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Technical Note

On Gaussian fuzzy Fibonacci numbers

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Abstract: This paper defines Gaussian fuzzy Fibonacci numbers and derives the Binet formula as well as the generating function for these numbers. Further, this note deals with Vajda's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity in the context of these numbers.

Keywords: Fibonacci numbers, Gaussian numbers, fuzzy numbers

INTRODUCTION

After its introduction by Zadeh [1], fuzzy logic has appeared in many fields such as mathematics, engineering and medicine. A fuzzy set is a generalisation of the concept of classical sets based on the degrees of membership of elements. In other words, with the help of membership functions, an element can be a certain proportion of the fuzzy set. If a fuzzy set is defined by real numbers with convex, normalised and bounded-continuous membership functions, it is called a fuzzy number. After Dubois and Prade [2] defined fuzzy numbers as fuzzy subsets on the real number line in 1978, various approaches to fuzzy numbers and their arithmetic operations have been developed throughout history.

Let (F_n) be the sequence of Fibonacci numbers given by

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0$$

with $F_0 = 0$ and $F_1 = 1$ [3]. Irmak and Demirtaş [4] defined fuzzy Fibonacci numbers and fuzzy Lucas numbers and studied some properties of these numbers. Later, Duman [5] showed some identities related to fuzzy Fibonacci numbers.

In 1832 Gauss [6] introduced Gaussian numbers. The set of these numbers is denoted by

$$\mathbb{Z}[i] = \{a + bi: \quad a, b \in \mathbb{Z}\}.$$

In 1963 Gaussian Fibonacci numbers were defined as

$$GF_{n+2} = GF_{n+1} + GF_n \quad \text{for } n \geq 0$$

with the initial conditions $GF_0 = i$ and $GF_1 = 1$ by Horadam [7]. It is easy to see that

$$GF_n = F_n + iF_{n-1} \quad \text{for } n \geq 0. \tag{1}$$

Later, many studies on Gaussian Fibonacci numbers, Gaussian Pell numbers, Gaussian balancing numbers and others were carried out by many researchers [8-16].

In this study Gaussian fuzzy Fibonacci numbers are defined and Binet formula and the generating function for these numbers are found. After, Vajda, Catalan, Cassini and d'Ocagne identities for Gaussian fuzzy Fibonacci numbers are discussed.

PRELIMINARIES

Let $A = (a_1, a_2, a_3)$ be a triangular fuzzy number. The membership function of the triangular fuzzy numbers is given by

$$\mu_A(x; a_1, a_2, a_3) = \begin{cases} \frac{x-a_1}{a_2-a_1}, a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, a_2 \leq x \leq a_3, \\ 0, x > a_3 \ \lor \ x < a_1 \end{cases}$$

where a_1 and a_3 are the lower and upper bound values of the fuzzy set and a_2 is the single point with exact membership [4]. Moreover, α- cut interval of the triangular fuzzy number is

$$A^\alpha = [a_1^\alpha, a_3^\alpha] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)],$$

where $\alpha \in [0,1]$ and $a_1^\alpha, a_3^\alpha \in \mathbb{R}$ [4]. Binet formula of the Fibonacci numbers is given by

$$F_n = \frac{\phi^n - \beta^n}{\sqrt{5}} \quad \text{for } n \geq 0,$$

where $\phi = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ [3]. Fuzzy Fibonacci numbers are defined as

$$F_{rn+l}^\alpha = \left[F_{r(n-1)+l} + \alpha\left(F_{rn+l} - F_{r(n-1)+l}\right), F_{r(n+1)+l} - \alpha\left(F_{r(n+1)+l} - F_{rn+l}\right)\right]$$

for $n \geq 0$ integer [4]. Here, $r, l \in \mathbb{Z}$ and $\alpha \in [0,1]$. If we take $r = 1$ and $l = 0$, then the above equality converts to

$$F_n^\alpha = [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}]. \tag{2}$$

