text{as } r \to 0.$$

Let $\alpha > 0$ be such that $\tilde{c} 2^{-n\alpha} = 1$. Then

$$\int_{B_{2R}} | u |^{2(p-1)} \le (\omega_n R^n)^\alpha \frac{1}{| B_{R2^{-k}} |^\alpha} \int_{B_{R2^{-k}}} | u |^{2(p-1)}.$$

Letting $k \to \infty$, we conclude that

$$u \equiv 0 \text{ in } B_{2R}.$$

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

APPLICATION: EXAMPLE WITH DEGENERATE ELLIPTIC EQUATION

In this section we present an example illustrating the applicability of the unique continuation results established in this paper. Let us consider the following degenerate elliptic problem in the unit ball $B_1 \subset \mathbb{R}^n$, $n \geq 3$:

$$-\text{div}(|u|^{p-2} \nabla u) + V(x) |u|^{p-2} u = 0 \text{ in } B_1,$$

where $p \geq 2$ and the potential V belongs to the Kato class K_n and is radial, i.e. $V(x) = V(|x|)$.

Sample Problem. Assume that

- $V(x) = \frac{1}{|x|^2 \ln^{1+\varepsilon}(x)}$ with $\varepsilon > 0$ sufficiently small so that $V \in K_n$,
- $u \in W^{2,2}_{loc}(B_1)$ is a weak solution,
- there exists $r_0 > 0$ such that

$$u(x) = 0 \text{ for all } x \in B_{r_0}.$$

Claim. Under these assumptions, the unique continuation property implies that

$$u \equiv 0 \text{ in } B_1.$$

Proof. Since $V(x) = \frac{1}{|x|^2 \ln^{1+\varepsilon}(x)}$ belongs to the Kato class for sufficiently small $\varepsilon > 0$, all the assumptions of Theorem 1 are satisfied. Moreover, the condition $u = 0$ in B_{r_0} implies that for any $r < r_0$,

$$\int_{B_r} |u|^{2(p-1)} \, dx = 0,$$

which satisfies the vanishing condition required in Theorem 1. Therefore, by applying Theorem 1, we conclude that

$$u \equiv 0 \text{ in a neighbourhood of the origin.}$$

By the connectedness of B_1 and standard continuation arguments, this implies

$$u \equiv 0 \text{ in the whole domain } B_1.$$

We note also, by using the inequality in Proposition 1,

$$\int_{B_r} \frac{u^2}{|x|^2} \, dx \leq c_n \left[\frac{1}{r} \int_{\partial B_r} u^2 \, d\sigma + \int_{B_r} |\nabla u|^2 \, dx\right],$$

we obtain the same results in Theorem 1 by taking $V(x) = \frac{1}{|x|^2} \notin K_n$.

Interpretation. This example shows that even in the presence of a singular potential such as $V(x) = \frac{1}{|x|^2 \ln^{1+\varepsilon}(x)} \in K_n$, $\varepsilon > 0$, the strong unique continuation property still holds. In practical terms, this means that if a solution vanishes in a small region, then it must vanish everywhere despite the degeneracy and singularity of the operator.

Such results are particularly important in inverse problems (uniqueness of recovered potentials), control theory (propagation of influence), and mathematical physics (models with singular potentials). This example clearly demonstrates the strength and applicability of the theoretical results obtained in this paper.

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

NUMERICAL ILLUSTRATION AND INTERPRETATION FOR MORE SINGULAR POTENTIAL : $V(x) = \frac{1}{|x|^2}$

In order to further illustrate the theoretical results, we consider a simple radial example and analyse the qualitative behaviour of solutions. As shown in Figure 1, the behaviour of the solutions differs significantly depending on the exponent.

