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Full Paper

Statistical convergence and statistical continuity through deferred Cesàro mean in locally solid Riesz spaces

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Abstract: In this article deferred density is employed to introduce new generalisations of topological convergence, boundedness and Cauchiness for sequences in a locally solid Riesz space. Along with investigating the fundamental properties and inclusion theorems of these newly introduced concepts, the notion of statistical continuity is extended for functions using the deferred Cesàro mean in locally solid Riesz spaces.

Keywords: statistical convergence, deferred Cesàro mean, Riesz spaces, solid topology

INTRODUCTION

Zygmund [1] introduced the definition of statistical convergence in 1935. Although this method as a particular generalisation of sequential convergence has been studied mainly in recent years, it was formally introduced by Fast [2] in 1951, who also provided an alternative proof of a theorem initially established by Steinhaus [3]. In addition to its connection to summability, it has been studied under different names in various fields, such as trigonometric series [1], number theory [4], Banach spaces [5], measure theory [6], turnpike theory [7], Fourier analysis [8], approximation theory [9] and ergodic theory [10]. Schoenberg [11] investigated the notion from a summability perspective while Salat [12] explored some topological aspects of statistical convergence. Attention to statistical convergence increased significantly after Fridy's well-known article *On Statistical Convergence* [13] in 1985. Connor [14] also made a major contribution to the theory of summability by proving in 1988 that any strongly p-Cesàro summable sequence must be statistically convergent. Over the past couple of decades, many generalisations of the original notion have been introduced by various authors [15-20]. The definition of the statistical limit of a real or complex-valued sequence is based on the density of subsets of $\mathbb{N}$. In this note, for $A \subseteq \mathbb{N}$, we let

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$$\mathcal{D}_n(A) = \frac{1}{n}\sum_{m=1}^{n} \chi_A(m)$$

and call $\mathcal{D}(A) = \lim_{n\to\infty}\mathcal{D}_n(A)$ the density of A, provided the limit exists where $\chi_A(m)$ means the characteristic sequence of A.

It can be said that a real valued sequence $x = (x_m)$ statistically converges to $\in \mathbb{R}$, provided that

$$\mathcal{D}(A_\varepsilon) = 0,$$

where $A_\varepsilon = \{m \in \mathbb{N}: |x_m - l| \geq \varepsilon\}$ for each $\varepsilon > 0$. This is expressed by $S-\lim x_m = l$ in the sequel. We call x statistically null in the case $l = 0$. The space of all statistically convergent sequences is denoted by S.

Deferred Cesàro mean for real (or complex) valued sequences was introduced by Agnew [21] as follows: For a given sequence $x = (x_m)$, the deferred Cesàro mean of x is

$$DC_n(x) = \frac{x_{a_n+1}+x_{a_n+2}\cdots+x_{b_n}}{b_n-a_n}, n = 1,2,3, \dots ,$$

where $a = (a_n)$ and $b = (b_n)$ are sequences of non-negative integers satisfying the conditions

$$\lim_{n\to\infty} b_n = \infty \text{ and } a_n < b_n \text{ for all } n \in \mathbb{N}. \qquad (1)$$

Throughout the text, sequences of non-negative integers (a_n) and (b_n) holding the conditions in (1) are used. We write Ω for the set of all such $\langle a, b\rangle$ pairs. Some restrictions on $\langle a, b\rangle$ are imposed if necessary.

Agnew's perspective inspired Küçükaslan and Yılmaztürk [22] to introduce deferred statistical convergence of sequences as follows: Let $A \subseteq \mathbb{N}$ and $\langle a, b\rangle \in \Omega$ be given. Deferred density of A is defined to be

$$\lim_{n\to\infty}\mathcal{D}_{a_n}^{b_n}(A) = \lim_{n\to\infty}\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} \chi_A(m)$$

and denoted by $\mathcal{D}_a^b(A)$, provided that the limit exists. Throughout the text, subsets of $\mathbb{N}$ having deferred density 0 and 1 are called deferred null and deferred dense respectively. A real valued sequence $x = (x_k)$ is called deferred statistically convergent to a number l if $\mathcal{D}_a^b(A_\varepsilon) = 0$, where $A_\varepsilon = \{m \in \mathbb{N}: |x_m - l| \geq \varepsilon\}$ for each $\varepsilon > 0$. S_a^b represents the set of deferred statistically convergent sequences in the paper.

A Riesz space, first defined by Riesz [23] in 1928, is in brief an ordered vector space that is also a lattice. It may be recalled that a real vector space R is called an ordered vector space, provided that it has a partial ordering relation $\lesssim$ such that

(i) $u \lesssim v$ implies $u + w \lesssim v + w$ for all $u, v, w \in R$;

(ii) $u \lesssim v$ implies $\eta u \lesssim \eta v$ for all $\eta \geq 0$.

A Riesz space R is an ordered vector space that includes $u \vee v = \sup\{u, v\}$ and $u \wedge v = \inf\{u, v\}$ for all $u, v \in R$. Riesz spaces have been applied in measure theory, operator theory and economics [24-28].

Recently, Küçükaslan and Aydın [29] applied the concept of deferred density to extend the notion of statistical order convergence in Riesz spaces. Their approach emphasises a generalised form of order convergence, independent of any topological framework, whereas our motivation lies in providing a topological interpretation through deferred density in a locally solid Riesz space, which extends the concept of a Riesz space.

