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Full Paper

Bicomplex approach to double sequences

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Abstract: In this study we introduce bicomplex versions of double sequences, extending the classical theory of sequences to the bicomplex number system. Also, we discuss formal definitions of bicomplex double sequences and investigate their fundamental properties pertaining to the hyperbolic valued norm $|.|_{\mathbb{k}}$ which serves as a natural tool for measuring the magnitude of bicomplex numbers while preserving their algebraic characteristics. Furthermore, we give some examples to show the usability of our findings and applicability of the theoretical results. The findings offer a foundational contribution to bicomplex analysis with potential implications for functional analysis, operator theory and other areas of applied mathematics.

Keywords: bicomplex number, hyperbolic valued norm, double sequence, convergence in Pringsheim's sense

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INTRODUCTION

In 1892 Segre [1] created a new number system called bicomplex numbers as a pioneering attempt in the development of special algebras, and the initial investigations into this new number system were conducted by Scorza [2] and Spampinato [3]. Unlike quaternions, this number system generalises complex numbers by four reals satisfying the commutative rule of multiplication. In this regard due to the fact that it is a ring having divisors of zero, the theory of bicomplex variables differs in many aspects from the analysis of one complex variable.

Subsequently, the book of Price [4] is a good resource of the bicomplex algebra and bicomplex functional analysis and it is considered the foundational work of this theory. In recent years the study of bicomplex numbers has become more popular and the interested reader is referred to an important book that develops the bicomplex counterpart of functional analysis with complex scalars by Alpay et al. [5]. Değirmen and Sağır [6] obtained a new and effective area of study by establishing $\mathbb{D}$-normed Banach bicomplex $\mathbb{BC}$-modules $l_p^{\mathbb{k}}(\mathbb{BC})$ with bicomplex scalars. Hereupon, bicomplex numbers have found prominent applications across various domains of mathematical sciences and different branches of science and technology such as digital image processing, fluid mechanics, neural networks, geometry, computer graphics for describing fractals and theoretical physics [7-13].

The sequence spaces have wide application areas ranging from economics to engineering. Due to their connections with the theory of selection principles which relates to game theory, combinatorics and function spaces, double sequences have recently attracted significant attention in both pure and applied analysis. The double sequences have more than one type of convergence specified, contrary to single sequences.

The notion of convergence for double sequences was introduced by Pringsheim [14]. Hardy [15] developed the idea of regular convergence because the convergence sequence in the sense of Pringsheim does not require being bounded. The convergence of the rows and columns of the double sequences, as well as the Pringsheim convergence, is required for this convergence concept. Móricz [16] investigated some properties of the space of convergent double sequences in the Pringsheim's sense, the space of convergent double sequences to zero in the Pringsheim's sense and the space of regularly convergent double sequences. Boos et al. [17] defined e-, be- and c-convergence concepts for double sequences and determined some topological properties of these convergence types. Zeltser [18] investigated the theory of topological double sequences and the theory of summability of double sequence spaces. Duyar and Oğur [19] defined the double sequence space $m^2(M, A, \phi, p)$ using the Orlicz function and matrix transformations and examined its general properties. Sağır et al. [20] delineated the space $m^2(F, \phi, p)$ by a double series of modulus functions and elucidated its significant analytical features comprehensively. Duyar and Oğur [21] created a Cesàro-Orlicz double sequence space utilising the Orlicz function and established some of its topological features.

Before explaining the main purpose of the article and its contribution to the literature, let us mention the topic of bicomplex sequence spaces. The topic was first studied by Sager and Sağır [22], who constructed the bicomplex sequence spaces $c(\mathbb{BC})$, $c_0(\mathbb{BC})$ and $l_p(\mathbb{BC})$ and investigated their completeness properties. In continuation, they discussed the topological [23] and geometric [24] properties of these spaces. In addition to topological and geometric properties of the fundamental spaces, some researchers also studied Köthe-Toeplitz duals, multiplier spaces and matrix transformations [25-27]. On the other hand, new studies have emerged that approach these spaces from different perspectives. In 2023 Raj et al. [28] constructed Orlicz-Lacunary bicomplex sequence spaces and presented their algebraic and topological properties. Işık and Duyar [29] defined bicomplex Lorentz sequence spaces. Bera and Tripathy [30] extended the concept of statistical bounded sequences to bicomplex numbers and constructed statistical bounded sequence spaces of bicomplex numbers, mentioning summability properties. Some authors also examined Cesàro convergence of sequences of bicomplex numbers using $\mathbb{BC}$-Orlicz function [31], statistically convergent difference sequences of bicomplex numbers [32] and some types of convergence and geometric properties of double sequences of bicomplex numbers with respect to the Euclidean norm on $\mathbb{BC}$ [33-36]. In 2025 Parajuli et al. [37] studied difference sequence spaces in the bicomplex sense.

Motivated essentially by all the work outlined above, in this study we consider the double sequences of bicomplex numbers as a new and exciting contribution to existing literature. Also, by using hyperbolic valued norm $|.|_{\Bbbk}$ which has quite good properties, we investigate bicomplex versions of some concepts and properties given by some researchers for classical double sequences [14-16, 18, 38]. Given the widespread usage of the theory of double sequences in fields ranging from engineering to economics, as well as the broad range of applications for bicomplex sequence spaces, our article offers fresh insights into how to solve various problems in these fields. Furthermore, we hope that the current article will become a starting point for the theory of bicomplex double sequences, bicomplex double sequence spaces and applications to many interesting problems in various aspects.

PRELIMINARIES

Now we provide a basic introduction to bicomplex numbers, which focuses on the nomenclature needed in this discussion. More comprehensive information can be obtained in the literature [4, 5, 13].

Bicomplex numbers represent one of the generalisations of the complex numbers by means of entities specified by four real numbers and any bicomplex number is defined by the base element $1,\ i, j, ij$, where $i^2 = j^2 = -1,\ ij = ji$, as follows:

$$z = z_1 + jz_2 = (x_1 + iy_1) + j(x_2 + iy_2),$$

$z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2 \in \mathbb{C},\ x_1, y_1, x_2, y_2 \in \mathbb{R}$. A set of bicomplex numbers is denoted by $\mathbb{BC}$ and it forms an algebra and $\mathbb{BC}$-module with respect to the addition, scalar multiplication and multiplication defined as

$$z + w = (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2)$$
$$\lambda \cdot w = \lambda \cdot (z_1 + jz_2) = \lambda z_1 + j\lambda z_2$$
$$z \times w = zw = (z_1 + jz_2)(w_1 + jw_2) = (z_1w_1 - z_2w_2) + j(z_1w_2 + z_2w_1)$$

for all $z = z_1 + jz_2,\ w = w_1 + jw_2 \in \mathbb{BC},\ \lambda \in \mathbb{R}$. Also, $\mathbb{D} = \{x + \Bbbk y: x, y \in \mathbb{R}, \Bbbk = ij\} \subset \mathbb{BC}$ is a set of hyperbolic numbers.