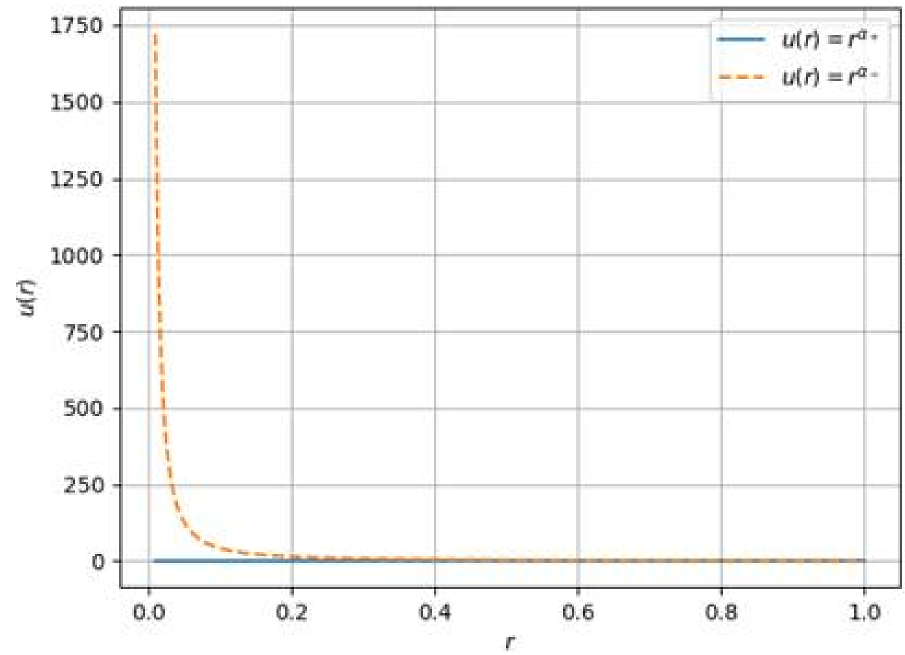

Figure 1. Qualitative behaviour of radial solutions

The figure illustrates the qualitative behaviour of solutions of the equation

$$-u''(r) - \frac{n-1}{r}u'(r) + \frac{\lambda}{r^2}u(r) = 0 \qquad (5)$$

for different values of the exponent α. The curve $u(r) = r^{\alpha_+}$(solid line) represents a regular solution that smoothly vanishes at $r = 0$. The curve $u(r) = r^{\alpha_-}$(dash line) represents a singular solution that blows up near the origin. This figure highlights the fact that any solution that is zero in a neighbourhood must correspond to the trivial case $u \equiv 0$, confirming the strong unique continuation property. (Note: The graph can be generated by plotting r^{α_+} and r^{α_-} for $r \in (0,1)$.)

Model Problem. Let us consider the equation:

$$-\text{div}(|\, u \,|^{p-2}\, \nabla u) + \frac{\lambda}{|\, x \,|^2} |\, u \,|^{p-2}\, u = 0 \text{ in } B_1 \subset \mathbb{R}^n,$$

with $p = 2$ (linear case for simplicity) and $\lambda > 0$. Assuming radial symmetry, i.e. $u(x) = u(r)$, $r = |\, x \,|$, the equation reduces to the ordinary differential equation:

$$-u''(r) - \frac{n-1}{r}u'(r) + \frac{\lambda}{r^2}u(r) = 0.$$

Explicit Solution Behaviour. We look for solutions of the form:

$$u(r) = r^{\alpha}.$$

Substituting into (5) gives

$$-\alpha(\alpha-1)r^{\alpha-2} - (n-1)\alpha r^{\alpha-2} + \lambda r^{\alpha-2} = 0.$$

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

Dividing by $r^{\alpha-2}$, we obtain

$$\alpha(\alpha + n - 2) = \lambda.$$

Thus, the exponents satisfy

$$\alpha = \frac{-(n-2) \pm \sqrt{(n-2)^2 + 4\lambda}}{2}.$$

Interpretation of Solutions. If $\lambda > 0$, then two real solutions exist. One solution behaves like r^{α_+}(regular near the origin) while the other behaves like r^{α_-}(singular near the origin).

Graphical Insight (Qualitative). The function $u(r) = r^{\alpha_+}$ approaches zero as $r \to 0$, whereas $u(r) = r^{\alpha_-}$ becomes unbounded near $r = 0$. Now suppose that

$$u(r) = 0 \text{ for } r < r_0.$$

From the representation $u(r) = Cr^{\alpha}$, it follows that $C = 0$. Hence

$$u(r) \equiv 0.$$

Connection with Unique Continuation. This explicit computation confirms the theoretical result: Even in the presence of a singular coefficient $\frac{1}{r^2}$, a solution cannot vanish in an open subset unless it is identically zero.