Let $F_0^\alpha = [1 - \alpha, 1 - \alpha]$ and $F_1^\alpha = [\alpha, 1]$. Fuzzy Fibonacci numbers recurrence relation

$$F_n^\alpha = F_{n-1}^\alpha + F_{n-2}^\alpha \quad \text{for } n \geq 2 \tag{3}$$

holds [4]. Considering the definitions of the addition, subtraction and multiplication given for two fuzzy numbers by Irmak and Demirtaş [4], we can make the following inferences for the addition, subtraction and multiplication on two fuzzy Fibonacci numbers.

$$F_n^\alpha \pm F_m^\alpha = [F_{n-1} \pm F_{m-1} + \alpha(F_{n-2} \pm F_{m-2}), F_{n+1} \pm F_{m+1} - \alpha(F_{n-1} \pm F_{m-1})], \tag{4}$$

$$\begin{aligned} F_n^\alpha \cdot F_m^\alpha \cong [&F_{n-1}F_{m-1} + \alpha(F_{n-2}F_{m-1} + F_{n-1}F_{m-2}) + \alpha^2(F_{n-2}F_{m-2}), \\ &F_{n+1}F_{m+1} - \alpha(F_{n-1}F_{m+1} + F_{n+1}F_{m-1}) + \alpha^2(F_{n-1}F_{m-1})]. \end{aligned} \tag{5}$$

Let us note that the product of two triangular fuzzy Fibonacci numbers is not a triangular fuzzy Fibonacci number. The multiplication of fuzzy Fibonacci numbers with a scalar and the definition of negative indices are as follows:

$$tF_n^\alpha = [t(F_{n-1} + \alpha F_{n-2}), t(F_{n+1} - \alpha F_{n-1})] \tag{6}$$

$$F_{-n}^\alpha = [F_{-n-1} + \alpha F_{-n-2}, F_{-n+1} - \alpha F_{-n-1}] \tag{7}$$

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The Binet formula of the Gaussian Fibonacci numbers is

$$GF_n = \frac{\phi^n - \beta^n}{\sqrt{5}} + i\frac{\phi^{n-1} - \beta^{n-1}}{\sqrt{5}} \quad \text{for } n \geq 0, \tag{8}$$

deduced from Jordan [8]. When we take $\hat{\phi} = 1 - \beta i$ and $\hat{\beta} = 1 - \phi i$, the equality (8) shows that

$$GF_n = \frac{\hat{\phi}\phi^n - \hat{\beta}\beta^n}{\sqrt{5}} \quad \text{for } n \geq 0. \tag{9}$$

MAIN RESULTS

Firstly, the Gaussian fuzzy Fibonacci numbers are defined as follows:

$$GF_n^\alpha = F_n^\alpha + iF_{n-1}^\alpha \text{ for } n \geq 0, \tag{10}$$

where F_n^α is *n-th* fuzzy Fibonacci number. The conjugate of these numbers is given by

$$\overline{GF_n^\alpha} = F_n^\alpha - iF_{n-1}^\alpha. \tag{11}$$

The addition, subtraction and multiplication of two Gaussian fuzzy Fibonacci numbers GF_n^α and GF_m^α are defined as

$$GF_n^\alpha \pm GF_m^\alpha = (F_n^\alpha \pm F_m^\alpha) + i(F_{n-1}^\alpha \pm F_{m-1}^\alpha), \tag{12}$$

$$GF_n^\alpha \cdot GF_m^\alpha \cong F_n^\alpha F_m^\alpha - F_{n-1}^\alpha F_{m-1}^\alpha + i(F_{n-1}^\alpha F_m^\alpha + F_n^\alpha F_{m-1}^\alpha). \tag{13}$$

Further, multiplication with a scalar $'t'$ and negative indices of the Gaussian fuzzy Fibonacci numbers GF_n^α are defined as

$$tGF_n^\alpha = tF_n^\alpha + tiF_{n-1}^\alpha \quad \text{for } n \geq 1, \tag{14}$$

$$GF_{-n}^\alpha = F_{-n}^\alpha + iF_{-n-1}^\alpha \quad \text{for } n \geq 1. \tag{15}$$

Proposition 1. For the Gaussian fuzzy Fibonacci numbers, there are the equalities:

(i) $GF_n^\alpha + \overline{GF_n^\alpha} = 2F_n^\alpha$

(ii) $(GF_n^\alpha)^2 \cong 2F_n^\alpha GF_n^\alpha - GF_n^\alpha \overline{GF_n^\alpha}$

(iii) $GF_n^\alpha \cdot \overline{GF_n^\alpha} = (F_n^\alpha)^2 + (F_{n-1}^\alpha)^2$

Proof (i) From the equalities (10) and (11), we get

$$GF_n^\alpha + \overline{GF_n^\alpha} = F_n^\alpha + iF_{n-1}^\alpha + F_n^\alpha - iF_{n-1}^\alpha = 2F_n^\alpha.$$

(ii) Taking into account the previous identity, we obtain

$$(GF_n^\alpha)^2 \cong GF_n^\alpha GF_n^\alpha = GF_n^\alpha\left(2F_n^\alpha - \overline{GF_n^\alpha}\right) = 2F_n^\alpha GF_n^\alpha - GF_n^\alpha \overline{GF_n^\alpha}.$$

(iii) The equalities (10), (11) and (13) give us

$$\begin{aligned} GF_n^\alpha \cdot \overline{GF_n^\alpha} &\cong (F_n^\alpha + iF_{n-1}^\alpha) \cdot (F_n^\alpha - iF_{n-1}^\alpha) \\ &\cong (F_n^\alpha)^2 + (F_{n-1}^\alpha)^2 + i(F_{n-1}^\alpha F_n^\alpha - F_n^\alpha F_{n-1}^\alpha) \\ &= (F_n^\alpha)^2 + (F_{n-1}^\alpha)^2. \end{aligned}$$

Proposition 2. For the Gaussian fuzzy Fibonacci numbers, the following equality holds:

$$GF_{n+2}^\alpha = GF_{n+1}^\alpha + GF_n^\alpha$$

Proof. Using the equalities (3), (10) and (12), we can write

$$\begin{aligned} GF_{n+1}^\alpha + GF_n^\alpha &= F_{n+1}^\alpha + iF_n^\alpha + F_n^\alpha + iF_{n-1}^\alpha \\ &= F_{n+1}^\alpha + F_n^\alpha + i(F_n^\alpha + F_{n-1}^\alpha) \\ &= F_{n+2}^\alpha + iF_{n+1}^\alpha \\ &= GF_{n+2}^\alpha. \end{aligned}$$

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Theorem 1. The Binet formula of the Gaussian fuzzy Fibonacci numbers is given by

$$GF_n^\alpha = \left[\frac{\hat{\phi}\phi^{n-1}-\hat{\beta}\beta^{n-1}}{\sqrt{5}} + \alpha\frac{\hat{\phi}\phi^{n-2}-\hat{\beta}\beta^{n-2}}{\sqrt{5}}, \frac{\hat{\phi}\phi^{n+1}-\hat{\beta}\beta^{n+1}}{\sqrt{5}} - \alpha\frac{\hat{\phi}\phi^{n-1}-\hat{\beta}\beta^{n-1}}{\sqrt{5}}\right],$$

where $\hat{\phi} = 1 - \beta i$ and $\hat{\beta} = 1 - \phi i$.

Proof. The equalities (1), (2), (4), (6) and (10) give us:

$$\begin{aligned} GF_n^\alpha &= F_n^\alpha + iF_{n-1}^\alpha = [F_{n-1} + \alpha F_{n-2}, F_{n+1} - \alpha F_{n-1}] + [F_{n-2} + \alpha F_{n-3}, F_n - \alpha F_{n-2}]i \\ &= [F_{n-1} + iF_{n-2} + \alpha(F_{n-2} + iF_{n-3}), F_{n+1} + iF_n - \alpha(F_{n-1} + iF_{n-2})] \\ &= [GF_{n-1} + \alpha GF_{n-2}, GF_{n+1} - \alpha GF_{n-1}]. \end{aligned}$$