There are three types of conjugates and three types of moduli of any bicomplex number in $\mathbb{BC}$ as follows:

$$z^{\dagger_1} = \overline{z_1} + j\overline{z_2}, z^{\dagger_2} = z_1 - jz_2, z^{\dagger_3} = \overline{z_1} - j\overline{z_2},$$
$$|z|_i^2 = zz^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(i),$$
$$|z|_j^2 = zz^{\dagger_1} = (|z_1|^2 - |z_2|^2) + j\big(2\Re(z_1 \cdot \overline{z_2})\big) \in \mathbb{C}(j) = \{x + jy: x, y \in \mathbb{R}\},$$
$$|z|_{\Bbbk}^2 = zz^{\dagger_3} = (|z_1|^2 + |z_2|^2) + \Bbbk\big(-\Im(z_1 \cdot \overline{z_2})\big) \in \mathbb{D}.$$

If $|z|_i \neq 0,\ z$ is called an invertible element and $z^{-1} = \frac{z^{\dagger}}{|z|_i^2}$. The fraction $\frac{z}{w}$ is defined for $z, w \in \mathbb{BC}$ if and only if w is invertible, and also $\frac{z}{w} = zw^{-1}$.

The set $\left\{e_1 = \frac{1+ij}{2}, e_2 = \frac{1-ij}{2}\right\}$ is the idempotent basis of $\mathbb{BC}$ and hence the idempotent representation of $z = z_1 + jz_2$ is uniquely written as $= \beta_1e_1 + \beta_2e_2$, where $\beta_1 = z_1 - iz_2$, $\beta_2 = z_1 + iz_2 \in \mathbb{C}$.

For two bicomplex numbers $z = \beta_1e_1 + \beta_2e_2$ and $= \gamma_1e_1 + \gamma_2e_2$, we have the following:

$$z \pm w = (\beta_1 \pm \gamma_1)e_1 + (\beta_2 \pm \gamma_2)e_2$$
$$zw = (\beta_1\gamma_1)e_1 + (\beta_2\gamma_2)e_2$$
$$\frac{z}{w} = \frac{\beta_1}{\gamma_1}e_1 + \frac{\beta_2}{\gamma_2}e_2, \text{ if } w \text{ is invertible.}$$

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The idempotent representation of any hyperbolic number $\alpha = x + \mathbb{k}y$ is $\alpha = \alpha_1 e_1 + \alpha_2 e_2 = (x+y)e_1 + (x-y)e_2$ and if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, then we say that α is positive hyperbolic. The set of such numbers is denoted by $\mathbb{D}^+$, i.e. $\mathbb{D}^+ = \{\alpha = \alpha_1 e_1 + \alpha_2 e_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}$. For $\alpha, \beta \in \mathbb{D}$, if $\beta - \alpha \in \mathbb{D}^+$, then we write $\alpha \precsim \beta$ and we have

$$\alpha \precsim \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2.$$
$$\alpha \prec \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2.$$

The relation $\precsim$ defines a partial order on $\mathbb{D}$. A map $\|.\|_{\mathbb{BC}}: \mathbb{BC} \to \mathbb{R}^+ \cup \{0\}$, $\|z\|_{\mathbb{BC}} = \|z_1 + jz_2\|_{\mathbb{BC}} = \sqrt{|z_1|^2 + |z_2|^2}$ is a real valued norm on $\mathbb{BC}$ and it satisfies the following properties:

i) $\|z\|_{\mathbb{BC}} = 0$ if and only if $z = 0$.
ii) $\|\mu z\|_{\mathbb{BC}} = |\mu| \cdot \|z\|_{\mathbb{BC}}$ for all $z \in \mathbb{BC}$, $\mu \in \mathbb{R}$.
iii) $\|z + w\|_{\mathbb{BC}} \leq \|z\|_{\mathbb{BC}} + \|w\|_{\mathbb{BC}}$ for all $z, w \in \mathbb{BC}$..

A sequence (z_n) in $\mathbb{BC}$ converges to $z_0 \in \mathbb{BC}$ with respect to the norm $\|.\|_{\mathbb{BC}}$ if for every $\varepsilon > 0$ there is a natural number n_0 such that $\|z_n - z_0\|_{\mathbb{BC}} < \varepsilon$ for all $n \geq n_0$.

The hyperbolic valued module $|z|_{\mathbb{k}}$ of a bicomplex number $z = \beta_1 e_1 + \beta_2 e_2$ is expressed as $|z|_{\mathbb{k}} = |\beta_1| e_1 + |\beta_2| e_2$. Also, $|.|_{\mathbb{k}} : \mathbb{BC} \to \mathbb{D}^+$ is the hyperbolic valued norm (or $\mathbb{D}$-norm) on the $\mathbb{BC}$-module $\mathbb{BC}$ because of the following properties:

i) $|z|_{\mathbb{k}} = 0$ if and only if $z = 0$.
ii) $|zw|_{\mathbb{k}} = |z|_{\mathbb{k}} |w|_{\mathbb{k}}$ for any $z, w \in \mathbb{BC}$.
iii) $|z + w|_{\mathbb{k}} \precsim |z|_{\mathbb{k}} + |w|_{\mathbb{k}}$.

A sequence (z_n) in $\mathbb{BC}$ converges to $z_0 \in \mathbb{BC}$ with respect to the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ if for every $0 \precsim \varepsilon$ there is a natural number n_0 such that $|z_n - z_0|_{\mathbb{k}} \precsim \varepsilon$ for all $n \geq n_0$. Throughout this paper we represent this convergence by ${}^{\mathbb{k}}\lim_{n\to\infty} z_n = z_0$ For any $,\beta, \gamma \in \mathbb{D}$, $z, w \in \mathbb{BC}$, the following hold:

i) $\left|\frac{z}{w}\right|_{\mathbb{k}} = \frac{|z|_{\mathbb{k}}}{|w|_{\mathbb{k}}}$ where w is invertible.
ii) If $\alpha \in \mathbb{D}^+$, then $|\alpha|_{\mathbb{k}} = \alpha$.
iii) If $\alpha \in \mathbb{D}^+$, then $|\alpha z|_{\mathbb{k}} = \alpha |z|_{\mathbb{k}}$.

Before starting our main results, we briefly review double sequences which will be used throughout the paper.

