Maejo Int. J. Sci. Technol. **2025**, *19*(01), 80-93

Maejo International Journal of Science and Technology

e-ISSN: 2697-4746
Available online at www.mijst.mju.ac.th

Full Paper

On a variation of co-coatomically supplemented modules

Emine Onal Kir

Department of Mathematics, Faculty of Science and Art, Kirsehir Ahi Evran University, Bagbasi, 40100 Kirsehir, Turkey
E-mail: emine.onal@ahievran.edu.tr

Received: 2 November 2024 / Accepted: 24 April 2025 / Published: 29 April 2025

Abstract: Co-coatomically *ss*-supplemented modules are identified by extending the concept of *ss*-supplemented modules, and certain characteristics of co-coatomically *ss*-supplemented modules are demonstrated. It is established that the sum of a finite set of co-coatomically *ss*-supplemented modules remains co-coatomically *ss*-supplemented. Moreover, it is proven that any quotient module of a co-coatomically *ss*-supplemented module is also co-coatomically *ss*-supplemented. It is observed that if G is a co-coatomically *ss*-supplemented submodule of a module E with the condition that E/G lacks a maximal submodule, then E is co-coatomically *ss*-supplemented. It is demonstrated that the ring S is semi-perfect and $Rad(S) \subseteq Soc({}_SS)$ if and only if S is semi-local and $Rad(S) \subseteq Soc({}_SS)$ if and only if each left S-module is co-coatomically *ss*-supplemented. Furthermore, specific characteristics of co-coatomically *ss*-supplemented modules over Dedekind rings are examined.

Keywords: co-coatomically *ss*-supplemented modules, co-coatomic submodules, semi-perfect rings, semi-local rings, left *ss*-perfect rings

---

INTRODUCTION

All along this text, S is considered an associative ring with a unit, and all modules are assumed to be unital left S-modules unless explicitly mentioned otherwise. Let E be an S-module. The notation ${}_SS$ is referred to the left S-module structure of the ring S. A submodule G of E is called *small* in E, denoted as $G \ll E$ if $E \neq G + G'$ for each proper submodule G' of E. $Rad(E)$ represents the intersection of all maximal submodules of E, equivalently the sum of all small submodules of E. A module E is called *radical* if E does not have any maximal submodule, i.e. $E = Rad(E)$. Moreover, $Soc(E)$ indicates the socle of a module E, i.e. the sum of all simple submodules of E. As discussed in Corollary 9.1.3 [1], it is widely known that $Soc(E)$ is the largest semi-simple submodule of a module E. Zhou and Zhang [2] introduced the concept of $Soc_s(E)$ for a module E as an extension of the idea of the socle of a module E by considering all simple submodules that are small within E instead of considering all simple submodules of E, i.e.

$$Soc_s(E) = \sum\{G \ll E \mid G \text{ is simple}\}.$$

A module is called *coatomic* when for each proper submodule, there exists a maximal submodule including it [3]. Assume that G is a submodule of a module E. G is called *(cofinite) co-coatomic* in E in the case the quotient module E/G is (finitely generated) coatomic [4, 5]. Examples of coatomic modules include local, semi-simple and finitely generated modules. It is well known that the property of being a coatomic module is transferred by quotient modules. So it can be concluded that each submodule of local modules and each submodule of finitely generated semi-simple modules are also co-coatomic [5]. Throughout this paper, the notations $G \leq E$ and $G \leq_{cc} E$ notify that G is a submodule of E and G is a co-coatomic submodule of E respectively.

A non-zero module E is called *local* when the sum of all proper submodules of E is also a proper submodule of E. A ring S is called *local* when ${}_SS$ is a local module. Kaynar et al. [6] defined strongly local modules and rings as follows. A module E is called *strongly local* when E is local and $Rad(E)$ is semi-simple. A ring S is called *left strongly local* when ${}_SS$ is a strongly local module.

Let E be a module and $G \leq E$. A submodule W is called an *ss-supplement* of G in E when $E = G + W$ and $G \cap W \leq Soc_s(W)$ [6]. It is shown in Lemma 3 [6] that W is an *ss*-supplement of G in E if and only if $E = G + W, G \cap W$ being semi-simple, and $G \cap W \ll W$ if and only if $E = G + W, G \cap W$ being semi-simple, and $G \cap W \leq Rad(W)$. Moreover, a module E is called *ss-supplemented* when each submodule of E has an *ss*-supplement in E [6]. A submodule W is called a *(weak) supplement* of G in E when $E = G + W$ and $G \cap W \ll W$ $(G \cap W \ll E)$. A module E is called *(weak) supplemented* when each submodule of E has a (weak) supplement in E. Semi-simple, artinian and local modules are supplemented [7, 8]. A submodule W is called a *Rad-supplement* of G in E when $E = G + W$ and $G \cap W \leq Rad(W)$. A module E is called *Rad-supplemented* when each submodule of E has a Rad-supplement in E [9]. Based on the provided definitions, the following implication regarding submodules of a module is observed:

direct summand ⟹ *ss*-supplement ⟹ supplement.

A module E is called *cofinitely supplemented* when each cofinite submodule has a supplement in E [4]. A module E is called *cofinitely ss-supplemented* in the case each cofinite submodule has an *ss*-supplement within E [10]. The same paper introduced different characteristics of cofinitely *ss*-supplemented modules.

A module E is called *co-coatomically supplemented* in the case each $G \leq_{cc} E$ has a supplement in E [5]. Moreover, a module E is called *co-coatomically weak supplemented* in the case each $G \leq_{cc} E$ has a weak supplement in E, in which case there exists a submodule W of E such that $E = G + W$ and $G \cap W \ll E$ [5]. Explicitly, the class of modules that are co-coatomically weakly supplemented includes modules that are co-coatomically supplemented, and also the class of modules that are cofinitely supplemented includes the modules that are co-coatomically supplemented. Sozen et al. [11] generalised co-coatomically supplemented modules to Rad-cc-supplemented modules and studied the module class S_{Rad-cc}. If for every co-coatomic submodule G of E, there is $W \leq E$ such that W is a Rad-supplement of G, then the module E is called belonging to the class S_{Rad-cc} [11]. Furthermore, in recent years more generalisations of co-coatomically supplemented modules have been studied [12, 13].