In view of the last equality and equality (9), we can write

$$\begin{aligned} GF_n^\alpha &= [GF_{n-1} + \alpha GF_{n-2}, GF_{n+1} - \alpha GF_{n-1}] \\ &= \left[\frac{\hat{\phi}\phi^{n-1}-\hat{\beta}\beta^{n-1}}{\sqrt{5}} + \alpha\frac{\hat{\phi}\phi^{n-2}-\hat{\beta}\beta^{n-2}}{\sqrt{5}}, \frac{\hat{\phi}\phi^{n+1}-\hat{\beta}\beta^{n+1}}{\sqrt{5}} - \alpha\frac{\hat{\phi}\phi^{n-1}-\hat{\beta}\beta^{n-1}}{\sqrt{5}}\right]. \end{aligned}$$

Theorem 2. The generating function of the Gaussian fuzzy Fibonacci numbers is given by

$$GF^\alpha(x) = \frac{[1-\alpha,1-\alpha]+[-1+2\alpha,\alpha]x+([-1+2\alpha,\alpha]+[2-3\alpha,1-2\alpha]x)i}{1-x-x^2}.$$

Proof. Let

$$GF^\alpha(x) = \sum_{n=0}^{\infty} GF_n^\alpha x^n = GF_0^\alpha + GF_1^\alpha x + GF_2^\alpha x^2 + \cdots \tag{16}$$

be the generating function of the Gaussian fuzzy Fibonacci numbers. Then we have

$$-xGF^\alpha(x) = -GF_0^\alpha x - GF_1^\alpha x^2 - GF_2^\alpha x^3 - \cdots \tag{17}$$

and

$$-x^2 GF^\alpha(x) = -GF_0^\alpha x^2 - GF_1^\alpha x^3 - GF_2^\alpha x^4 - \cdots \tag{18}$$

From the equalities (16), (17) and (18), we conclude that

$$(1 - x - x^2)GF^\alpha(x) = GF_0^\alpha + (GF_1^\alpha - GF_0^\alpha)x.$$

Thanks to Proposition 2, we know that $GF_{-1}^\alpha = GF_1^\alpha - GF_0^\alpha$. Thus, the above equality implies that

$$GF^\alpha(x) = \frac{GF_0^\alpha + GF_{-1}^\alpha x}{1-x-x^2}.$$

Considering the above equality together with the equalities (7), (10), (14) and (15), we find that

$$\begin{aligned} GF^\alpha(x) &= \frac{F_0^\alpha + iF_{-1}^\alpha + (F_{-1}^\alpha + iF_{-2}^\alpha)x}{1-x-x^2} \\ &= \frac{F_0^\alpha + F_{-1}^\alpha x + (F_{-1}^\alpha + F_{-2}^\alpha x)i}{1-x-x^2} \\ &= \frac{[1-\alpha,1-\alpha]+[-1+2\alpha,\alpha]x+([-1+2\alpha,\alpha]+[2-3\alpha,1-2\alpha]x)i}{1-x-x^2}. \end{aligned}$$

Lemma 1. Let $\hat{\phi} = 1 - \beta i$ and $\hat{\beta} = 1 - \phi i$. So we have

$$-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r = (2 - i)(\phi^r - \beta^r).$$

Proof. Let us recall that $\phi\beta = -1$, $\phi + \beta = 1$ and $i^2 = -1$. Thus, we have the equality

$$\begin{aligned} -\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r &= -(1 - \beta i)(1 - \phi i)\beta^r + (1 - \phi i)(1 - \beta i)\phi^r \\ &= (-1 + \beta i + \phi i + \phi\beta)\beta^r + (1 - \beta i - \phi i - \phi\beta)\phi^r \\ &= (-1 + (\beta + \phi)i - 1)\beta^r + (1 - (\beta + \phi)i + 1)\phi^r \\ &= (-2 + i)\beta^r + (2 - i)\phi^r \\ &= (2 - i)(\phi^r - \beta^r). \end{aligned}$$

Theorem 3 (Vajda's identity). For $n, m, r \in \mathbb{Z}$, there is the equality

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha} - GF_{n}^{\alpha}GF_{n+m+r}^{\alpha} \cong (2-i)F_mF_r(F_0^{\alpha}F_2^{\alpha} - (F_1^{\alpha})^2).$$