A double sequence of complex numbers is a function $x: \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ where $\mathbb{N}$ is the set of all natural numbers. The notation $(x(n,m))$ or simply (x_{nm}) shall be used. The value of double sequence space $x = (x_{nm})$ at a point $(n,m) \in \mathbb{N}^2$ is a complex number x_{nm} and it is called the (n,m)-th term of x. So we can think of the elements x_{nm} of the double sequence (x_{nm}) as a table:

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} & \cdots \\ x_{21} & x_{22} & \cdots & x_{2n} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}$$

The double sequence $x = (x_{nm})$ converges to l in Pringsheim's sense or simply that (x_{nm}) is p-convergent to l if for any $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{nm} - l| < \varepsilon$ for all $n, m \geq n_0$ and it is denoted by ${}^{p}\lim x_{nm} = l$. The space of all convergent double sequences in the Pringsheim's sense is stated by C_p. A double sequence of complex numbers can have at most one

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limit in Pringsheim's sense. A double sequence (x_{nm}) is called bounded if there exists a real number $M > 0$ such that $|x_{nm}| \leq M$ for all $n, m \in \mathbb{N}$. It is known that sequences in the space C_p may not be bounded. If the double sequence $x = (x_{nm})$ converges in the sense of Pringsheim, and in addition, the limits $\lim_n x_{nm}$ and $\lim_m x_{nm}$ exist, then it is called regularly convergent and denoted by ${}^r\lim x_{nm} = l$. The space of all regularly convergent double sequences is denoted by C_r. From the definition, all regularly convergent double sequences are convergent in Pringsheim's sense. Also, every regularly convergent double sequence is bounded.

For a double sequence x_{nm}, the limits $\lim_n \left(\lim_m x_{nm}\right)$ and $\lim_m \left(\lim_n x_{nm}\right)$ are called iterated limits. Let $\lim_{nm} x_{nm} = a$; in this case,

i) $\lim_m \left(\lim_n x_{nm}\right) = a$ if and only if $\lim_n x_{nm}$ exists for all $m \in \mathbb{N}$.

ii) $\lim_n \left(\lim_m x_{nm}\right) = a$ if and only if $\lim_m x_{nm}$ exists for all $n \in \mathbb{N}$.

iii) The iterated limits $\lim_m \left(\lim_n x_{nm}\right)$ and $\lim_n \left(\lim_m x_{nm}\right)$ exist and both are equal to a if and only if $\lim_n x_{nm}$ exists for all $m \in \mathbb{N}$ and $\lim_m x_{nm}$ exists for all $n \in \mathbb{N}$.

A double sequence (x_{nm}) of complex numbers is called a Cauchy sequence if and only if for any $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{nm} - x_{pq}| < \varepsilon$ for all $n, m \geq n_0$ and $p, q \geq n_0$. A double sequence of complex numbers converges in Pringsheim's sense if and only if it is Cauchy sequence.

Let (x_{nm}) be a double sequence of complex numbers and let $(k_1, r_1) < (k_2, r_2) < \cdots < (k_n, r_n) < \cdots$ be a strictly increasing sequence of pairs of natural numbers. Then the sequence $\left(x_{k_n r_m}\right)$ is called a subsequence of (x_{nm}). If a double sequence (x_{nm}) of complex numbers is p-convergent to a complex number a, then any subsequence of (x_{nm}) is also p-convergent to a. If the iterated limits of a double sequence (x_{nm}) exit and $\lim_{n\to\infty} \left(\lim_{m\to\infty} x_{nm}\right) = \lim_{m\to\infty} \left(\lim_{n\to\infty} x_{nm}\right) = a$, then iterated limits for any subsequence $\left(x_{k_n r_m}\right)$ exist and $\lim_{n\to\infty} \left(\lim_{m\to\infty} x_{k_n r_m}\right) = \lim_{m\to\infty} \left(\lim_{n\to\infty} x_{k_n r_m}\right) = a$.

BICOMPLEX DOUBLE SEQUENCES

In this part we define the concept of a bicomplex double sequence with a new perspective on the concept of a double sequence and we give fundamentals of the theory of bicomplex double sequences. We first give the concept of a bicomplex double sequence, which is the basis of this paper:

Definition 1. The function $\zeta: \mathbb{N} \times \mathbb{N} \to \mathbb{BC}, (n, m) \to \zeta(n, m) = \zeta_{nm}$ is called a bicomplex double sequence.

Let (ζ_{nm}) be a bicomplex double sequence. Then there exist double sequences $(z_{nm}), (w_{nm})$ of complex numbers such that $\zeta_{nm} = z_{nm} + jw_{nm}$ for all $n, m \in \mathbb{N}$. On the other hand, the bicomplex number ζ_{nm} has the idempotent representation $\zeta_{nm} = s_{nm}e_1 + t_{nm}e_2$ where $s_{nm} = z_{nm} - iw_{nm}$, $t_{nm} = z_{nm} + iw_{nm}$ and also $(s_{nm}), (t_{nm})$ are double sequences of complex numbers.

Example 1. Consider the function

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$$\zeta:\mathbb{N}\times\mathbb{N}\to\mathbb{BC},(n,m)\to\zeta(n,m)=\zeta_{nm}=\begin{cases}(m-i)^2e_1+mie_2, \text{ if } n=1 \text{ for all } m\in\mathbb{N}\\ nie_1+n^2ie_2, \text{ if } n=m\\ nme_1+nmie_2, \text{ otherwise}\end{cases}.$$

So (ζ_{nm}) is a bicomplex double sequence. Also, the bicomplex number ζ_{nm} for all $n,m\in\mathbb{N}$ can be described by the idempotent representation $\zeta_{nm}=s_{nm}e_1+t_{nm}e_2$ where

$$s_{nm}=\begin{cases}(m-i)^2, & \text{if } n=1 \text{ for all } m\in\mathbb{N}\\ ni, & \text{if } n=m\\ nm, & \text{otherwise}\end{cases} \quad \text{and} \quad t_{nm}=nmi$$

for all $n,m\in\mathbb{N}$ and (s_{nm}) and (t_{nm}) are complex double sequences.

We continue with the concept of a subsequence of a bicomplex double sequence.

Definition 2. Let $\zeta:\mathbb{N}\times\mathbb{N}\to\mathbb{BC},(n,m)\to\zeta(n,m)=\zeta_{nm}$ be a bicomplex double sequence. Define $\eta:\mathbb{N}\times\mathbb{N}\to\mathbb{N}\times\mathbb{N},(n,m)\to\eta(n,m)=(\lambda_n,\mu_m)$ where $\lambda:\mathbb{N}\to\mathbb{N},n\to\lambda(n)=\lambda_n$ and $\mu:\mathbb{N}\to\mathbb{N},m\to\mu(m)=\mu_m$ are increasing functions. Then the function $\zeta\circ\eta:\mathbb{N}\times\mathbb{N}\to\mathbb{BC},(n,m)\to(\zeta\circ\eta)(n,m)=\zeta_{\lambda_n\mu_m}$ is called a subsequence of the bicomplex double sequence (ζ_{nm}).