In this paper firstly a module E is called *co-coatomically ss-supplemented* when each $G \leq_{cc} E$ has an *ss*-supplement in E as a proper generalisation of *ss*-supplemented modules. It can be observed that each co-coatomically *ss*-supplemented module is cofinitely *ss*-supplemented. An example of a module that is co-coatomically *ss*-supplemented but not *ss*-supplemented is given. In continuation of the study it is demonstrated that if E is an *ss*-semi-local module such that $Soc_s(E) \ll E$, then E being a co-coatomically *ss*-supplemented module is equivalent to it being an *ss*-supplemented module. It is established that quotient modules and finite sums of co-coatomically *ss*-supplemented modules remain unchanged. It is proven that when G is a co-coatomically *ss*-supplemented submodule of a module E and E/G does not have a maximal submodule, then E is co-coatomically *ss*-supplemented. Later, it is determined that a necessary and sufficient condition for every left S-module to be co-coatomically *ss*-supplemented is that the ring S is semi-perfect such that $Rad(S) \subseteq Soc({}_SS$. Equivalently S is a semi-local ring such that $Rad(S) \subseteq Soc({}_SS)$. Moreover, a novel characterisation of left *ss*-perfect rings via co-coatomically *ss*-supplemented modules is presented.

In the rest of this paper, a module E is called *co-coatomically ss-semi-local* when each $G \leq_{cc} E$ has a weak supplement W in E such that $G \cap W$ is semi-simple, and the rings whose left modules are co-coatomically *ss*-semi-local are determined. It is proven that when the ring has a semi-simple radical, each projective cover of a co-coatomically *ss*-semi-local module is co-coatomically *ss*-semi-local. It is established that the quotient modules of co-coatomically *ss*-semi-local modules continue to be co-coatomically *ss*-semi-local. Furthermore, attention is directed to specific algebraic properties of the modules defined in this paper, particularly when they are over Dedekind domains. Specifially, it is proven that over a non-local Dedekind domain, a torsion co-coatomically *ss*-semi-local module is a co-coatomically *ss*-supplemented module. It is also shown that over a Dedekind domain which is not a field, a torsion-free co-coatomically *ss*-supplemented module is a divisible module.

CO-COATOMICALLY *SS*-SUPPLEMENTED MODULES

It is explicit that each *ss*-supplemented module is co-coatomically *ss*-supplemented. Nevertheless, it is crucial to emphasise that the reverse of this statement is generally not valid.

Consider a commutative domain denoted as S with an S-module E. Let $T(E)$ be the set including all elements e within E for which there exists a non-zero element s in S, leading to $se = 0$; in other words, $Ann(e) \neq 0$. As a submodule of E, $T(E)$ is called the *torsion submodule* of E. If E coincides with $T(E)$, E is called a *torsion module*. Additionally, E is called *torsion-free* when $T(E)$ is equal to 0.

Example 1. Consider $\mathbb{Q}$, the set of rational numbers, as a $\mathbb{Z}$-module. It is co-coatomically *ss*-supplemented, given that the only co-coatomic submodule is $\mathbb{Q}$ itself. Besides that, $\mathbb{Q}$ is not *ss*-supplemented since it is not supplemented due to its torsion-free property by Theorem 3.1 [14].

A module E is called *ss-semi-local* in the case the quotient module $E/Soc_s(E)$ is semi-simple, or equivalently each submodule of E has (is) a weak *ss*-supplement in E. That is, for each submodule G of E, E has a submodule W such that $E = G + W$ and $G \cap W \leq Soc_s(E)$, or equivalently $E = G + W$, $G \cap W$ is semi-simple and $G \cap W \ll E$ [15].

Proposition 1. Let E be an *ss*-semi-local module with $Soc_s(E) \ll E$. Then E is a co-coatomically *ss*-supplemented module if and only if E is an *ss*-supplemented module.

Proof. Suppose that E is a co-coatomically *ss*-supplemented module and $G \leq E$. Due to the *ss*-semi-locality of the module E, we conclude that $E/Soc_s(E)$ is coatomic as it is semi-simple. Since

$$E/(G + Soc_s(E)) = [E/Soc_s(E)]/[(G + Soc_s(E))/Soc_s(E)]$$

and each quotient module of a coatomic module is coatomic, then $E/(G + Soc_s(E))$ is also coatomic. By the hypothesis, $G + Soc_s(E)$ has an *ss*-supplement in E, say W. Then $E = (G + Soc_s(E)) + W$, $(G + Soc_s(E)) \cap W$ is semi-simple and $(G + Soc_s(E)) \cap W \ll W$. Since $Soc_s(E) \ll E$ and $G \cap W \leq (G + Soc_s(E)) \cap W$, we have $E = G + W$ and $G \cap W \ll W$ by Section 19.3 [8]. Moreover, $G \cap W$ is semi-simple as a submodule of $(G + Soc_s(E)) \cap W$ from Corollary 8.1.5 [1]. Hence E is an *ss*-supplemented module. The rest of the proof is obvious.

A module E is called *semi-local* in the case its quotient module $E/Rad(E)$ is semi-simple [7]. Each *ss*-semi-local module is obviously semi-local.

Corollary 1. For a semi-local module E with small radical, E is a co-coatomically *ss*-supplemented module if and only if E is an *ss*-supplemented module.

Proof. The proof can be made with a similar method for the proof of Proposition 1.

Now we provide an example showing that a cofinitely supplemented module may not necessarily be co-coatomically *ss*-supplemented.

It is recalled that an ideal J of the ring S is called *left t-nilpotent* when for any sequence of elements $a_1, a_2, \ldots$ belonging to J, there is a $k \in \mathbb{Z}^+$ with $a_k a_{k-1} \ldots a_1 = 0$. A ring S is called *left perfect* in the case that S is semi-local and $Rad(S)$ is left *t*-nilpotent [8].

Example 2. Assuming that $t \in \mathbb{Z}$ is prime, let us consider the local Dedekind domain below:

$$S = \mathbb{Z}_{(t)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, (b, t) = 1\}.$$

Let us say $E = {}_S S^{(\mathbb{N})}$. ${}_S S$ is a supplemented module as S is a local ring. Thus, E is cofinitely supplemented according to Corollary 2.4 [4]. Let us put $G = Rad({}_S S^{(\mathbb{N})})$. The ring S is semi-local; however it is not a left perfect ring, as $Rad(S)$ is not left *t*-nilpotent by Section 43.9 [8]. It should be noted that $G \leq_{cc} E$ and G does not have an *ss*-supplement in E because it neither has a supplement in E. This is due to the fact that S is not a left perfect ring, as stated in Theorem 1 [16]. Therefore, it follows that E is not a co-coatomically *ss*-supplemented module.