Proof. According to Theorem 1, we can write

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha} - GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}$$

$$\cong \left[\frac{\hat{\phi}\phi^{n+m-1}-\hat{\beta}\beta^{n+m-1}}{\sqrt{5}} + \alpha\frac{\hat{\phi}\phi^{n+m-2}-\hat{\beta}\beta^{n+m-2}}{\sqrt{5}}, \frac{\hat{\phi}\phi^{n+m+1}-\hat{\beta}\beta^{n+m+1}}{\sqrt{5}} - \alpha\frac{\hat{\phi}\phi^{n+m-1}-\hat{\beta}\beta^{n+m-1}}{\sqrt{5}}\right]$$
$$\times \left[\frac{\hat{\phi}\phi^{n+r-1}-\hat{\beta}\beta^{n+r-1}}{\sqrt{5}} + \alpha\frac{\hat{\phi}\phi^{n+r-2}-\hat{\beta}\beta^{n+r-2}}{\sqrt{5}}, \frac{\hat{\phi}\phi^{n+r+1}-\hat{\beta}\beta^{n+r+1}}{\sqrt{5}} - \alpha\frac{\hat{\phi}\phi^{n+r-1}-\hat{\beta}\beta^{n+r-1}}{\sqrt{5}}\right]$$
$$- \left[\frac{\hat{\phi}\phi^{n-1}-\hat{\beta}\beta^{n-1}}{\sqrt{5}} + \alpha\frac{\hat{\phi}\phi^{n-2}-\hat{\beta}\beta^{n-2}}{\sqrt{5}}, \frac{\hat{\phi}\phi^{n+1}-\hat{\beta}\beta^{n+1}}{\sqrt{5}} - \alpha\frac{\hat{\phi}\phi^{n-1}-\hat{\beta}\beta^{n-1}}{\sqrt{5}}\right]$$
$$\times \left[\frac{\hat{\phi}\phi^{n+m+r-1}-\hat{\beta}\beta^{n+m+r-1}}{\sqrt{5}} + \alpha\frac{\hat{\phi}\phi^{n+m+r-2}-\hat{\beta}\beta^{n+m+r-2}}{\sqrt{5}}, \frac{\hat{\phi}\phi^{n+m+r+1}-\hat{\beta}\beta^{n+m+r+1}}{\sqrt{5}} - \alpha\frac{\hat{\phi}\phi^{n+m+r-1}-\hat{\beta}\beta^{n+m+r-1}}{\sqrt{5}}\right].$$

When the necessary mathematical operations are made, we can write

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha} - GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}$$