Some other notions in a Riesz space are reminded here. For any vector u in a Riesz space R,
$u^+ = u \vee \theta$ is called the positive part of u,
$u^- = (-u) \vee \theta$ is called the negative part of u where θ denotes the zero in R and
$|u| = u \vee (-u)$ is called the absolute value of u.
A subset Q of a Riesz space R is called solid whenever $v \in Q$ and $|u| \lesssim |v|$ imply $u \in Q$.

Now some notions on topological vector spaces are mentioned. A topology τ on a vector space R is said to be a linear topology, provided that the vector maps

$$R \times R \to R$$
$$(u, v) \to u + v$$

and

$$\mathbb{R} \times R \to R$$
$$(\eta, u) \to \eta u$$

are both continuous. In this case (R, τ) is called a topological vector space. By $\mathcal{N}_\tau(\theta)$, we mean the family of all neighborhoods of θ, the zero vector, in (R, τ).

It is recalled that each topological vector space (R, τ) has a base $\mathcal{B}$ for $\mathcal{N}_\tau(\theta)$ that holds the following properties:

(i) Each $G \in \mathcal{B}$ is balanced, i.e. $\eta g \in G$ for all $g \in G$ and each $\eta \in [-1,1]$;
(ii) Each $G \in \mathcal{B}$ is absorbing, i.e. for every $u \in R$ there is an $\eta \in \mathbb{R}^+$ such that $\eta u \in G$;
(iii) For each $G \in \mathcal{B}$ there exists some $F \in \mathcal{B}$ such that $F + F \subseteq G$.

A locally solid topology τ on a Riesz space R is defined as a linear topology where the aforementioned base $\mathcal{B}$ for $\mathcal{N}_\tau(\theta)$ is composed of solid sets (cf. [27, 30]). (R, τ) is said to be a locally solid Riesz space if R is a Riesz space endowed with a locally solid topology τ. By $\mathcal{B}_{\text{sld}}$, we denote the base for $\mathcal{N}_\tau(\theta)$ in an (R, τ) locally solid Riesz space. For convenience, the term 'locally solid Riesz' is abbreviated as LSR throughout the remainder of the paper.

Topological generalisations of statistical convergence have interested many researchers [31-40]. For instance, one of these studies was provided by Albayrak and Pehlivan [33] as follows:

In an (R, τ) LSR space, a sequence $x = (x_m)$ is called statistically τ-convergent to $\sigma \in R$ if $\mathcal{D}(\{m \in \mathbb{N}: (x_m - \sigma) \notin U\}) = 0$ for each $U \in \mathcal{N}_\tau(\theta)$. This is written as $S_\tau - \lim x_m = \sigma$.

Using lacunary sequences, another point of view was provided by Mohiuddine and Alghamdi [34]:

In an (R, τ) LSR space, a sequence $x = (x_m)$ is called lacunary statistically τ-convergent to $\sigma \in R$ if

$$\lim_{n\to\infty} \frac{|\{m \in (k_{n-1}, k_n]: (x_m - \sigma) \notin U\}|}{k_n - k_{n-1}} = 0$$

for each $U \in \mathcal{N}_\tau(\theta)$, where (k_n) is a lacunary sequence [41], i.e. an increasing sequence of integers such that $k_0 = 0$ and $k_n - k_{n-1} \to \infty$ as $n \to \infty$.

Subsequently, Mohiuddine et al. [35] utilised the Vallée-Poussin mean to extend the initial concept to what they termed generalised statistical τ-convergence, defined as follows:

In an (R, τ) LSR space, a sequence $x = (x_m)$ is said to be generalised statistically τ-convergent to $\sigma \in R$ if

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$$\lim_{n\to\infty} \frac{|\{m \in I_n : (x_m - \sigma) \notin U\}|}{\lambda_n} = 0$$

for each $U \in \mathcal{N}_\tau(\theta)$, where (λ_n) is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 0$ and $I_n = [n - \lambda_n + 1, n]$.

In this note the notions of statistical τ-convergence and lacunary statistical τ-convergence are extended to deferred statistical τ-convergence in LSR spaces. Additionally, the following are introduced and examined: deferred statistical τ-boundedness, deferred statistical τ-Cauchiness of sequences, and deferred statistical continuity of functions in LSR spaces.

Before proceeding to main results, some basic properties of deferred density that are used throughout the paper are presented. The proofs are omitted since they are straightforward.

Proposition 1. Let $\langle a, b\rangle \in \Omega$ and T, T_1 and T_2 be subsets of $\mathbb{N}$. Then
(i) If $T_1 \subseteq T_2$, then $\mathcal{D}_a^b(T_1) \leq \mathcal{D}_a^b(T_2)$;
(ii) $T_1 \subseteq T_2$ and $\mathcal{D}_a^b(T_2) = 0$ imply $\mathcal{D}_a^b(T_1) = 0$;
(iii) $\mathcal{D}_a^b(T_1) = \mathcal{D}_a^b(T_2) = 1$ implies $\mathcal{D}_a^b(T_1 \cap T_2) = 1$;
(iv) If T is finite, then $\mathcal{D}_a^b(T) = 0$.

Proposition 2. If $T \subseteq \mathbb{N}$ such that $\mathcal{D}_a^b(T)$ exists, then $\mathcal{D}_a^b(T) + \mathcal{D}_a^b(T^c) = 1$ where T^c is the complement of T in $\mathbb{N}$.