In Section 19.4 [8], it is defined that a *projective cover* of a module X as a module E is equipped with a homomorphism $h : E \to X$, where E itself is a projective module and h is a small epimorphism, meaning that the kernel of h (denoted as *Ker*(h)) is a small submodule of E.

A ring S is called *semi-perfect* in the case each finitely generated left S-module has a projective cover in Section 42.6 [8]. Example 2 also illustrates that co-coatomically *ss*-supplemented modules and cofinitely supplemented modules do not necessarily have the same characteristics over semi-perfect rings and discrete valuation rings.

Proposition 2. Co-coatomically *ss*-supplemented modules exhibit transfer properties through their quotient modules.

Proof. Suppose that E is a co-coatomically *ss*-supplemented module and $G \leq E$. If we take any co-coatomic submodule in the quotient module E/G, it can be represented as a submodule of the form H/G, where $H \leq_{cc} E$. By the assumption, H has an *ss*-supplement W in E. Therefore, $E = H + W$, $H \cap W$ is semi-simple and $H \cap W \ll W$. Then we have $E/G = H/G + (W + G)/G$. Now consider the canonical projection $\pi : E \to E/G$. Therefore, $\pi(H \cap W) = (H \cap (W + G))/G$ is semi-simple from Corollary 8.1.5 [1], since $H \cap W$ is semi-simple. Moreover, $(H \cap (W + G))/G = \pi(H \cap W) \ll \pi(W) = (W + G)/G$ from Section 19.3 [8]. Hence $(W + G)/G$ is an *ss*-supplement of H/G in E/G.

Proposition 3. Suppose that E is a co-coatomically *ss*-supplemented module. In that case each co-coatomic submodule of the quotient module $E/Soc_s(E)$ is a direct summand.

Proof. The quotient module $E/Soc_s(E)$ has co-coatomic submodules of the form $G/Soc_s(E)$, where $G \leq_{cc} E$ and $Soc_s(E) \leq G$. Then by the assumption, there is a submodule W of E such that $E = G + W$, $G \cap W \leq Soc_s(W)$. This yields $G \cap W \leq Soc_s(E)$. Thus,

$$E/Soc_s(E) = G/Soc_s(E) + (W + Soc_s(E))/Soc_s(E) \text{ and}$$
$$(G/Soc_s(E)) \cap ((W + Soc_s(E))/Soc_s(E)) = ((G \cap W) + Soc_s(E))/Soc_s(E) = 0.$$

Consequently,

$$E/Soc_s(E) = (G/Soc_s(E)) \oplus ((W + Soc_s(E))/Soc_s(E)).$$

Corollary 2. For a co-coatomically *ss*-supplemented module E, the conditions below are verified.

(1) For each $G/Rad(E) \leq_{cc} E/Rad(E), G/Rad(E)$ is a direct summand of $E/Rad(E)$.

(2) For each $G/Soc(E) \leq_{cc} E/Soc(E), G/Soc(E)$ is a direct summand of $E/Soc(E)$.

Proof. The proofs for these statements can be carried out in a similar manner as that of Proposition 3.

In the next step it is aimed to demonstrate that any finite sum of modules that are co-coatomically *ss*-supplemented is also co-coatomically *ss*-supplemented. To begin, the validity of the following commonly used lemma is established.

Lemma 1. Suppose that E is a module, $H \leq E$ and $G \leq_{cc} E$. When H is co-coatomically *ss*-supplemented and $G + H$ has an *ss*-supplement in *E*, G has an *ss*-supplement in *E*.

Proof. Let W be an *ss*-supplement of $G + H$ in *E*. Thus, $E = W + G + H$, $W \cap (G + H)$ is semi-simple and $W \cap (G + H) \ll W$. It is denoted that $H/H \cap (G + W) \cong E/(G + W)$. Since $E/(G + W) \cong (E/G)/((G + W)/G)$ and E/G is coatomic, then $E/(G + W)$ is coatomic. Thus, $H \cap (G + W)$ has an *ss*-supplement X in H by assumption, i.e. $H = X + H \cap (G + W)$, $X \cap (G + W)$ is semi-simple and $X \cap (G + W) \ll X$. Therefore, $E = W + G + H = W + G + (X + H \cap (G + W)) = G + W + X$. Also, we obtain

$$\begin{aligned} G \cap (X + W) &\leq X \cap (G + W) + W \cap (G + X) \\ &\leq X \cap (G + W) + W \cap (G + H) \ll X + W \end{aligned}$$

by Section 19.3 [8]. Moreover, since $X \cap (G + W)$ and $W \cap (G + H)$ are semi-simple, then $G \cap (X + W)$ is semi-simple by Corollary 8.1.5 [1]. Hence $X + W$ is an *ss*-supplement of G in E.

Theorem 1. The sum of finitely many co-coatomically *ss*-supplemented modules is also co-coatomically *ss*-supplemented.

Proof. Consider a finite collection of co-coatomically *ss*-supplemented modules, denoted as $E_1, E_2, \ldots, E_n$, and put $E = E_1 + E_2 + \cdots + E_n$. To demonstrate the claim, we can limit our proof to the case when there are only two modules, namely E_1 and E_2, both of which are co-coatomically *ss*-supplemented. We show that if $E = E_1 + E_2$, then the result holds for any finite collection. Suppose that $G \leq_{cc} E$. Then $E = E_1 + E_2 + G$. Note that $E/(E_2 + G)$ is coatomic as a quotient module of the coatomic module E/G, and hence $E_2 + G \leq_{cc} E$. Since E_1 is co-coatomically *ss*-supplemented, $E_2 + G \leq_{cc} E$ and E has an *ss*-supplement 0 in E. Then $E_2 + G$ has an *ss*-supplement in E by Lemma 1. By using Lemma 1 once more, we conclude that G has an *ss*-supplement in E since E_2 is co-coatomically *ss*-supplemented, $G \leq_{cc} E$ and $E_2 + G$ has an *ss*-supplement in E. Hence $E_1 + E_2$ is a co-coatomically *ss*-supplemented module.