For $\zeta_{nm}=s_{nm}e_1+t_{nm}e_2$ and $\zeta_{\lambda_n\mu_m}=s_{\lambda_n\mu_m}e_1+t_{\lambda_n\mu_m}e_2$, the bicomplex double sequence $\left(\zeta_{\lambda_n\mu_m}\right)$ is a subsequence of (ζ_{nm}) if and only if the complex double sequences $\left(s_{\lambda_n\mu_m}\right)$ and $\left(t_{\lambda_n\mu_m}\right)$ are subsequences of (s_{nm}) and (t_{nm}) respectively.

Example 2. Consider the bicomplex double sequence (ζ_{nm}) defined by

$$\zeta_{nm}=\frac{i}{n+m}e_1+\left[(-1)^m\left(\frac{1}{n}+\frac{1}{m}\right)+i\right]e_2$$

where $s_{nm}=\frac{i}{n+m}$ and $t_{nm}=(-1)^m\left(\frac{1}{n}+\frac{1}{m}\right)+i$ for all $n,m\in\mathbb{N}$. Define $\eta:\mathbb{N}\times\mathbb{N}\to\mathbb{N}\times\mathbb{N},(n,m)\to\eta(n,m)=(2n,m^3)$ where $\lambda:\mathbb{N}\to\mathbb{N},n\to\lambda(n)=2n$ and $\mu:\mathbb{N}\to\mathbb{N},m\to\mu(m)=m^3$. Then

$$\zeta\circ\eta:\mathbb{N}\times\mathbb{N}\to\mathbb{BC},(n,m)\to(\zeta\circ\eta)(n,m)=\zeta_{2n,m^3}=\frac{i}{2n+m^3}e_1+\left[(-1)^{m^3}\left(\frac{1}{2n}+\frac{1}{m^3}\right)+i\right]e_2$$

is a subsequence of (ζ_{nm}).

Some Fundamentals as per Hyperbolic Valued Norm $|.|_{\mathbb{k}}$

We begin with the bicomplex versions of convergence in Pringsheim's sense and regular convergence for double sequences and evince their relationships with each other as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ in analogy with some well-known results for classical double sequences.

Definition 3. Let (ζ_{nm}) be a bicomplex double sequence and $\zeta_0\in\mathbb{BC}$. If for every $0\prec\varepsilon\in\mathbb{D}$ there exists $n_0\in\mathbb{N}$ such that $|\zeta_{nm}-\zeta_0|_{\mathbb{k}}\prec\varepsilon$ for all $n,m\geq n_0$, then we say that (ζ_{nm}) converges to ζ_0 in Pringsheim's sense as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ or simply that it is ${}^{\mathbb{k}}p$-convergent to ζ_0 and denoted by ${}^{\mathbb{k}p}\lim\zeta_{nm}=\zeta_0$. The bicomplex number ζ_0 is called the Pringsheim bicomplex limit as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ or ${}^{\mathbb{k}}p$-limit of (ζ_{nm}). If no such $\zeta_0\in\mathbb{BC}$ exists, then (ζ_{nm}) is said to be a divergent bicomplex double sequence in Pringsheim's sense as per the $\mathbb{D}$-norm $|\cdot|_{\mathbb{k}}$ or simply ${}^{p}\mathbb{D}$-divergent.

Let ${}^{\mathbb{k}p}\lim\zeta_{nm}=\zeta_0$ and let $\zeta_0=s_0e_1+t_0e_2$, $\varepsilon=\varepsilon_1e_1+\varepsilon_2e_2$, where $s_0,t_0\in\mathbb{C}$ and $\varepsilon_1,\varepsilon_2\in\mathbb{R}^+$. Since we have

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$$\begin{aligned}|\zeta_{nm}-\zeta_0|_{\mathbb{k}} &= |(s_{nm}e_1+t_{nm}e_2)-(s_0e_1+t_0e_2)|_{\mathbb{k}}\\ &= |(s_{nm}-s_0)e_1+(t_{nm}-t_0)e_2|_{\mathbb{k}}\\ &= |s_{nm}-s_0|e_1+|t_{nm}-t_0|e_2,\end{aligned}$$

the statement $|\zeta_{nm}-\zeta_0|_{\mathbb{k}} \prec \varepsilon$ is equivalent to $|s_{nm}-s_0| < \varepsilon_1$ and $|t_{nm}-t_0| < \varepsilon_2$. So we can write that for every $\varepsilon_1, \varepsilon_2 > 0$ there exists $n_0 \in \mathbb{N}$ such that $|s_{nm}-s_0| < \varepsilon_1$ and $|t_{nm}-t_0| < \varepsilon_2$ for all $n, m \geq n_0$. This implies that (s_{nm}) and (t_{nm}) are p-convergent to s_0 and t_0 respectively. Consequently, the ${}^{\mathbb{k}}p$-convergence of bicomplex double sequence (ζ_{nm}) to ζ_0 is identical to the p-convergence of complex double sequences (s_{nm}) and (t_{nm}) to s_0 and t_0 respectively.

Example 3. Consider the bicomplex double sequence (ζ_{nm}) defined as in Example 2. For every $\varepsilon_1, \varepsilon_2 > 0$ if we choose $n_1 = \left[\left|\frac{1}{\varepsilon_1}\right|\right]+1, n_2 = \left[\left|\frac{2}{\varepsilon_2}\right|\right]+1 \in \mathbb{N}$, we get

$$\left|\frac{i}{n+m}-0\right| = \frac{1}{n+m} < \frac{1}{n} \leq \frac{1}{n_1} < \varepsilon_1$$

for all $n, m \geq n_1$ and

$$\left|\left((-1)^m\left(\frac{1}{n}+\frac{1}{m}\right)+i\right)-i\right| = \frac{1}{n}+\frac{1}{m} \leq \frac{1}{n_2}+\frac{1}{n_2} = \frac{2}{n_2} < \varepsilon_2$$

for all $n, m \geq n_2$. This implies that the complex double sequences (s_{nm}) and (t_{nm}) are p-convergent to the complex numbers 0 and i. Therefore, the bicomplex double sequence (ζ_{nm}) is ${}^{\mathbb{k}}p$-convergent to $0e_1 + ie_2 = \frac{i+j}{2}$.

Example 4. Consider the bicomplex double sequence (ζ_{nm}) defined by

$$\zeta_{nm} = (-1)^{n+2m}ie_1 + \left(\frac{1}{n}+\frac{1}{m}+i\right)e_2$$

where $s_{nm} = (-1)^{n+2m}i$ and $t_{nm} = \frac{1}{n}+\frac{1}{m}+i$ for all $n, m \in \mathbb{N}$. Since the complex double sequence (s_{nm}) does not converge to any bicomplex number s_0 in Pringsheim's sense, the bicomplex double sequence (ζ_{nm}) is ${}^{p}\mathbb{D}$-divergent.