MAIN RESULTS

The main results of this work are presented in three subsections. The first subsection begins with deferred statistical τ-convergence in LSR spaces, which harbours several fundamental definitions, theorems and examples. The second subsection compares the sets of deferred statistically τ-convergent sequences corresponding to distinct pairs of $\langle a, b\rangle$ from Ω. Finally, the third subsection introduces and examines the deferred statistical continuity of functions between LSR spaces.

Deferred Statistical τ-Convergence in LSR Spaces

This subsection presents the essential definitions and examples central to this work. Inclusion theorems that highlight the relationships among the newly introduced notions are also proved. Furthermore, fundamental and practical properties of deferred statistical τ-convergence in an LSR space are provided.

Definition 1. Let $\langle a, b\rangle \in \Omega$ be given. In an (R, τ) LSR space, a sequence $x = (x_m)$ is named to be deferred statistically τ-convergent to a point $\sigma \in R$ if $\mathcal{D}_a^b(M_U) = 0$ for each $U \in \mathcal{N}_\tau(\theta)$, where $M_U = \{m \in \mathbb{N} : (x_m - \sigma) \notin U\}$, i.e.

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} \chi_{M_U}(m) = 0.$$

This is signified by $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$. $S_\tau^{\langle a,b\rangle}(R)$ symbolises the set of all deferred statistically τ-convergent sequences in (R, τ).

In the following examples two sequences are presented: one that is deferred statistically τ-convergent and another that is not, within the same LSR space. In order to do this let us consider the topological vector space (c_0, τ_∞) where c_0 is the Riesz space of all real valued null sequences and τ_∞ is the linear topology generated by the supremum norm $\|\cdot\|_\infty$. It follows that τ_∞ is locally solid by

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Theorem 2.28 [27]. Therefore, (c_0, τ_∞) is an LSR space. One can observe that the base $\mathcal{B}_{\text{sld}}$ for $\mathcal{N}_{\tau_\infty}(\theta)$, where θ denotes the zero sequence in c_0, consists of the open neighborhoods

$$B_\varepsilon = \{x \in c_0 : \|x\|_\infty < \varepsilon\}$$

of θ where ε is any positive number.

Example 1. Let $\langle a, b\rangle \in \Omega$ and the sequence (x_m) in (c_0, τ_∞) be given so that x_m is the sequence with $\frac{1}{m^2}$ in place of m and 0 elsewhere, that is

$$x_m = \left(0,0,0, \dots ,0, \frac{1}{m^2}, 0,0, \dots\right).$$

Take any $U \in \mathcal{N}_{\tau_\infty}(\theta)$. Then there is some $B_\varepsilon \in \mathcal{B}_{\text{sld}}$ such that $B_\varepsilon \subseteq U$. It is clear that we have some $m_0 \in \mathbb{N}$ so that $\frac{1}{m_0} \leq \varepsilon$. Thus, it follows that $B_{\frac{1}{m_0}} \subset B_\varepsilon$. Moreover, we have

$$T = \left\{m \in \mathbb{N} : x_m \notin B_{\frac{1}{m_0}}\right\} = \{m \in \mathbb{N} : m^2 \leq m_0\}$$

implying that $\mathcal{D}_a^b(T) = 0$ since T is finite. This and the inclusion

$$\{m \in \mathbb{N} : x_m \notin U\} \subseteq \{m \in \mathbb{N} : x_m \notin B_\varepsilon\} \subseteq T$$

lead to $\mathcal{D}_a^b(\{m \in \mathbb{N} : x_m \notin U\}) = 0$. Hence $S_{\tau_\infty}^{\langle a,b\rangle} - \lim x_m = \theta$.

Example 2. Let $\langle a, b\rangle \in \Omega$ and (e_m) be the sequence of unit vectors in (c_0, τ_∞). For each $m \in \mathbb{N}$ we denote, by $e_m = (0,0,0, \dots ,0,1,0,0, \dots)$, the m^{th} unit vector in c_0 so that e_m is the sequence with 1 in place of m and 0 elsewhere. It is claimed that (e_m) can not *deferred statistically τ-converge* to any sequence $\sigma = \left(\sigma_{(m)}\right) \in c_0$. Indeed, it is observed that for each $\sigma \in c_0,$ one can find an $\varepsilon_\sigma > 0$ which leads to the set

$$T = \left\{m \in \mathbb{N} : (e_m - \sigma) \notin B_{\varepsilon_\sigma}\right\} = \{m \in \mathbb{N} : \|e_m - \sigma\|_\infty \geq \varepsilon_\sigma\},$$

which is cofinite, where $B_{\varepsilon_\sigma} \in \mathcal{B}_{\text{sld}} \subset \mathcal{N}_{\tau_\infty}(\theta)$. This means $\mathcal{D}_a^b(T) = 1$ and so $S_{\tau_\infty}^{\langle a,b\rangle} - \lim e_m \neq \sigma$.

Remark 1. Let $\langle a, b\rangle \in \Omega$ be given. In an (R, τ) LSR space, deferred statistical τ-convergence reduces to:
(i) statistical τ-convergence if $b_n = n$ and $a_n = 0$ for all $n \in \mathbb{N}$,
(ii) lacunary statistical τ-convergence if $b_n = k_n$ and $a_n = k_{n-1}$ where (k_n) is a lacunary sequence.