$$\cong \frac{1}{5}[-\hat{\phi}\phi^{n+m-1}\hat{\beta}\beta^{n+r-1} - \hat{\beta}\beta^{n+m-1}\hat{\phi}\phi^{n+r-1} + \hat{\phi}\phi^{n-1}\hat{\beta}\beta^{n+m+r-1} + \hat{\beta}\beta^{n-1}\hat{\phi}\phi^{n+m+r-1}$$
$$+\alpha(-\hat{\phi}\phi^{n+m-2}\hat{\beta}\beta^{n+r-1} - \hat{\beta}\beta^{n+m-2}\hat{\phi}\phi^{n+r-1} + \hat{\phi}\phi^{n-2}\hat{\beta}\beta^{n+m+r-1} + \hat{\beta}\beta^{n-2}\hat{\phi}\phi^{n+m+r-1}$$
$$-\hat{\phi}\phi^{n+m-1}\hat{\beta}\beta^{n+r-2} - \hat{\beta}\beta^{n+m-1}\hat{\phi}\phi^{n+r-2} + \hat{\phi}\phi^{n-1}\hat{\beta}\beta^{n+m+r-2} + \hat{\beta}\beta^{n-1}\hat{\phi}\phi^{n+m+r-2})$$
$$+\alpha^2(-\hat{\phi}\phi^{n+m-2}\hat{\beta}\beta^{n+r-2} - \hat{\beta}\beta^{n+m-2}\hat{\phi}\phi^{n+r-2} + \hat{\phi}\phi^{n-2}\hat{\beta}\beta^{n+m+r-2} + \hat{\beta}\beta^{n-2}\hat{\phi}\phi^{n+m+r-2}),$$
$$-\hat{\phi}\phi^{n+m+1}\hat{\beta}\beta^{n+r+1} - \hat{\beta}\beta^{n+m+1}\hat{\phi}\phi^{n+r+1} + \hat{\phi}\phi^{n+1}\hat{\beta}\beta^{n+m+r+1} + \hat{\beta}\beta^{n+1}\hat{\phi}\phi^{n+m+r+1}$$
$$-\alpha(-\hat{\phi}\phi^{n+m+1}\hat{\beta}\beta^{n+r-1} - \hat{\beta}\beta^{n+m+1}\hat{\phi}\phi^{n+r-1} + \hat{\phi}\phi^{n+1}\hat{\beta}\beta^{n+m+r-1} + \hat{\beta}\beta^{n+1}\hat{\phi}\phi^{n+m+r-1}$$
$$-\hat{\phi}\phi^{n+m-1}\hat{\beta}\beta^{n+r+1} - \hat{\beta}\beta^{n+m-1}\hat{\phi}\phi^{n+r+1} + \hat{\phi}\phi^{n-1}\hat{\beta}\beta^{n+m+r+1} + \hat{\beta}\beta^{n-1}\hat{\phi}\phi^{n+m+r+1})$$
$$+\alpha^2(-\hat{\phi}\phi^{n+m-1}\hat{\beta}\beta^{n+r-1} - \hat{\beta}\beta^{n+m-1}\hat{\phi}\phi^{n+r-1} + \hat{\phi}\phi^{n-1}\hat{\beta}\beta^{n+m+r-1} + \hat{\beta}\beta^{n-1}\hat{\phi}\phi^{n+m+r-1})].$$

If we make manipulations, we obtain

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha} - GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}$$

$$= \frac{1}{5}[-\hat{\phi}\hat{\beta}(-1)^{n-1}\beta^r(\phi^m - \beta^m) + \hat{\beta}\hat{\phi}(-1)^{n-1}\phi^r(\phi^m - \beta^m)$$
$$+\alpha\left(-\hat{\phi}\hat{\beta}(-1)^{n-2}\beta^r(\phi^m - \beta^m)(\beta + \phi) + \hat{\beta}\hat{\phi}(-1)^{n-2}\phi^r(\phi^m - \beta^m)(\phi + \beta)\right)$$
$$+\alpha^2\left(-\hat{\phi}\hat{\beta}(-1)^{n-2}\beta^r(\phi^m - \beta^m) + \hat{\beta}\hat{\phi}(-1)^{n-2}\phi^r(\phi^m - \beta^m)\right),$$
$$-\hat{\phi}\hat{\beta}(-1)^{n+1}\beta^r(\phi^m - \beta^m) + \hat{\beta}\hat{\phi}(-1)^{n+1}\phi^r(\phi^m - \beta^m)$$
$$-\alpha\left(-\hat{\phi}\hat{\beta}(-1)^{n-1}\beta^r(\phi^m - \beta^m)\left(\phi^2 + \beta^2\right) + \hat{\beta}\hat{\phi}(-1)^{n-1}\phi^r(\phi^m - \beta^m)\left(\beta^2 + \phi^2\right)\right)$$
$$+\alpha^2\left(-\hat{\phi}\hat{\beta}(-1)^{n-1}\beta^r(\phi^m - \beta^m) + \hat{\beta}\hat{\phi}(-1)^{n-1}\phi^r(\phi^m - \beta^m)\right)]$$

i.e.