Maejo Int. J. Sci. Technol. **2025**, *19*(01), 80-93

A module X is called *finitely E-generated* in the case there is an epimorphism $h : E^{(\Lambda)} \to X$ where Λ is a finite set.

Corollary 3. For a co-coatomically ss-supplemented module E, any finitely E-generated module is co-coatomically ss-supplemented.

Proof. The proof can be seen by Proposition 2 and Theorem 1.

The left $\mathbb{Z}$-module $E = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is a co-coatomically ss-supplemented module, although it is not semi-simple. When each simple left S-module is injective, the ring S is called *left* V *−ring*. It is widely known that $Rad(E) = 0$ for each left S-module E if and only if the ring S is left V −ring [8].

Proposition 4. Suppose that S is a left V −ring and E is a left S-module. Then E is a semi-simple module if and only if E is a co-coatomically ss-supplemented module.

Proof. (⟹) This is explicit.
(⟸) Let E be co-coatomically ss-supplemented S-module. Then each $G \leq_{cc} E$ has an ss-supplement W in E and thus, $G \cap W \leq Rad(W)$. Since S is a left V −ring, then $Rad(W) = 0$. Therefore, we conclude that $E = G \oplus W$. Thus, $E/Soc(E)$ does not have a maximal submodule from Theorem 2.1 [5]. By Section 23.1 [8], we reach the conclusion that $E/Soc(E) = Rad(E/Soc(E)) = 0$ since S is a left V −ring. Hence E is a semi-simple module.

Corollary 4. Over a left V −ring, a module that is a direct sum of co-coatomically ss-supplemented modules is co-coatomically ss-supplemented.

Proof. According to Proposition 4, we arrive at the conclusion that over left V −rings, semi-simple modules and co-coatomically ss-supplemented modules coincide. This completes the proof.

While a quotient module of a module is co-coatomically ss-supplemented, the module itself does not necessarily have to be co-coatomically ss-supplemented.

Example 3. Let us assume that S signifies the ring $Q[[x]]$ of all power series $\sum_{\lambda=0}^{\infty} k_\lambda x^\lambda$ where x is an indeterminate and coefficients belong to a field Q. The ring S is local [1]. Hence the module ${}_S S$ is supplemented, and so S is a semi-perfect ring by Section 42.6 [8]. Note that

$$Rad(S) = \{\sum_{\lambda=1}^{\infty} k_\lambda x^\lambda \mid k_\lambda \in Q\} = Sx$$

is not left t-nilpotent [1]. Thus, S is not a left perfect ring by Section 43.9 [8]. Since S is semi-perfect, $S/Rad(S)$ is semi-simple. If $E = {}_S S^{(\mathbb{N})}$ and $G = Rad({}_S S^{(\mathbb{N})})$, E/G is a co-coatomically ss-supplemented module as it is semi-simple. It is noted that $G \leq_{cc} E$ as E/G is semi-simple. According to Theorem 1 [16], G does not have a supplement in E and thus, it does not have an ss-supplement in E. Thus, E is not a co-coatomically ss-supplemented module.

Theorem 2. Suppose that $G \leq E$. When G is a co-coatomically ss-supplemented module and E/G does not have any maximal submodule, E is a co-coatomically ss-supplemented module.

Proof. Let $H \leq_{cc} E$. Then $E/(G + H) = (E/H)/((G + H)/H)$ is coatomic since E/H is coatomic. Since E/G gets no maximal submodule, then $E/(G + H)$ also gets no maximal submodule. Therefore, we conclude that $E = G + H$. Since G is a co-coatomically ss-supplemented module and $H \leq_{cc} E$, then H has an ss-supplement in E according to Lemma 1. Hence E is a co-coatomically ss-supplemented module.

Corollary 5. For a module E, when $E/Soc(E)$ does not include a maximal submodule, E is a co-coatomically ss-supplemented module.

Proof. Explicitly, $Soc(E)$ is a co-coatomically ss-supplemented submodule of E. By the hypothesis, E is a co-coatomically ss-supplemented module according to Theorem 2.

Proposition 5. When a co-coatomically ss-supplemented module E includes a maximal submodule, E also includes a strongly local submodule.

Proof. Suppose that $H \leq E$ is maximal. Then $H \leq_{cc} E$. By assumption, there is a submodule W of E such that $E = H + W, H \cap W$ is semi-simple and $H \cap W \ll W$. Hence W is a strongly local submodule of E by Proposition 12 [6].

RINGS WHOSE MODULES ARE CO-COATOMICALLY *SS*-SUPPLEMENTED

In this section it is firstly aimed to provide a characterisation of rings whose modules are co-coatomically ss-supplemented. However, initially, we need to articulate a lemma. The remainder of the section will examine the behaviour of co-coatomically ss-supplemented modules over various rings.

Lemma 2. Given a coatomic module E, the statements below are equivalent:

(1) E is the sum of all strongly local submodules.
(2) E is an ss-supplemented module.
(3) E is a co-coatomically ss-supplemented module.
(4) Each co-coatomic (cofinite, maximal) submodule of E has an ss-supplement in E.

Proof. (1) $\Leftrightarrow$ (2): By Corollary 31 [6].
The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are explicit.
(4) $\Rightarrow$ (1): Suppose that G is the sum of all strongly local submodules of E and $G \neq E$. Because of coatomic E, there is a maximal submodule H of E such that $G \leq H$. Note that $H \leq_{cc} E$. By (4), H has an ss-supplement W in E. Thus, according to Proposition 12 [6], we conclude that W is a strongly local module. Therefore, the inclusions $W \leq G \leq H$ arise, and this is a contradiction.

G is called *co-closed submodule* in a module E when G does not have any proper submodule K for which $G/K \ll E/K$ [9]. It is remembered that a co-closed submodule of a coatomic module is coatomic as given in Lemma 4.1 [17]. Thus, we obtain the direct consequence below.

Corollary 6. Suppose that E is a coatomic module and G is a co-closed submodule of E. In that case G is a co-coatomically ss-supplemented module if and only if G is an ss-supplemented module.