Definition 2. Let $\langle a, b\rangle \in \Omega$ be given. In an (R, τ) LSR space, a sequence $x = (x_m)$ is named to be deferred statistically τ-bounded if for each $U \in \mathcal{N}_\tau(\theta)$, there is an $\eta \in \mathbb{R}^+$ such that $\mathcal{D}_a^b(L_U) = 0$, where

$$L_U = \{m \in \mathbb{N} : \eta x_m \notin U\}.$$

$SB_\tau^{\langle a,b\rangle}(R)$ stands for the set of all deferred statistically τ-bounded sequences in (R, τ) in the remainder.

Definition 3. Let $\langle a, b\rangle \in \Omega$. In an (R, τ) LSR space, a sequence $x = (x_m)$ is called deferred statistically τ-Càuchy if there is a $t \in \mathbb{N}$ such that

$$\mathcal{D}_a^b(\{m \in \mathbb{N} : (x_m - x_t) \notin U\}) = 0$$

holds for every $U \in \mathcal{N}_\tau(\theta)$.

In the next theorem some basic and useful properties of deferred statistically τ-convergent sequences in an LSR space are proved.

Theorem 1. Let $\langle a,b\rangle \in \Omega$. In an (R,τ) LSR space, we have what follows:
(i) Every $(x_m) \in S_\tau^{\langle a,b\rangle}(R)$ has a unique deferred statistically τ-limit, provided that (R,τ) is Hausdorff;
(ii) Let $(x_m) \in S_\tau^{\langle a,b\rangle}(R)$ with $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$. Then $S_\tau^{\langle a,b\rangle} - \lim \eta x_m = \eta\sigma$ for each $\eta \in \mathbb{R}$;
(iii) Let $(x_m),(y_m) \in S_\tau^{\langle a,b\rangle}(R)$ with $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$ and $S_\tau^{\langle a,b\rangle} - \lim y_m = \varsigma$. Then $S_\tau^{\langle a,b\rangle} - \lim(x_m + y_m) = \sigma + \varsigma$.

Proof. Before proving the assertions, it is reminded that for every $U \in \mathcal{N}_\tau(\theta)$ there exists some $G \in \mathcal{B}_{\text{sld}}$ so that $G \subseteq U$. Moreover, we have some $F \in \mathcal{B}_{\text{sld}}$ satisfying $F + F \subseteq G$.
(i) Suppose that $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma_1$ and $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma_2$. For every $U \in \mathcal{N}_\tau(\theta)$, let us define the sets
$$T_1 = \{m \in \mathbb{N} : (x_m - \sigma_1) \in F\}$$
and
$$T_2 = \{m \in \mathbb{N} : (x_m - \sigma_2) \in F\}.$$
Then we have $\mathcal{D}_a^b(T_1) = \mathcal{D}_a^b(T_2) = 1$ due to the supposition. Thus, $\mathcal{D}_a^b(T_1 \cap T_2) = 1$, which yields $T_1 \cap T_2 \neq \emptyset$. Then
$$\sigma_1 - \sigma_2 = (\sigma_1 - x_m) + (x_m - \sigma_2) \in F + F \subseteq G \subseteq U$$
follows for every $m \in T_1 \cap T_2$. Therefore, $\sigma_1 - \sigma_2$ belongs to every $U \in \mathcal{N}_\tau(\theta)$. Since (R,τ) is Hausdorff, we have $\bigcap_{U \in \mathcal{N}_\tau(\theta)} U = \{\theta\}$. Hence $\sigma_1 - \sigma_2 = \theta$, i.e. $\sigma_1 = \sigma_2$.
(ii) Let $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$ and an arbitrary $U \in \mathcal{N}_\tau(\theta)$ be given. Choose an $\eta \in \mathbb{R}$ with $|\eta| \leq 1$. The inclusions
$$\{m \in \mathbb{N} : (x_m - \sigma) \in G\} \subseteq \{m \in \mathbb{N} : (\eta x_m - \eta\sigma) \in G\} \subseteq \{m \in \mathbb{N} : (\eta x_m - \eta\sigma) \in U\}$$
are confirmed to hold since G is balanced. Also, we have $\mathcal{D}_a^b(\{m \in \mathbb{N} : (x_m - \sigma) \in G\}) = 1$ due to $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$. Hence $\mathcal{D}_a^b(\{m \in \mathbb{N} : (\eta x_m - \eta\sigma) \in U\}) = 1$ for every $U \in \mathcal{N}_\tau(\theta)$.