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha} - GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}$$

$$= \frac{1}{5}[(-1)^{n-1}(\phi^m - \beta^m)\left(-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r\right) + \alpha\left((-1)^{n-2}(\phi^m - \beta^m)\left(-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r\right)\right)$$
$$+\alpha^2\left((-1)^{n-2}(\phi^m - \beta^m)\left(-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r\right)\right), (-1)^{n+1}(\phi^m - \beta^m)\left(-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r\right)$$
$$-\alpha\left(3(-1)^{n-1}(\phi^m - \beta^m)\left(-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r\right)\right) + \alpha^2\left((-1)^{n-1}(\phi^m - \beta^m)\left(-\hat{\phi}\hat{\beta}\beta^r + \hat{\beta}\hat{\phi}\phi^r\right)\right)].$$

By Lemma 1,

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha} - GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}$$

$$\cong \frac{1}{5}[(-1)^{n-1}(\phi^m-\beta^m)(2-i)(\phi^r-\beta^r)+\alpha\big((-1)^{n-2}(\phi^m-\beta^m)(2-i)(\phi^r-\beta^r)\big)$$
$$+\alpha^2\left((-1)^{n-2}(\phi^m-\beta^m)(2-i)(\phi^r-\beta^r)\right),(-1)^{n+1}(\phi^m-\beta^m)(2-i)(\phi^r-\beta^r)$$
$$-\alpha\left(3(-1)^{n-1}(\phi^m-\beta^m)(2-i)(\phi^r-\beta^r)\right)+\alpha^2\left((-1)^{n-1}(\phi^m-\beta^m)(2-i)(\phi^r-\beta^r)\right)]$$

and so

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha}-GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}$$
$$=[(-1)^{n-1}F_m(2-i)F_r+\alpha((-1)^{n-2}F_m(2-i)F_r)+\alpha^2((-1)^{n-2}F_m(2-i)F_r),$$
$$(-1)^{n+1}F_m(2-i)F_r-\alpha(3(-1)^{n-1}F_m(2-i)F_r)+\alpha^2((-1)^{n-1}F_m(2-i)F_r)].$$

Thus, we get

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha}-GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}\cong(2-i)F_mF_r[1-\alpha-\alpha^2,1-3\,\alpha+\alpha^2]$$

or

$$GF_{n+m}^{\alpha}GF_{n+r}^{\alpha}-GF_{n}^{\alpha}GF_{n+m+r}^{\alpha}\cong(2-i)F_mF_r(F_0^{\alpha}F_2^{\alpha}-(F_1^{\alpha})^2).$$

Corollary 1 (Catalan identity). For $n,r\in\mathbb{Z}$, we obtain

$$GF_{n-r}^{\alpha}GF_{n+r}^{\alpha}-(GF_{n}^{\alpha})^2\cong(-1)^{r+1}(2-i)(F_r)^2(F_0^{\alpha}F_2^{\alpha}-(F_1^{\alpha})^2).$$

Proof. If $m=-r$ is taken in Vajda's identity, the corollary is proven.

Corollary 2 (Cassini identity). For $n\in\mathbb{Z}$, we have

$$GF_{n-1}^{\alpha}GF_{n+1}^{\alpha}-(GF_{n}^{\alpha})^2\cong(2-i)(F_0^{\alpha}F_2^{\alpha}-(F_1^{\alpha})^2).$$

Proof. If $m=-1$ and $r=1$ are taken in Vajda's identity, the proof is completed.

Corollary 3 (d' Ocagne identity). For $n,k\in\mathbb{Z}$, we get

$$GF_{k}^{\alpha}GF_{n+1}^{\alpha}-GF_{n}^{\alpha}GF_{k+1}^{\alpha}\cong(2-i)F_{k-n}(F_0^{\alpha}F_2^{\alpha}-(F_1^{\alpha})^2).$$

Proof. If $m+n=k$ and $r=1$ are taken in Vajda's identity, the desired result is obtained.

CONCLUSIONS

This study has explored the interplay between Gaussian numbers, fuzzy numbers and Fibonacci numbers. Specifically, the concept of Gaussian fuzzy Fibonacci numbers has been introduced and several key properties of these numbers have been established. It is suggested that future researchers can extend this work by defining fuzzy variants of other number sequences such as Gaussian fuzzy Lucas, Gaussian fuzzy Pell and Gaussian fuzzy Pell-Lucas numbers, further exploring the properties associated with these new constructs.

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