Theorem 3. For any ring S, the conditions below are equivalent:

(1) ${}_SS$ is a co-coatomically ss-supplemented module.
(2) S is a semi-perfect ring and $Rad(S) \subseteq Soc({}_SS)$.
(3) S is a semi-local ring and $Rad(S) \subseteq Soc({}_SS)$.
(4) Each projective left S-module is co-coatomically ss-supplemented.
(5) Each left S-module is co-coatomically ss-supplemented.
(6) Each left S-module is cofinitely ss-supplemented.
(7) Each left S-module is the sum of all strongly local submodules.
(8) ${}_SS$ is a finite sum of strongly local submodules.
(9) Each maximal left ideal of S has an ss-supplement in S.

Proof. (1) $\Longrightarrow$ (2): Since ${}_SS$ is a co-coatomically *ss*-supplemented module, then each left ideal J of S has an *ss*-supplement in S. Thus, ${}_SS$ is a supplemented module and so S is a semi-perfect ring by Section 42.6 [8]. Moreover, since S is an *ss*-supplement of $Rad(S)$ in S, then we conclude that $Rad(S) \subseteq Soc({}_SS)$.
(2) $\Longrightarrow$ (3): By Section 42.6 [8].
(3) $\Longrightarrow$ (4): By Theorem 41 [6], the result holds.
(4) $\Longrightarrow$ (5): The claim holds from Section 18.6 [8] and Proposition 2.
The implications (5) $\Longrightarrow$ (6) and (7) $\Longrightarrow$ (8) are explicit.
(6) $\Longrightarrow$ (7): By Lemma 5 [10].
(8) $\Longrightarrow$ (9): By Corollary 31 [6].
(9) $\Longrightarrow$ (1): The assertion can be seen from Lemma 2.

Now an example of a module that is co-coatomically supplemented but not co-coatomically *ss*-supplemented is provided. The notation $P(E)$ is used to represent the sum of all radical submodules of a module E, defined as $P(E) = \sum\{G \le E \mid Rad(G) = G\}$. It is obvious that $P(E)$ is the largest radical submodule of E. A module E is called *reduced* when $P(E) = 0$.

Example 4 [18]**.** Consider the polynomial ring $Q[x_1, x_2, \ldots]$ with countably many indeterminates x_k where Q is a field and $k \in \mathbb{Z}^+$. Let $J = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ be the ideal generated by x_1^2 and $x_{k+1}^2 - x_k$ for each $k \in \mathbb{Z}^+$. Then the quotient ring $S = Q[x_1, x_2, \ldots]/J$ is local with the unique maximal ideal $M = (x_1, x_2, \ldots)/J$ generated by all $\overline{x_k} = x_k + J$, $k \in \mathbb{Z}^+$. Note that $Rad(M) = M \ne 0$ by Example 6.2 [18]. If $E = {}_SS$, then E is a co-coatomically supplemented module because local modules are supplemented. However, since $P(E) = Rad(M) = M \ne 0$, then E is not reduced. Thus, E is not strongly local, which means that $Rad(E)$ is not semi-simple by Proposition 6 [6]. Consequently, E is not a co-coatomically *ss*-supplemented module by Theorem 3.

As an extension of the concept of left V-rings, a ring S is called a *left weakly* V*-ring* (abbreviated WV*-ring*) when each simple S-module is S/J-injective for any left ideal J of S such that S/J is a proper quotient ring [19].

Proposition 6. Suppose that E is an S-module over the left WV-ring S. E is a co-coatomically *ss*-supplemented module if and only if E is a co-coatomically supplemented module.

Proof. Assume that E is a co-coatomically supplemented module and $G \le_{cc} E$. Then by the hypothesis, G has a supplement W in E, i.e. $E = G + W$ and $G \cap W \ll W$. Note that $G \cap W \le Rad(W)$. Now two cases arise:

Case 1: When S is a left V-ring, according to Section 23.1 [8] we conclude that $Rad(W) = 0$, and so $E = G \oplus W$. Hence E is a co-coatomically *ss*-supplemented module.

Case 2: When S is not a left V-ring, $Rad(S)$ is simple and $S/Rad(S)$ is a left V-ring [19]. Therefore, we infer that S is a left good ring from Section 23.7 [8]. Thus, we have $G \cap W \le Rad(W) = Rad(S)W \le Soc({}_SS)W \le Soc(W)$ from Section 23.7 [8]. Hence E is a co-coatomically *ss*-supplemented module. The other part of the proof is explicit.

The following theorem belongs to Zöschinger [20]. Using this theorem, the aim is to illustrate an alternative characterisation of left perfect rings through co-coatomically *ss*-supplemented modules. A module E is called $\sum$*-self-projective* when for each index set Λ, the direct sum module $E^{(\Lambda)}$ is self-projective.

Maejo Int. J. Sci. Technol. **2025**, *19*(01), 80-93

Theorem 4 [20]. When the module E is $\sum$-self-projective and $G \leq Rad(E)$, G has a supplement in E, so G is a small submodule of E.

Theorem 5. Each left S-module is co-coatomically ss-supplemented if and only if the ring S is a left perfect ring with $Rad(S) \subseteq Soc({}_SS)$.

Proof. The sufficiency is obviously seen by Theorem 41 [6]. To demonstrate the necessity, assume that each left *S*-module is co-coatomically ss-supplemented. Then each left *S*-module is cofinitely ss-supplemented. Therefore, according to Theorem 3 [10], S is a semi-perfect ring with $Rad(S) \subseteq Soc({}_SS)$. By Section 42.6 [8] we conclude that the quotient ring $S/Rad(S)$ is semi-simple, and hence ${}_SS^{(\mathbb{N})}/Rad({}_SS^{(\mathbb{N})})$ is semi-simple as an $S/Rad(S)$ -module. Thus, we obtain $Rad({}_SS^{(\mathbb{N})}) \leq_{cc} {}_SS^{(\mathbb{N})}$. By the hypothesis, $Rad({}_SS^{(\mathbb{N})})$ has an ss-supplement in ${}_SS^{(\mathbb{N})}$. By applying Theorem 4 we conclude that $Rad({}_SS^{(\mathbb{N})}) \ll {}_SS^{(\mathbb{N})}$. Therefore, by deducing that $S/Rad(S)$ is a left semi-simple ring and $Rad({}_SS^{(\mathbb{N})}) \ll {}_SS^{(\mathbb{N})}$, ${}_SS$ is left perfect by Section 43.9 [8]. As a result, *S* is a left perfect ring.