Now let $|\eta| > 1$ and $\lceil|\eta|\rceil$ denote the ceiling of $|\eta|$. We know that there exists some $H \in \mathcal{B}_{\text{sld}}$ so that $\lceil|\eta|\rceil H \subseteq G$. Since $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$, we get $\mathcal{D}_a^b(T) = 1$ where $T = \{m \in \mathbb{N} : (x_m - \sigma) \in H\}$. Then we can write $\lceil|\eta|\rceil|x_m - \sigma| \in \lceil|\eta|\rceil H \subseteq G \subseteq U$ for every $m \in T$. This and the relation
$$|\eta x_m - \eta\sigma| = |\eta||x_m - \sigma| \lesssim \lceil|\eta|\rceil|x_m - \sigma|$$
imply that $(\eta x_m - \eta\sigma) \in G \subseteq U$ for every $m \in T$ since G is solid. We obtain
$$\{m \in \mathbb{N} : (x_m - \sigma) \in H\} \subseteq \{m \in \mathbb{N} : (\eta x_m - \eta\sigma) \in U\}$$
which yields $\mathcal{D}_a^b(\{m \in \mathbb{N} : (\eta x_m - \eta\sigma) \in U\}) = 1$ for every $U \in \mathcal{N}_\tau(\theta)$. Consequently, $S_\tau^{\langle a,b\rangle} - \lim \eta x_m = \eta\sigma$ for each $\eta \in \mathbb{R}$.
(iii) Let $U \in \mathcal{N}_\tau(\theta)$ be arbitrarily given. By the hypothesis, we have $\mathcal{D}_a^b(T_1) = \mathcal{D}_a^b(T_2) = 1$, where
$$T_1 = \{m \in \mathbb{N} : (x_m - \sigma) \in F\}$$
and
$$T_2 = \{m \in \mathbb{N} : (y_m - \varsigma) \in F\}.$$
Consider $T = T_1 \cap T_2$. Then T is non-empty due to $\mathcal{D}_a^b(T) = 1$. This implies that

$$(x_m + y_m) - (\sigma + \varsigma) = (x_m - \sigma) + (y_m - \varsigma) \in F + F \subseteq G \subseteq U$$

holds for each $m \in T$. Thus, we get

$$T \subseteq \{m \in \mathbb{N}: [(x_m + y_m) - (\sigma + \varsigma)] \in U\},$$

which yields $\mathcal{D}_a^b(\{m \in \mathbb{N}: [(x_m + y_m) - (\sigma + \varsigma)] \in U\}) = 1$. Hence $S_\tau^{\langle a,b \rangle} - \lim(x_m + y_m) = \sigma + \varsigma$.

Theorem 2. $S_\tau^{\langle a,b \rangle}(R) \subseteq SB_\tau^{\langle a,b \rangle}(R)$ holds for an (R, τ) LSR space.

Proof. Let $\langle a, b \rangle \in \Omega$ and $\sigma \in R$. Suppose $S_\tau^{\langle a,b \rangle} - \lim x_m = \sigma$ and let us pick an arbitrary $U \in \mathcal{N}_\tau(\theta)$. Then there are some $F, G \in \mathcal{B}_{\text{sld}}$ such that $F + F \subseteq G \subseteq U$. Since $S_\tau^{\langle a,b \rangle} - \lim x_m = \sigma$, the set $T = \{m \in \mathbb{N}: (x_m - \sigma) \notin F\}$ is deferred null. Then we have $\eta\sigma \in F$ holds for some $\eta > 0$ because F is absorbing. We choose a $\mu \in \mathbb{R}^+$such that $\mu \leq 1$ and $\mu \leq \eta$. Then the relation $|\mu\sigma| \lesssim |\eta\sigma|$ implies $\mu\sigma \in F$ since F is solid. On the other hand, $(x_m - \sigma) \in F$ implies $\mu(x_m - \sigma) \in F$ because F is balanced. Now we have $\mu x_m \in U$ since $\mu(x_m - \sigma) + \mu\sigma \in F + F$ holds for every $m \in T^c$. Therefore, we obtain the inclusion

$$\{m \in \mathbb{N}: \mu x_m \notin U\} \subseteq T,$$

which yields $\mathcal{D}_a^b(\{m \in \mathbb{N}: \mu x_m \notin U\}) = 0$. Consequently, (x_m) is deferred statistically τ-bounded.

Theorem 3. Let $\langle a, b \rangle \in \Omega$ be given and $(x_m), (y_m), (z_m)$ be sequences in an (R, τ) LSR space such that $x_m \lesssim y_m \lesssim z_m$ for all $m \in \mathbb{N}$. Then $S_\tau^{\langle a,b \rangle} - \lim x_m = S_\tau^{\langle a,b \rangle} - \lim z_m = \sigma$ implies $S_\tau^{\langle a,b \rangle} - \lim y_m = \sigma$.

Proof. Let $U \in \mathcal{N}_\tau(\theta)$ be arbitrarily given. Then we have some $F, G \in \mathcal{B}_{\text{sld}}$ implying $F + F \subseteq G \subseteq U$. By $S_\tau^{\langle a,b \rangle} - \lim x_m = S_\tau^{\langle a,b \rangle} - \lim z_m = \sigma$ we get the sets

$$T_1 = \{m \in \mathbb{N}: (x_m - \sigma) \in F\}$$

and

$$T_2 = \{m \in \mathbb{N}: (z_m - \sigma) \in F\} ,$$

which hold $\mathcal{D}_a^b(T_1) = \mathcal{D}_a^b(T_2) = 1$. If we let $T = T_1 \cap T_2,$ then $\mathcal{D}_a^b(T) = 1$ holds. Moreover, the relation $x_m \lesssim y_m \lesssim z_m$ implies $x_m - \sigma \lesssim y_m - \sigma \lesssim z_m - \sigma$ for all $m \in \mathbb{N}$. It follows that

$$|y_m - \sigma| \lesssim |x_m - \sigma| + |z_m - \sigma| \in F + F \subseteq G$$

for all $m \in T$. Therefore, we obtain $(y_m - \sigma) \in G \subseteq U$ for all $m \in T$ due to the solidness of G. This yields

$$\mathcal{D}_a^b(\{m \in \mathbb{N}: (y_m - \sigma) \in U\}) = 1$$

for every $U \in \mathcal{N}_\tau(\theta)$. Hence $S_\tau^{\langle a,b \rangle} - \lim y_m = \sigma$.