Numerical Perspective. If one approximates the solution numerically (for example using a finite difference method), any non-zero initial condition leads to a non-zero solution throughout the domain, whereas only the trivial initial condition yields the zero solution. Thus, numerical observation also supports the strong unique continuation property.

Conclusion of Example. This example demonstrates that the theoretical results are consistent with explicit solutions. Moreover, the behaviour can be visualised and approximated numerically, and the unique continuation property remains valid even in the presence of degeneracy and singularity.

CONCLUSIONS

We have established strong unique continuation properties for a class of degenerate elliptic operators involving non-linear structures and potentials belonging to the Kato class. The results extend and generalise several known contributions in the literature, particularly under weak regularity assumptions. The main contribution is to show that solutions vanishing on a non-trivial subset must vanish identically, even in the presence of singular potentials. The illustrative example confirms that the theoretical findings are consistent with explicit and qualitative solution behaviour. The numerical interpretation further supports the analytical results.

Future research directions include extending these results to more general non-linear operators, investigating time-dependent problems, studying quantitative unique continuation estimates, and developing numerical simulations. Applications to inverse problems also represent a promising direction for further study.

REFERENCES

1. B. Simon, "Schrödinger semigroups", *Bull. Amer. Math. Soc. (N.S.)*, **1982**, *7*, 447-526.
2. E. B. Fabes, N. Garofalo and F.-H. Lin, "A partial answer to a conjecture of B. Simon concerning unique continuation", *J. Funct. Anal.*, **1990**, *88*, 194-210.

Maejo Int. J. Sci. Technol. **2026**, *20*(02), 138-152

3. J. Ling, "Unique continuation for a class of degenerate elliptic operators", *J. Math. Anal. Appl.*, **1992**, *168*, 511-517.
4. M. Aizenman and B. Simon, "Brownian motion and Harnack inequality for Schrödinger operators", *Comm. Pure Appl. Math.*, **1982**, *35*, 209-273.
5. F. Chiarenza, E. Fabes and N. Garofalo, "Harnack's inequality for Schrödinger operators and the continuity of solutions", *Proc. Amer. Math. Soc.*, **1986**, *98*, 415-425.
6. G. Citti and G. Di Fazio, "Hölder-continuity of the solutions for operators which are a sum of squares of vector fields plus a potential", *Proc. Amer. Math. Soc.*, **1994**, *122*, 741-750.
7. D. G. de Figuereido and J.-P. Gossez, "Strict monotonicity of eigenvalues and unique continuation", *Comm. Partial Diff. Equ.*, **1992**, *17*, 339-346.
8. G. Di Fazio, "On Dirichlet problem in Morrey spaces", *Diff. Integral Equ.*, **1993**, *6*, 383-391.
9. G. Di Fazio, "Dirichlet problem characterization of regularity", *Manuscripta Math.*, **1994**, *84*, 47-56.
10. J.-P. Gossez and A. Loulit, "A note on two notions of unique continuation", *Bull. Soc. Math. Belg. Sér. B*, **1993**, *45*, 257-268.
11. A. Loulit, "Carleman estimate, UCP and the Morrey class $F^{\alpha,p}, \frac{2n}{n+1} \leq \alpha \leq 2$", *Commun. Optim. Theory*, **2018**, *2018*, Art.no.11.
12. L. Abatangelo, A. Ferrero and P. Luzzini, "On solutions to a class of degenerate equations with the Grushin operator", *J. Diff. Equ.*, **2025**, *445*, Art.no.113666.
13. D. Yang, W. Wu and B.-Z. Guo, "The strong unique continuation property of elliptic operators with interior single-point degeneration", *J. Math. Anal. Appl.*, **2026**, *556*, Art.no.130209.
14. N. Garofalo and F.-H. Lin, "Monotonicity properties of variational integrals, A_p weights and unique continuation", *Indiana Univ. Math. J.*, **1986**, *35*, 245-268.
15. N. Garofaloand and F.-H. Lin, "Unique continuation for elliptic operators: A geometric-variational approach", *Comm. Pure Appl. Math.*, **1987**, *40*, 347-366.
16. I. P. Natanson, "Theory of Functions of a Real Variable", Frederick Ungar Publishing Co., New York, **1955**.