A specific class of the left perfect rings is introduced via *ss*-semi-local modules as follows. A ring S is called *left ss-perfect*, provided that ${}_SS$ is an ss-semi-local module, or equivalently each left *S*-module is *ss*-semi-local [15].

Corollary 7. Each left *S*-module is co-coatomically ss-supplemented if and only if the ring *S* is left ss-perfect.

Proof. To prove the necessity, assume that each left *S*-module is co-coatomically ss-supplemented. Then by Theorem 5, *S* is a left perfect ring with semi-simple radical. Thus, each left *S*-module is *ss*-supplemented by Theorem 41 [6], and so is *ss*-semi-local. Hence according to Theorem 2.15 [15], *S* is a left *ss*-perfect ring. The sufficiency can be seen by Theorem 5 and Proposition 2.17 [15].

An S-module *E* is called *radical supplemented* when $Rad(E)$ has a supplement in *E* [20].

Proposition 7. Suppose that S is a discrete valuation ring whose maximal ideal is St where $t \in S$ is the unique prime element and *E* is an *S*-module. İt is assumed that the radical of each coatomic *S*-module is semi-simple. Then the basic submodule of *E* is coatomic if and only if *E* is a co-coatomically *ss*-supplemented module.

Proof. (⟹) Let $H \leq_{cc} E$ and *G* be the basic submodule of *E*. Then $E/(H + G)$ is also coatomic. Thus, $E/(H + G)$ is reduced by Lemma 2.1 [14]. On the other hand, since E/G is divisible, then $E/(H + G)$ is divisible. Therefore, $E/(H + G) = 0$, i.e. $E = H + G$. By the hypothesis, G is coatomic and hence it is supplemented by Lemma 2.1 [14]. Thus, by assumption G is an *ss*-supplemented module according to Theorem 20 [6]. By applying Lemma 1, *H* has an *ss*-supplement in *E*. Hence *E* is a co-coatomically *ss*-supplemented module.
(⟸) Since *S* is a discrete valuation ring, $E/Rad(E) = E/tE$ is semi-simple, and so it is coatomic. By the hypothesis, since E is a co-coatomically ss-supplemented module, then tE has an ss-supplement in *E*. Thus, *E* is a radical supplemented module. Hence according to Theorem 3.1 [20], the basic submodule of *E* is coatomic.

A module *E* is called *ss-radical supplemented* when $Rad(E)$ has an *ss*-supplement in *E* [21].

Corollary 8. Suppose that E is an S-module over discrete valuation ring S and that the radical of each coatomic *S*-module is semi-simple. Then the conditions below are equivalent:

(1) E is a co-coatomically *ss*-supplemented module.

(2) E is a radical supplemented module.
(3) E is an *ss*-radical supplemented module.

Proof. $(1) \Leftrightarrow (2)$: This can be seen by Proposition 7 and Theorem 3.1 [20].
$(1) \Longrightarrow (3)$: Since S is a discrete valuation ring, $E/Rad(E)$ is coatomic as it is semi-simple. By the hypothesis, since E is a co-coatomically *ss*-supplemented module, then $Rad(E)$ has an *ss*-supplement in E. Thus, E is an *ss*-radical supplemented module.
$(3) \Longrightarrow (2)$: Obvious.

Corollary 9. Suppose that E is an S-module over discrete valuation ring S and that the radical of each coatomic S-module is semi-simple. Then E is a co-coatomically *ss*-supplemented module if and only if $E = T(E) \oplus X$ where $T(E)$ is the torsion part of E, the reduced part of $T(E)$ is bounded and $X/Rad(X)$ is finitely generated.

Proof. This can be proved by Corollary 8 and Theorem 3.1 [20].

In Lemma 3.2 [20], certain properties below were presented for radical supplemented modules over discrete valuation rings. Through the application of Corollary 8, it has been established that radical supplemented modules coincide with co-coatomically *ss*-supplemented modules over discrete valuation rings, provided a specific condition is met. Therefore, co-coatomically *ss*-supplemented modules exhibit these properties under the specified condition over discrete valuation rings.

Corollary 10. Suppose that S is a discrete valuation ring, E is an S-module and that the radical of each coatomic S-module is semi-simple. Then the assertions below hold.

(1) Suppose that E is a co-coatomically *ss*-supplemented module and $G \leq E$ is pure. In that case G is a co-coatomically *ss*-supplemented module.
(2) Suppose that E and E'/E are co-coatomically *ss*-supplemented modules. Then E' is a co-coatomically *ss*-supplemented module.
(3) When E is a co-coatomically *ss*-supplemented module and E/G is reduced for some submodule G of E, then G is also a co-coatomically *ss*-supplemented module.
(4) Each submodule of E is a co-coatomically *ss*-supplemented module if and only if $T(E)$ is a supplemented module and $E/T(E)$ has finite rank where $T(E)$ is the torsion part of E.

Proof. (1) E is a radical supplemented module according to Corollary 8. Thus, G is a radical supplemented module by Lemma 3.2 [20]. Hence G is a co-coatomically *ss*-supplemented module according to Corollary 8.
(2) E and E'/E are radical supplemented modules according to Corollary 8. Therefore, E' is a radical supplemented module by Lemma 3.2 [20]. Hence E' is a co-coatomically *ss*-supplemented module according to Corollary 8.
(3) Since E is a co-coatomically *ss*-supplemented S-module, then according to Corollary 8, E is a radical supplemented module. By the hypothesis, since E/G is reduced, then G is a radical supplemented module by Lemma 3.2 [20]. Hence G is a co-coatomically *ss*-supplemented module according to Corollary 8.
(4) $(\Longrightarrow)$ Since each submodule of E is co-coatomically *ss*-supplemented, then each submodule of E is a radical supplemented module according to Corollary 8. Therefore, $T(E)$ is a supplemented module and $E/T(E)$ has finite rank where $T(E)$ is the torsion part of E by Lemma 3.2 [20].