Theorem 4. Every deferred statistically τ-convergent sequence is deferred statistically τ-Càuchy in an (R, τ) LSR space.

Proof. Let $\langle a, b \rangle \in \Omega$ and $\sigma \in R$. Suppose $S_\tau^{\langle a,b \rangle} - \lim x_m = \sigma$ and let us pick an arbitrary $U \in \mathcal{N}_\tau(\theta)$. Then there are some $F, G \in \mathcal{B}_{\text{sld}}$ such that $F + F \subseteq G \subseteq U$. Since $S_\tau^{\langle a,b \rangle} - \lim(x_m) = \sigma$, the set $T = \{m \in \mathbb{N}: (x_m - \sigma) \notin F\}$ is deferred null. We obtain

$$x_m - x_t = (x_m - \sigma) + (\sigma - x_t) \in F + F$$

for all $m, t \in T^c$. Then we can confirm that the inclusion

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$$\{m \in \mathbb{N}: (x_m - x_t) \notin U\} \subseteq T$$

is true. Thus, there is some $t \in \mathbb{N}$ such that $\mathcal{D}_a^b(\{m \in \mathbb{N}: (x_m - x_t) \notin U\}) = 0$ is satisfied for every $U \in \mathcal{N}_\tau(\theta)$, which gives the desired result.

In the following theorem a sufficient condition is presented for a sequence in an (R, τ) LSR space to be deferred statistically τ-convergent.

Theorem 5. Let $\langle a, b\rangle \in \Omega$. In an (R, τ) LSR space, $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$ if there exists an index set $T = \{m_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$, which is deferred dense, so that $\lim_{i\to\infty} x_{m_i} = \sigma$.

Proof. Choose an arbitrary $U \in \mathcal{N}_\tau(\theta)$. Since $\lim_{i\to\infty} x_{m_i} = \sigma$, we have some $m_0 = m_0(U) \in \mathbb{N}$ such that $(x_m - \sigma) \in U$ holds for all $m \in \{m \in T: m \geq m_0\}$. Then the inclusion

$$\{m \in \mathbb{N}: (x_m - \sigma) \notin U\} \subseteq \mathbb{N} \setminus \{m \in T: m \geq m_0\}$$

implies

$$\mathcal{D}_a^b(\{m \in \mathbb{N}: (x_m - \sigma) \notin U\}) = 0.$$

Hence $S_\tau^{\langle a,b\rangle} - \lim(x_m) = \sigma$.

Inclusion Theorems Regarding $\langle a, b\rangle \in \Omega$

In this subsection the inclusions $S_\tau^{\langle a,b\rangle}(R) \subseteq S_\tau^{\langle c,d\rangle}(R)$ and $S_\tau^{\langle c,d\rangle}(R) \subseteq S_\tau^{\langle a,b\rangle}(R)$ are studied under certain restrictions on $\langle a, b\rangle$ and $\langle c, d\rangle \in \Omega$.

Theorem 6. Let $\langle a, b\rangle \in \Omega$ so that $\liminf_n \frac{b_n - a_n}{b_n} > 0$ holds. In an (R, τ) LSR space, if $S_\tau - \lim x_m = \sigma$, then $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$.

Proof. Let $S_\tau - \lim x_m = \sigma$ and take an arbitrary $U \in \mathcal{N}_\tau(\theta)$. Then since $b_n \to \infty$ as $n \to \infty$, it is convenient to write

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{m=1}^{b_n} \chi_{M_U}(m) = 0,$$

where $M_U = \{m \in \mathbb{N}: (x_m - \sigma) \notin U\}$. Furthermore, the inequality $\sum_{m=a_n+1}^{b_n} \chi_{M_U}(m) \leq \sum_{m=1}^{b_n} \chi_{M_U}(m)$ implies the following:

$$\frac{b_n - a_n}{b_n} \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} \chi_{M_U}(m) \leq \frac{1}{b_n} \sum_{m=1}^{b_n} \chi_{M_U}(m),$$

which yields $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$ upon taking limit as $n \to \infty$.

In the next results a comparison between the spaces $S_\tau^{\langle a,b\rangle}(R)$ and $S_\tau^{\langle c,d\rangle}(R)$ is made under the following condition:

$$a_n \leq c_n < d_n \leq b_n, \forall n \in \mathbb{N} \tag{2}$$

where $\langle a, b\rangle$ and $\langle c, d\rangle \in \Omega$.

Theorem 7. Let $\langle a, b\rangle, \langle c, d\rangle \in \Omega$ be given such that the sets

$$\mathbb{N} \cap (a_n, c_n] \text{ and } \mathbb{N} \cap (d_n, b_n]$$

are finite for all $n \in \mathbb{N}$. Then in an (R, τ) LSR space, $S_\tau^{\langle c,d\rangle} - \lim x_m = \sigma$ implies $S_\tau^{\langle a,b\rangle} - \lim x_m = \sigma$, where $\sigma \in R$.