The next theorem is a bicomplex version of the uniqueness of p-limits for complex double sequences as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Theorem 1. A bicomplex double sequence can have at most one Pringsheim bicomplex limit as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ or ${}^{\mathbb{k}}p$-limit.

Proof. Let (ζ_{nm}) be a bicomplex double sequence and let $\zeta_1 = s_1e_1 + t_1e_2, \zeta_2 = s_2e_1 + t_2e_2$ be ${}^{\mathbb{k}}p$-limits. Then s_1 and s_2 are Pringsheim limits of complex double sequence (s_{nm}). So by uniqueness of Pringsheim limits of complex double sequences we get $s_1 = s_2$. Similarly, we get $t_1 = t_2$ for complex double sequence (t_{nm}). Thus, we write $\zeta_1 = \zeta_2$ implying that (ζ_{nm}) can have at most one ${}^{\mathbb{k}}p$-limit.

Definition 4. Let (ζ_{nm}) be a bicomplex double sequence. If for every $0 \prec \varepsilon \in \mathbb{D}$ there exists $n_0, m_0 \in \mathbb{N}$ such that $|\zeta_{nm}|_{\mathbb{k}} \succ \varepsilon$ for all $n \geq n_0$ and for all $m \geq m_0$, then we say that (ζ_{nm}) diverges to ∞ in Pringsheim's sense as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ and denote it by ${}^{\mathbb{k}p}\lim \zeta_{nm} = \infty$.

The case ${}^{\mathbb{k}p}\lim \zeta_{nm} = \infty$ is stated by ${}^{p}\mathbb{D}$-divergence of (ζ_{nm}) to ∞. The ${}^{p}\mathbb{D}$-divergence of bicomplex double sequence (ζ_{nm}) to ∞ is equivalent to the divergence of real double sequences $(|s_{nm}|)$ and $(|t_{nm}|)$ to ∞ in Pringsheim's sense.

Example 5. Consider the bicomplex double sequence (ζ_{nm}) defined by

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$$\zeta_{nm} = (n^2 + im)e_1 + (nm + 3i)e_2$$

where $s_{nm} = n^2 + im$ and $t_{nm} = nm + 3i$ for all $n, m \in \mathbb{N}$. For every $\varepsilon_1, \varepsilon_2 > 0$ if we choose $n_1 = \left[\left|\frac{\varepsilon_1}{\sqrt{2}}\right|\right] + 1, n_2 = [|\sqrt{\varepsilon_2}|] + 1 \in \mathbb{N}$, we get

$$|n^2 + im| = \sqrt{n^4 + m^2} \geq \sqrt{n_1^4 + n_1^2} \geq \sqrt{2n_1^2} = \sqrt{2}n_1 > \varepsilon_1$$

for all $n, m \geq n_1$ and

$$|nm + 3i| = \sqrt{(nm)^2 + 9} > \sqrt{(nm)^2} = nm \geq n_2^2 > \varepsilon_2$$

for all $n, m \geq n_2$. So since the real double sequences $(|s_{nm}|)$ and $(|t_{nm}|)$ diverge to ∞ in Pringsheim's sense, we can write ${}^{\mathbb{k}p}\lim \zeta_{nm} = \infty$.

Example 6. Consider the bicomplex double sequence (ζ_{nm}) as defined by

$$\zeta_{nm} = \frac{-n-m}{2} ie_1 + (-nm + i)e_2$$

where $s_{nm} = \frac{-n-m}{2} i$ and $t_{nm} = -nm + i$ for all $n, m \in \mathbb{N}$. For every $\varepsilon_1, \varepsilon_2 > 0$ if we choose $n_1 = [|\varepsilon_1|] + 1, n_2 = [|\sqrt{\varepsilon_2}|] + 1 \in \mathbb{N}$, we get

$$\left|\frac{-n-m}{2} i\right| = \frac{n+m}{2} \geq n_1 > \varepsilon_1$$

for all $n, m \geq n_1$ and

$$|-nm + i| = \sqrt{(nm)^2 + 1} > n_2^2 > \varepsilon_2$$

for all $n, m \geq n_2$. Thus, since the real double sequences $(|s_{nm}|)$ and $(|t_{nm}|)$ diverge to ∞ in Pringsheim's sense, we can write ${}^{\mathbb{k}p}\lim \zeta_{nm} = \infty$ **.**

Definition 5. Let (ζ_{nm}) be a bicomplex double sequence, $\zeta_0 \in \mathbb{BC}$ and ${}^{\mathbb{k}p}\lim \zeta_{nm} = \zeta_0$. If (ζ_{nm}) is a single bicomplex convergent sequence as per each index with the $\mathbb{D}$-norm $|\cdot|_{\mathbb{k}}$, i.e. there exist $a_m, b_n \in \mathbb{BC}$ such that ${}^{\mathbb{k}}\lim_{n\to\infty} \zeta_{nm} = a_m$ for all $m \in \mathbb{N}$ and ${}^{\mathbb{k}}\lim_{m\to\infty} \zeta_{nm} = b_n$ for all $n \in \mathbb{N}$, then we say that (ζ_{nm}) regularly converges to ζ_0 as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ or simply (ζ_{nm}) is ${}^{\mathbb{k}}r$-convergent to ζ_0 and denote it by ${}^{\mathbb{k}r}\lim \zeta_{nm} = \zeta_0$. The bicomplex number ζ_0 is called the regular bicomplex limit as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ or ${}^{\mathbb{k}}r$-limit of (ζ_{nm}).

Let $a_m = a_{m1}e_1 + a_{m2}e_2$ for all $m \in \mathbb{N}$ and $b_n = b_{n1}e_1 + b_{n2}e_2$ for all $n \in \mathbb{N}$. Thus, ${}^{\mathbb{k}}\lim_{n\to\infty} \zeta_{nm} = a_m$ if and only if $\lim_{n\to\infty} s_{nm} = a_{m1}$ and $\lim_{n\to\infty} t_{nm} = a_{m2}$, and also, ${}^{\mathbb{k}}\lim_{m\to\infty} \zeta_{nm} = b_n$ if and only if $\lim_{m\to\infty} s_{nm} = b_{n1}$ and $\lim_{m\to\infty} t_{nm} = b_{n2}$. Therefore, the ${}^{\mathbb{k}}r$-convergence of bicomplex double sequence (ζ_{nm}) to ζ_0 is equivalent to the regular convergence of complex double sequences (s_{nm}) and (t_{nm}) to s_0 and t_0 respectively.