Maejo Int. J. Sci. Technol. **2025**, *19*(01), 80-93

($\Leftarrow$) By the hypothesis, each submodule of E is a radical supplemented module by Lemma 3.2 [20]. Thus, each submodule of E is a co-coatomically ss-supplemented module according to Corollary 8.

CO-COATOMICALLY *SS*-SEMI-LOCAL MODULES

For a module E in the subsequent discussion, we call it *co-coatomically ss-semi-local* when each $G \leq_{cc} E$ has a weak ss-supplement W within E, i.e. $E = G + W$, $G \cap W$ is semi-simple and $G \cap W \ll E$. Explicitly, each co-coatomically ss-supplemented module is a co-coatomically ss-semi-local module. However, as demonstrated in Example 1, the $\mathbb{Z}$-module $\mathbb{Q}$ is a co-coatomically ss-semi-local module despite not being ss-semi-local as indicated in Example 2.2 [15].

Lemma 3. Suppose that E is a coatomic module. Then E is a co-coatomically ss-semi-local module if and only if E is an ss-semi-local module.

Proof. To demonstrate the necessity, assume that the coatomic module E is co-coatomically ss-semi-local and $G \leq E$. Then the quotient module E/G is coatomic. Therefore, G has a weak ss-supplement in E by assumption. Hence E is an ss-semi-local module.

Theorem 6. The statements below for a ring S are equivalent:

(1) ${}_SS$ is an ss-semi-local module.
(2) Each left S-module is an ss-semi-local module.
(3) Each left S-module is a co-coatomically ss-semi-local module.
(4) S is a semi-local ring and $Rad(S) \subseteq Soc({}_SS)$.

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (4)$: By Theorem 2.15 [15].
$(2) \Rightarrow (3)$: Obvious.
$(3) \Rightarrow (4)$: By (3), the coatomic module ${}_SS$ is a co-coatomically ss-semi-local module. Then according to Lemma 3, ${}_SS$ is an ss-semi-local module. Thus, ${}_SS$ is a semi-local module and so S is a semi-local ring. Moreover, as ${}_SS$ is a weak ss-supplement of $Rad(S)$ in ${}_SS$, $Rad(S)$ is semi-simple.

Proposition 8. Suppose that E is a projective module over a ring S with semi-simple radical. When the quotient module E/G is a co-coatomically ss-semi-local module with $G \ll E$, E is a co-coatomically ss-semi-local module.

Proof. Let $L \leq_{cc} E$. Thus, $(L+G)/G \leq_{cc} E/G$ because $L + G \leq_{cc} E$. By the hypothesis, there exists a weak ss-supplement W/G of $(L + G)/G$ in E/G, i.e. $E/G = (L + G)/G + W/G$, $((L + G) \cap W)/G$ is semi-simple and $((L + G) \cap W)/G \ll E/G$. Since $G \ll E$, then we have $(W \cap L) + G = W \cap (L + G) \ll E$ from Section 2.2 [9]. Therefore, we obtain $E = L + W$ and $L \cap W \ll E$. Thus, $L \cap W \leq Rad(E)$. Since E is a projective module, then by assumption, $L \cap W \leq Rad(S)E \leq Soc({}_SS)E = Soc(E)$. Hence W is a weak ss-supplement of L in E, and so E is a co-coatomically ss-semi-local module.

Corollary 11. Suppose that S is a ring with semi-simple radical and that E is a co-coatomically ss-semi-local S-module. Then the projective cover of E is a co-coatomically ss-semi-local module.

Proof. By Proposition 8.

Proposition 9. Co-coatomically ss-semi-local modules exhibit transfer properties through their quotient modules.

Maejo Int. J. Sci. Technol. **2025**, *19*(01), 80-93

Proof. Suppose that E is a co-coatomically *ss*-semi-local module and $G \leq E$. Then any co-coatomic submodule of E/G has the form L/G, where $L \leq_{cc} E$. By the hypothesis, L has a weak *ss*-supplement W in E, i.e. $E = L + W$, $L \cap W$ is semi-simple and $L \cap W \ll E$. For this reason, $E/G = L/G + (W + G)/G$. Also by considering the canonical projection $\pi : E \to E/G$, we conclude that $((W + G) \cap L)/G = ((L \cap W) + G)/G = \pi(L \cap W) \ll \pi(E) = E/G$ from Section 2.2 [9]. Moreover, $((W + G) \cap L)/G$ is semi-simple from Corollary 8.1.5 [1]. Hence L/G has a weak *ss*-supplement $(W + G)/G$ in E/G, and so E/G is a co-coatomically *ss*-semi-local module.

Let E be a module and $G, H \leq E$. G is called a *complement submodule* in E of H when it is maximal element in the set of whole submodules L of E such that $H \cap L = 0$ [9]. By Section 1.10 [9] it is known that a submodule of E is a complement if and only if it is closed. Over a Dedekind domain, closed submodules and co-closed submodules coincide as indicated in Lemma 3.3 [14]. Consequently, a torsion submodule $T(E)$ of a module E is a co-closed submodule of E over a Dedekind domain, as it is closed as mentioned in Example 6.34 [22].

Proposition 10. Suppose that *S* is a non-local Dedekind domain and E is a torsion *S*-module. Then E is a co-coatomically *ss*-semi-local module if and only if E is a co-coatomically *ss*-supplemented module.

Proof. (⟹) Let $G \leq_{cc} E$. By the hypothesis, G has a weak *ss*-supplement W in E. Thus, we have $E = G + W$, $G \cap W$ is semi-simple and $G \cap W \ll E$. Since E is a torsion module, then W is too. Hence $W \leq E$ is co-closed. Therefore, $G \cap W \ll W$ by Section 3.7 [9]. Hence E is a co-coatomically *ss*-supplemented module.
(⟸) Obvious.

Proposition 11. Suppose that E is a reduced *S*-module where *S* is non-local Dedekind domain. When E is a co-coatomically *ss*-supplemented module and $T(E)$ has a weak *ss*-supplement in E, $E/T(E)$ is divisible and $T(E)/(T(E) \cap G)$ is co-coatomically *ss*-semi-local for any submodule *G* of *E*.