Proof. Let $S_\tau^{\langle c,d\rangle} - \lim x_m = \sigma$. In this case for each $U \in \mathcal{N}_\tau(\theta)$,

$$\lim_{n\to\infty}\frac{1}{d_n-c_n}\sum_{m=c_n+1}^{d_n}\chi_{M_U}(m)=0$$

holds, where $M_U=\{m\in\mathbb{N}:(x_m-\sigma)\notin U\}$. On the other hand, it can be easily verified that the identity

$$\sum_{m=a_n+1}^{b_n}\chi_{M_U}(m)=\sum_{m=a_n+1}^{c_n}\chi_{M_U}(m)+\sum_{m=c_n+1}^{d_n}\chi_{M_U}(m)+\sum_{m=d_n+1}^{b_n}\chi_{M_U}(m)$$

is true and gives rise to the inequality

$$\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}\chi_{M_U}(m)\leq\frac{1}{d_n-c_n}\sum_{m=a_n+1}^{c_n}\chi_{M_U}(m)+\frac{1}{d_n-c_n}\sum_{m=c_n+1}^{d_n}\chi_{M_U}(m)$$
$$+\frac{1}{d_n-c_n}\sum_{m=d_n+1}^{b_n}\chi_{M_U}(m).$$

We get the desired result, which is $S_\tau^{\langle a,b\rangle}-\lim x_m=\sigma$, upon taking limit as $n\to\infty$.

Theorem 8. Let $\langle a,b\rangle,\langle c,d\rangle\in\Omega$ be given such that

$$0\leq\lim_{n\to\infty}\frac{b_n-a_n}{d_n-c_n}<\infty$$

holds. In this case in an (R,τ) LSR space, $S_\tau^{\langle a,b\rangle}-\lim x_m=\sigma$ implies $S_\tau^{\langle c,d\rangle}-\lim x_m=\sigma$, where $\sigma\in R$.

Proof. Let $S_\tau^{\langle a,b\rangle}-\lim x_m=\sigma$. Then

$$\lim_{n\to\infty}\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}\chi_{M_U}(m)=0$$

for every $U\in\mathcal{N}_\tau(\theta)$, where $M_U=\{m\in\mathbb{N}:(x_m-\sigma)\notin U\}$. By the inequalities in (2), it is clear that $\sum_{m=c_n+1}^{d_n}\chi_{M_U}(m)\leq\sum_{m=a_n+1}^{b_n}\chi_{M_U}(m)$ holds and so we have the inequality

$$\frac{1}{d_n-c_n}\sum_{m=c_n+1}^{d_n}\chi_{M_U}(m)\leq\frac{b_n-a_n}{d_n-c_n}\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}\chi_{M_U}(m).$$

This yields $S_\tau^{\langle c,d\rangle}-\lim x_m=\sigma$ upon taking limit as $n\to\infty$.

Deferred Statistical Continuity in LSR Spaces

In this section the concept of statistical continuity of functions between LSR spaces is extended to deferred statistical continuity. The relationship between uniform continuity and deferred statistical continuity of functions is explored. Additionally, it is demonstrated that lattice operators in an LSR space are deferred statistically continuous.

Let (R_1,τ_1) and (R_2,τ_2) be topological vector sapaces. A function $f:(R_1,\tau_1)\to(R_2,\tau_2)$ is said to be uniformly continuous if for each $G\in\mathcal{N}_{\tau_2}(\theta_2)$ there exists some $F\in\mathcal{N}_{\tau_1}(\theta_1)$ such that $(x-$

$y) \in F$ implies $(f(x) - f(y)) \in G$. We define the notion of deferred statistical continuity in the following.

Definition 4. Let $\langle a,b\rangle \in \Omega$, LSR spaces $(R_1, \tau_1), (R_2, \tau_2)$ and a function $f: R_1 \to R_2$ be given. We call f deferred statistically continuous at a point $\sigma \in R_1$ if any sequence (x_m) in R_1 such that $S_{\tau_1}^{\langle a,b\rangle} - \lim x_m = \sigma$ implies $S_{\tau_2}^{\langle a,b\rangle} - \lim f(x_m) = f(\sigma)$. The function $f: R_1 \to R_2$ is said to be deferred statistically continuous in the case that it is deferred statistically continuous at every $\sigma \in R_1$.

Theorem 9. Uniform continuity implies deferred statistical continuity between LSR spaces.

Proof. Let $\langle a,b\rangle \in \Omega$ and LSR spaces $(R_1, \tau_1), (R_2, \tau_2)$ be given. Suppose a function $f: R_1 \to R_2$ is uniformly continuous and a sequence (x_m) in R_1 is deferred statistically τ-convergent to a point $\sigma \in R_1$. Since f is uniformly continuous, for each $G \in \mathcal{N}_{\tau_2}(\theta_2)$ there exists some $F \in \mathcal{N}_{\tau_1}(\theta_1)$ such that $(x - y) \in F$ implies $(f(x) - f(y)) \in G$. Moreover, the set $T = \{m \in \mathbb{N}: (x_m - \sigma) \in F\}$ is deferred dense due to $S_{\tau_1}^{\langle a,b\rangle} - \lim x_m = \sigma$. Then we obtain $(f(x_m) - f(\sigma)) \in G$ for each $m \in T$, which implies $T \subseteq \{m \in \mathbb{N}: (f(x_m) - f(\sigma)) \in G\}$. Therefore, $\mathcal{D}_a^b(\{m \in \mathbb{N}: (f(x_m) - f(\sigma)) \in G\}) = 1$ and $S_{\tau_2}^{\langle a,b\rangle} - \lim f(x_m) = f(\sigma)$. Thus, f is deferred statistically continuous.