Remark 1. We draw attention to the fact that if (ζ_{nm}) is a bicomplex double sequence and it is ${}^{\mathbb{k}}r$-convergent to ζ_0, it is ${}^{\mathbb{k}}p$-convergent to ζ_0. However, the converse is not always true as seen in the following example. Because of this relationship, it is of great importance to study convergence in Pringsheim's sense.

Example 7. Consider the bicomplex double sequence (ζ_{nm}) defined as in Example 2. Since

$$\lim_{n\to\infty} s_{nm} = \lim_{n\to\infty} \frac{i}{n+m} = 0 = a_{m1}$$

and

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$$\lim_{n\to\infty} t_{nm} = \lim_{n\to\infty}\left[(-1)^m\left(\frac{1}{n}+\frac{1}{m}\right)+i\right] = \frac{(-1)^m}{m}+i = a_{m2},$$

we have

$$^{\mathbb{k}}\lim_{n\longrightarrow\infty} \zeta_{nm} = 0e_1 + \left(\frac{(-1)^m}{m}+i\right)e_2 = a_m$$

for all $m \in \mathbb{N}$. But since

$$\lim_{m\to\infty} t_{nm} = \lim_{m\to\infty}\left[(-1)^m\left(\frac{1}{n}+\frac{1}{m}\right)+i\right]$$

does not exist for any $n \in \mathbb{N}$, there is no b_n for any $n \in \mathbb{N}$ such that $^{\mathbb{k}}\lim_{m\longrightarrow\infty} \zeta_{nm} = b_n$. So although (ζ_{nm}) is $^{\mathbb{k}}p$-convergent to $\frac{i+j}{2}$, it is not $^{\mathbb{k}}r$-convergent to $\frac{i+j}{2}$.

We are ready to define the boundedness of bicomplex double sequences and examine the relationship between two types of convergence given above and as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Definition 6. Let (ζ_{nm}) be a bicomplex double sequence. If there exists $0 \prec M \in \mathbb{D}$ such that $|\zeta_{nm}|_{\mathbb{k}} \precsim M$ for all $n, m \in \mathbb{N}$, then (ζ_{nm}) is said to be bounded as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

The boundedness of a bicomplex double sequence (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ is equivalent to the boundedness of complex double sequences (s_{nm}) and (t_{nm}). Indeed, there exist $M_1, M_2 \in \mathbb{R}^+$such that $|s_{nm}| \leq M_1$ and $|t_{nm}| \leq M_2$ for all $n, m \in \mathbb{N}$ if and only if there exists $0 \prec M \in \mathbb{D}$ such that $|\zeta_{nm}|_{\mathbb{k}} \precsim M$ for all $n, m \in \mathbb{N}$ where $M = M_1e_1 + M_2e_2$.

Remark 2. We note that a bounded bicomplex double sequence need not be $^{\mathbb{k}}p$-convergent. The following example simply explains it.

Example 8. Consider the $^{p}\mathbb{D}$-divergent bicomplex double sequence (ζ_{nm}) defined as in Example 4. Since

$$|s_{nm}| = |(-1)^{n+2m}i| = 1$$

and

$$|t_{nm}| = \left|\frac{1}{n}+\frac{1}{m}+i\right| = \sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)^2+1} \leq 5,$$

complex double sequences (s_{nm}) and (t_{nm}) are bounded. So if we choose $M = 1e_1 + 5e_2$, it follows that $|\zeta_{nm}|_{\mathbb{k}} \precsim M$ for all $n, m \in \mathbb{N}$. This means that (ζ_{nm}) is bounded as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Remark 3. Notice that the statement "Every convergent bicomplex sequence is bounded", which is valid for single bicomplex sequences, is not always true for bicomplex double sequences. The following example emphasises that $^{\mathbb{k}}p$-convergent bicomplex double sequences may not be bounded as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Example 9. Consider the bicomplex double sequence (ζ_{nm}) defined by

$$\zeta_{nm} = \begin{cases} nie_1 + \dfrac{i}{n}e_2, \text{ if } m = 1 \\ -2me_1 + \dfrac{i}{m}e_2, \text{ if } n = 1. \\ \dfrac{i}{n+m}e_2, \text{ otherwise} \end{cases}$$

We can write $\zeta_{nm} = s_{nm}e_1 + t_{nm}e_2$, where

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$$s_{nm} = \begin{cases} ni, \text{ if } m = 1 \\ -2m, \text{ if } n = 1 \\ 0, \text{ otherwise} \end{cases}$$

and

$$t_{nm} = \frac{i}{n+m}$$

for all $n, m \in \mathbb{N}$. It can be easily shown that complex double sequences (s_{nm}) and (t_{nm}) converge to the complex number 0 in Pringsheim's sense. However, (ζ_{nm}) is not bounded as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ since (s_{nm}) is not bounded even though (t_{nm}) is bounded.

Theorem 2. Every ${}^{\mathbb{k}}r$-convergent bicomplex double sequence is bounded as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Proof. Let (ζ_{nm}) be a ${}^{\mathbb{k}}r$-convergent bicomplex double sequence. Then complex double sequences (s_{nm}) and (t_{nm}) are regularly convergent to s_0 and t_0 respectively. So they are bounded. This guarantees the boundedness of the bicomplex double sequence (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$. The proof is completed.

Remark 4. We see from Example 7 and Example 8 that any bounded bicomplex double sequence need not be ${}^{\mathbb{k}}r$-convergent.

Now we define a new concept as follows.

Definition 7. Let (ζ_{nm}) be a bicomplex double sequence. If for every $0 \prec \varepsilon \in \mathbb{D}$ there exists $n_0 \in \mathbb{N}$ such that $\left|\zeta_{nm} - \zeta_{pq}\right|_{\mathbb{k}} \prec \varepsilon$ for all $n, m, p, q \geq n_0$, then we say that (ζ_{nm}) is a Cauchy sequence in Pringsheim's sense as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ or or simply ${}^{\mathbb{k}}p$-Cauchy sequence.

Since we have $\left|\zeta_{nm} - \zeta_{pq}\right|_{\mathbb{k}} = \left|s_{nm} - s_{pq}\right|e_1 + \left|t_{nm} - t_{pq}\right|e_2$, the statement $\left|\zeta_{nm} - \zeta_{pq}\right|_{\mathbb{k}} \prec \varepsilon$ is equivalent to $\left|s_{nm} - s_{pq}\right| < \varepsilon_1$ and $\left|t_{nm} - t_{pq}\right| < \varepsilon_2$. Thus, the bicomplex double sequence (ζ_{nm}) is a ${}^{\mathbb{k}}p$-Cauchy sequence if and only if the complex double sequences (s_{nm}) and (t_{nm}) are Cauchy sequences in Pringsheim's sense.