Proof. We claim that $E/T(E)$ is a radical module. To demonstrate this, assume on the contrary that there is a maximal $H \leq E$ including $T(E)$. By the hypothesis, H has an *ss*-supplement W in E. Since H is a maximal submodule, then W is a strongly local module according to Proposition 12 [6], so that W is cyclic, and for some left ideal J of *S*, we deduce that $W \cong S/J$. However, $J \neq 0$ as *S* is not local. Hence W is a torsion module. From this, we reach the contradiction that $W \subseteq T(E)$. So $E/T(E)$ does not have any maximal submodule, i.e. $E/T(E)$ is divisible by Lemma 4.4 [4]. On the other hand, $T(E)$ is closed from Example 6.34 [22]. Thus, $T(E)$ is co-closed according to Lemma 3.3 [14]. Since $T(E)$ has a weak *ss*-supplement, then we can conclude that $T(E)$ is a supplement submodule in E according to Section 20.2 [9]. Hence there is a submodule G of E such that $E = T(E) + G$, $T(E) \cap G \ll T(E)$. Note here that $E/G \cong T(E)/(T(E) \cap G)$. Since E is also co-coatomically *ss*-semi-local module, then $T(E)/(T(E) \cap G)$ is a co-coatomically *ss*-semi-local module by Proposition 9.

Proposition 12. Suppose that *S* is a Dedekind domain and E is an *S*-module. When $T(E) = E_1 \oplus E_2$, where E_1 is semi-simple and E_2 and $E/T(E)$ are divisible, E is a co-coatomically *ss*-supplemented module.

Proof. By the hypothesis, co-coatomic submodules of E are direct summands from Theorem 4.1 [5]. Hence E is a co-coatomically *ss*-supplemented module.

Proposition 13. Suppose that S is a Dedekind domain which is not a field and E is a torsion-free left S-module. If E is a co-coatomically ss-supplemented module, then E is a divisible module.

Proof. Let E be a co-coatomically ss-supplemented module and $H \leq E$ be maximal. Then H has an ss-supplement W in E by assumption. Note that $H \cap W \leq Soc(E) \leq T(E)$ due to S being a domain, not a field. Then $E = H \oplus W$ as $T(E) = 0$. This implies that E is a divisible module according to Lemma 6.10 [17].

CONCLUSIONS

In this note, ss-supplemented modules defined by Kaynar et al. [6] are viewed from the same point of view and co-coatomic submodules defined by Alizade and Güngör [5] having an ss-supplement are considered instead of each submodule of the module. Over a semi-perfect ring whose radical is semi-simple, each module is co-coatomically ss-supplemented. In addition, co-coatomically ss-semi-local modules are defined by weakening ss-semi-local module structure defined by Olgun and Türkmen [15]. A torsion module over a non-local Dedekind domain being co-coatomically ss-supplemented is equivalent to being co-coatomically ss-semi-local.

ACKNOWLEDGEMENTS

The author would like to thank the referees and editors for careful reading and useful comments that improved this manuscript.

REFERENCES

1. F. Kasch, "Modules and Rings", Academic Press, New York, **1982**.
2. D. X. Zhou and X. R. Zhang, "Small-essential submodules and Morita duality", *Southeast Asian Bull. Math.*, **2011**, *35*, 1051-1062.
3. H. Zöschinger and F. A. Rosenberg, "Coatomic modules", *Math. Z.*, **1980**, *170,* 221-232 (in German).
4. R. Alizade, G. Bilhan and P. F. Smith, "Modules whose maximal submodules have supplements", *Commun. Algebra*, **2001**, *29*, 2389-2405.
5. R. Alizade and S. Güngör, "Co-coatomically supplemented modules", *Ukr. Math. J.*, **2017**, *69*, 1007-1018.
6. E. Kaynar, H. Çalışıcı and E. Türkmen, "$ss-$Supplemented modules", *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **2020**, *69*, 473-485.
7. C. Lomp, "On semilocal modules and rings", *Commun. Algebra*, **1999**, *27*, 1921-1935.
8. R. Wisbauer, "Foundations of Modules and Rings Theory", Gordon and Breach Science Publishers, Philadelphia, **1991**.
9. J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, "Lifting Modules: Supplements and Projectivity in Module Theory", Birkhauser-Verlag, Basel, **2006**.
10. B. N. Türkmen and B. Kılıç, "On cofinitely ss-supplemented modules", *Algebra Discrete Math.*, **2022**, *34*, 141-151.
11. E. O. Sözen, F. Eryılmaz and B. N. Türkmen, "On module classes of generalized semiperfect modules", *Mat. Bohemica*, **2025**, doi: 10.21136/MB.2025.0053-24.
12. F. Eryılmaz and E. O. Sözen, "On a generalization of $\oplus$-co-coatomically supplemented modules", *Honam Math. J.*, **2023**, *45*, 146–159.

Maejo Int. J. Sci. Technol. **2025**, *19*(01), 80-93

13. İ. Soydan and E. Türkmen, "Co-coatomically ps-supplemented modules", *Erzincan Univ. J. Sci, Technol.*, **2025**, *18*, 140-148.
14. H. Zöschinger, "Supplemented modules over Dedeking rings", *J. Algebra*, **1974**, *29*, 42-56 (in German).
15. A. Olgun and E. Türkmen, "On a class of perfect rings", *Honam Math. J.*, **2020**, *42*, 591-600.
16. E. Büyükaşık and C. Lomp, "Rings whose modules are weakly supplemented are perfect. Application to certain ring extensions", *Math. Scand.*, **2009**, *105*, 25-30.
17. E. Büyükaşık and D. Pusat Yılmaz, "Modules whose maximal submodules are supplements", *Hacettepe J. Math. Stat.*, **2010**, *39*, 477-487.
18. E. Büyükaşık, E. Mermut and S. Özdemir, "Rad-supplemented modules", *Rend. Semin. Mat. Univ. Padova*, **2010**, *124*, 157-177.
19. C. J. Holston, S. K. Jain and A. Leroy, "Rings over which cyclics are direct sums of projective and CS or Noetherian", *Glasg. Math. J.*, **2010**, *52*, 103-110.
20. H. Zöschinger, "Modules that have a supplement in every extension", *Math. Scand.*, **1974**, *35*, 267-287 (in German).
21. İ. Soydan and E. Türkmen, "Generalizations of *ss*-supplemented modules", *Carpathian Math. Publ.*, **2021**, *13*, 119-126.
22. T. Y. Lam, "Lectures on Modules and Rings", Springer, New York, **1999**.