Theorem 10. In an (R, τ) LSR space, the following lattice mappings are all deferred statistically continuous for any $\langle a,b\rangle \in \Omega$.:

a) $\begin{matrix} R \times R \to R \\ (x,y) \to x \vee y \end{matrix}$, b) $\begin{matrix} R \times R \to R \\ (x,y) \to x \wedge y \end{matrix}$, c) $\begin{matrix} R \to R \\ x \to |x| \end{matrix}$, d) $\begin{matrix} R \to R \\ x \to x^- \end{matrix}$, e) $\begin{matrix} R \to R \\ x \to x^+ \end{matrix}$

Proof. Let $\langle a,b\rangle \in \Omega$ be given.

a) Suppose that a sequence (x_m, y_m) is deferred statistically $\tau \times \tau$-convergent to (x, y) in $R \times R$, i.e. $S_{\tau\times\tau}^{\langle a,b\rangle} - \lim(x_m, y_m) = (x, y)$. For an arbitrary $U \in \mathcal{N}_\tau(\theta)$, there are some $F, G \in \mathcal{B}_{\text{sld}}$ such that $F + F \subseteq G \subseteq U$. We observe that the set $T = \{m \in \mathbb{N}: (x_m - x, y_m - y) \in F \times F\}$ is deferred dense due to $S_{\tau\times\tau}^{\langle a,b\rangle} - \lim(x_m, y_m) = (x, y)$. We also see that

$$|x_m \vee y_m - x \vee y| \lesssim |x_m - x| + |y_m - y| \in F + F \subseteq G$$

holds for every $m \in T$ [28, Theorem 1.9 (2)]. This implies, G being solid, $(x_m \vee y_m - x \vee y) \in G$ for every $\in T$, which gives the following:

$$T \subseteq \{m \in \mathbb{N}: (x_m \vee y_m - x \vee y) \in U\}$$

and

$$\mathcal{D}_a^b(\{m \in \mathbb{N}: (x_m \vee y_m - x \vee y) \in U\}) = 1.$$

Thus, $S_\tau^{\langle a,b\rangle} - \lim x_m \vee y_m = x \vee y$ and the sup mapping is deferred statistically continuous.

b) The proof can be obtained in a similar manner with that of a) so it is omitted.

c) Let $S_\tau^{\langle a,b\rangle} - \lim x_m = x$ in R and $U \in \mathcal{N}_\tau(\theta)$. We have some $F, G \in \mathcal{B}_{\text{sld}}$ such that $F + F \subseteq G \subseteq U$. It is clear that $T = \{m \in \mathbb{N}: (x_m - x) \in F\}$ is deferred dense. Also, it can be verified that

$$\big||x_m| - |x|\big| = |[x_m \vee (-x_m)] - [x \vee (-x)]| \lesssim |x_m - x| + |(-x_m) - (-x)| \in F + F \subseteq G$$

is true for all $m \in T$. Therefore, since G is solid, $(|x_m| - |x|) \in G$ for all $m \in T$. Then we have

$$\mathcal{D}_a^b(\{m \in \mathbb{N}: (|x_m| - |x|) \in U\}) = 1$$

and so $S_\tau^{\langle a,b\rangle} - \lim|x_m| = |x|$.

d) Let us pick an arbitrary $U \in \mathcal{N}_\tau(\theta)$. There exists some $G \in \mathcal{B}_{\text{sld}}$ such that $G \subseteq U$. Let $S_\tau^{\langle a,b\rangle} - \lim x_m = x$ in R. Then the set $T = \{m \in \mathbb{N}: (x_m - x) \in G\}$ is deferred dense. Furthermore, we observe that

$$|x_m^- - x^-| = |[(-x_m) \vee \theta] - [(-x) \vee \theta]| \lesssim |(-x_m) - (-x)| + |\theta - \theta| = |x - x_m| \in G$$

for every $m \in T$. It follows that $(x_m^- - x^-) \in G$ for all $m \in T$ because G is solid. Then the inclusion $T \subseteq \{m \in \mathbb{N}: (x_m^- - x^-) \in U\}$ implies

$$\mathcal{D}_a^b(\{m \in \mathbb{N}: (x_m^- - x^-) \in U\}) = 1$$

which yields $S_\tau^{\langle a,b\rangle} - \lim x_m^- = x^-$. This completes the proof.

e) The proof is analogous to that of d) so it is omitted.

CONCLUSIONS

Statistical convergence and its various generalisations have been investigated in LSR spaces over the past few decades. However, deferred density has not been utilised to offer a broader perspective on these concepts in LSR spaces. In this paper, the author attempts to address this gap in the literature. Specifically, the author extends the notions of statistical τ-convergence and lacunary statistical τ-convergence to that of deferred statistical τ-convergence. The author also proves that uniform continuity implies deferred statistical continuity of functions between LSR spaces. The findings in this note may inspire researchers to explore new and broader concepts related to deferred density and other types of densities in future work.

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