The following theorem reveals the relationship between definitions of convergent sequence and Cauchy sequence in Pringsheim's sense as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Theorem 3. A bicomplex double sequence is ${}^{\mathbb{k}}p$-convergent if and only if it is a ${}^{\mathbb{k}}p$-Cauchy sequence.

Proof. Let (ζ_{nm}) be a ${}^{\mathbb{k}}p$-convergent bicomplex double sequence. Then complex double sequences (s_{nm}) and (t_{nm}) are p-convergent. So (s_{nm}) and (t_{nm}) are Cauchy sequences in Pringsheim's sense. This implies that (ζ_{nm}) is a ${}^{\mathbb{k}}p$-Cauchy sequence.

On the contrary, let (ζ_{nm}) be a ${}^{\mathbb{k}}p$-Cauchy sequence. So (s_{nm}) and (t_{nm}) are Cauchy sequences in Pringsheim's sense. Thus, since (s_{nm}) and (t_{nm}) are p-convergent, we say that (ζ_{nm}) is a ${}^{\mathbb{k}}p$-convergent bicomplex double sequence, which proves the claim.

Example 10. In Example 3 we derived that the bicomplex double sequence (ζ_{nm}) is ${}^{\mathbb{k}}p$-convergent to $\frac{i+j}{2}$. So it is a ${}^{\mathbb{k}}p$-Cauchy sequence by Theorem 3. On the other hand, in Example 4 we showed that the bicomplex double sequence (ζ_{nm}) is ${}^{p}\mathbb{D}$-divergent. Thus, it is not a ${}^{\mathbb{k}}p$-Cauchy sequence by Theorem 3.

We shall introduce bicomplex iterated limits as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

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Definition 8. Let (ζ_{nm}) be a bicomplex double sequence. The limits $\lim_{n\to\infty}\left(\lim_{m\to\infty}\zeta_{nm}\right)$ and $\lim_{m\to\infty}\left(\lim_{n\to\infty}\zeta_{nm}\right)$ are called bicomplex iterated limits of (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$. It is clear that

$$^{\mathbb{k}}\lim_{n\to\infty}\left(^{\mathbb{k}}\lim_{m\to\infty}\zeta_{nm}\right)=\left[\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{nm}\right)\right]e_1+\left[\lim_{n\to\infty}\left(\lim_{m\to\infty}t_{nm}\right)\right]e_2$$

and

$$^{\mathbb{k}}\lim_{m\to\infty}\left(^{\mathbb{k}}\lim_{n\to\infty}\zeta_{nm}\right)=\left[\lim_{m\to\infty}\left(\lim_{n\to\infty}s_{nm}\right)\right]e_1+\left[\lim_{m\to\infty}\left(\lim_{n\to\infty}t_{nm}\right)\right]e_2$$

where $\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{nm}\right)$ and $\lim_{m\to\infty}\left(\lim_{n\to\infty}s_{nm}\right)$ are iterated limits of (s_{nm}) and also, $\lim_{n\to\infty}\left(\lim_{m\to\infty}t_{nm}\right)$ and $\lim_{m\to\infty}\left(\lim_{n\to\infty}t_{nm}\right)$ are iterated limits of (t_{nm}).

Example 11. 1) Consider the bicomplex double sequence (ζ_{nm}) defined as in Example 2. Then we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{nm}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{i}{n+m}\right)=\lim_{n\to\infty}0=0$$
$$\lim_{m\to\infty}\left(\lim_{n\to\infty}s_{nm}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{i}{n+m}\right)=\lim_{m\to\infty}0=0$$

and

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}t_{nm}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left((-1)^m\left(\tfrac{1}{n}+\tfrac{1}{m}\right)+i\right)\right)\text{ does not exist,}$$
$$\lim_{m\to\infty}\left(\lim_{n\to\infty}t_{nm}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}\left((-1)^m\left(\tfrac{1}{n}+\tfrac{1}{m}\right)+i\right)\right)=\lim_{m\to\infty}\left(\tfrac{(-1)^m}{m}+i\right)=i.$$

Thus, $^{\mathbb{k}}\lim_{n\to\infty}\left(^{\mathbb{k}}\lim_{m\to\infty}\zeta_{nm}\right)$ does not exist and $^{\mathbb{k}}\lim_{m\to\infty}\left(^{\mathbb{k}}\lim_{n\to\infty}\zeta_{nm}\right)=0e_1+ie_2=\frac{i+j}{2}$.

2) Consider the bicomplex double sequence (ζ_{nm}) defined as in Example 4. Then we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{nm}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}(-1)^{n+2m}i\right)=\lim_{n\to\infty}((-1)^{n+2m}i)\text{ does not exist ,}$$
$$\lim_{m\to\infty}\left(\lim_{n\to\infty}s_{nm}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}(-1)^{n+2m}i\right)\text{ does not exist, i.e. the sequence }((-1)^n i)$$

does not converge to any finite value,
and

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}t_{nm}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{1}{n}+\frac{1}{m}+i\right)\right)=\lim_{n\to\infty}\left(\frac{1}{n}+i\right)=i$$
$$\lim_{m\to\infty}\left(\lim_{n\to\infty}t_{nm}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{m}+i\right)\right)=\lim_{m\to\infty}\left(\frac{1}{m}+i\right)=i$$

Thus, there are no bicomplex iterated limits of (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

3) Consider the bicomplex double sequence (ζ_{nm}) defined by $\zeta_{nm}=\frac{n+5mi}{2n+3m}e_1+\left(\frac{1}{n}+\frac{i}{m}\right)e_2$, where $s_{nm}=\frac{n+5mi}{2n+3m}$ and $t_{nm}=\frac{1}{n}+\frac{i}{m}$ for all $n,m\in\mathbb{N}$. Then we have

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{nm}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{n+5mi}{2n+3m}\right)=\lim_{n\to\infty}\frac{5i}{3}=\frac{5i}{3}$$
$$\lim_{m\to\infty}\left(\lim_{n\to\infty}s_{nm}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{n+5mi}{2n+3m}\right)=\lim_{m\to\infty}\frac{1}{2}=\frac{1}{2}$$

and

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$$\lim_{n\to\infty}\left(\lim_{m\to\infty} t_{nm}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\left(\frac{1}{n}+\frac{i+m}{m}\right)\right)=\lim_{n\to\infty}\left(\frac{1}{n}+1\right)=1$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} t_{nm}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\frac{1}{n}+\frac{i+m}{m}\right)\right)=\lim_{m\to\infty}\frac{i+m}{m}=1$$

Thus, ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\zeta_{nm}\right)=\frac{5i}{3}e_1+1e_2$ and ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}\right)=\frac{1}{2}e_1+1e_2$ are bicomplex iterated limits of (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Theorem 4. Let (ζ_{nm}) be a bicomplex double sequence and ${}^{\mathbb{k}p}\lim\zeta_{nm}=\zeta_{0.}$ Then ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}\right)=\zeta_0$ if and only if there exists a bicomplex sequence (a_m) such that ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}=a_m$ for each $m\in\mathbb{N}$.

Proof. Let (ζ_{nm}) be a bicomplex double sequence and ${}^{\mathbb{k}p}\lim\zeta_{nm}=\zeta_0$. Suppose that ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}\right)=\zeta_0.$. Then we have $\lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty}s_{nm}\right)=s_0$ and $\lim\limits_{n\to\infty}\left(\lim\limits_{m\to\infty}t_{nm}\right)=t_0$. So there exist complex sequences (a_{m1}) and (a_{m2}) such that $\lim\limits_{n\to\infty}s_{nm}=a_{m1}$ and $\lim\limits_{n\to\infty}t_{nm}=a_{m2}$ for each $m\in\mathbb{N}$. This implies that ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}=\left(\lim\limits_{n\to\infty}s_{nm}\right)e_1+\left(\lim\limits_{n\to\infty}t_{nm}\right)e_2=a_{m1}e_1+a_{m2}e_2=a_m$ as desired.

Conversely, assume that there exists a bicomplex sequence (a_m) such that ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}=a_m$ for each $m\in\mathbb{N}$. In this case there exist complex sequences (a_{m1}) and (a_{m2}) such that $\lim\limits_{n\to\infty}s_{nm}=a_{m1}$ and $\lim\limits_{n\to\infty}t_{nm}=a_{m2}$ for each $m\in\mathbb{N}$. Therefore, it follows that $\lim\limits_{m\to\infty}\left(\lim\limits_{n\to\infty}s_{nm}\right)=s_0$ and $\lim\limits_{n\to\infty}\left(\lim\limits_{m\to\infty}t_{nm}\right)=t_0$, which implies ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}\right)=\zeta_0$. The proof is then completed.

Theorem 5. Let (ζ_{nm}) be a bicomplex double sequence and ${}^{\mathbb{k}p}\lim\zeta_{nm}=\zeta_0$. Then ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\zeta_{nm}\right)=\zeta_0$ if and only if there exists a bicomplex sequence (b_n) such that ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\zeta_{nm}=b_n$ for each $n\in\mathbb{N}$.

Proof. The proof is obtained in a similar manner as that of Theorem 4.

Corollary 1. Let (ζ_{nm}) be a bicomplex double sequence and ${}^{\mathbb{k}p}\lim\zeta_{nm}=\zeta_0$. Then ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}\right)={}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\zeta_{nm}\right)=\zeta_0$ if and only if there exist bicomplex sequences (a_m) and (b_n) such that ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}=a_m$ for each $m\in\mathbb{N}$ and ${}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\zeta_{nm}=b_n$ for each $n\in\mathbb{N}$.

Now we focus on properties of subsequences of bicomplex double sequences as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$.

Theorem 6. If a bicomplex double sequence is ${}^{\mathbb{k}}p$-convergent, then every subsequence of it is ${}^{\mathbb{k}}p$-convergent.

Proof. Let (ζ_{nm}) be a ${}^{\mathbb{k}}p$-convergent bicomplex double sequence. Then complex double sequences (s_{nm}) and (t_{nm}) are p-convergent. So every subsequence of (s_{nm}) and (t_{nm}) is p-convergent. This means that every subsequence of (ζ_{nm}) is ${}^{\mathbb{k}}p$-convergent. The proof is completed.

Theorem 7. If bicomplex iterated limits of (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ exist and ${}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\zeta_{nm}\right)={}^{\mathbb{k}}\lim\limits_{m\longrightarrow\infty}\left({}^{\mathbb{k}}\lim\limits_{n\longrightarrow\infty}\zeta_{nm}\right)=\zeta_0$, then bicomplex iterated limits as per the $\mathbb{D}$-norm

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$|.|_{\mathbb{k}}$for any subsequence $(\zeta_{\lambda_n\mu_m})$ of (ζ_{nm}) exist and $^{\mathbb{k}}\lim_{n\to\infty}\left(^{\mathbb{k}}\lim_{m\to\infty}\zeta_{\lambda_n\mu_m}\right) =$ $^{\mathbb{k}}\lim_{m\to\infty}\left(^{\mathbb{k}}\lim_{n\to\infty}\zeta_{\lambda_n\mu_m}\right) = \zeta_0$.

Proof. Suppose that bicomplex iterated limits of (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ exist and $^{\mathbb{k}}\lim_{n\to\infty}\left(^{\mathbb{k}}\lim_{m\to\infty}\zeta_{nm}\right) = {}^{\mathbb{k}}\lim_{m\to\infty}\left(^{\mathbb{k}}\lim_{n\to\infty}\zeta_{nm}\right) = \zeta_0$. Then iterated limits of (s_{nm}) and (t_{nm}) exist and satisfy the equalities $\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{nm}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}s_{nm}\right) = s_0$ and $\lim_{n\to\infty}\left(\lim_{m\to\infty}t_{nm}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}t_{nm}\right) = t_0$. This shows that iterated limits for all subsequences $(s_{\lambda_n\mu_m})$ and $(t_{\lambda_n\mu_m})$ of (s_{nm}) and (t_{nm}) exist and $\lim_{n\to\infty}\left(\lim_{m\to\infty}s_{\lambda_n\mu_m}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}s_{\lambda_n\mu_m}\right) = s_0$ and $\lim_{n\to\infty}\left(\lim_{m\to\infty}t_{\lambda_n\mu_m}\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty}t_{\lambda_n\mu_m}\right) = t_0$. Therefore, bicomplex iterated limits for any subsequence $(\zeta_{\lambda_n\mu_m})$ of (ζ_{nm}) as per the $\mathbb{D}$-norm $|.|_{\mathbb{k}}$ exist and $^{\mathbb{k}}\lim_{n\to\infty}\left(^{\mathbb{k}}\lim_{m\to\infty}\zeta_{\lambda_n\mu_m}\right) = {}^{\mathbb{k}}\lim_{m\to\infty}\left(^{\mathbb{k}}\lim_{n\to\infty}\zeta_{\lambda_n\mu_m}\right) = \zeta_0$. This completes the proof.

CONCLUSIONS

We have established bicomplex double sequences as new versions of double sequences. Also, we construct some of their important properties as per the hyperbolic valued norm $|.|_{\mathbb{k}}$ that lay the groundwork for bicomplex double sequence spaces. Our future work will focus on the definitions of the bicomplex double sequence spaces $\mathcal{L}_{\infty}^{\mathbb{k}}(\mathbb{BC})$, $C^{\mathbb{k}}(\mathbb{BC})$, $C_0^{\mathbb{k}}(\mathbb{BC})$ and $\mathcal{L}_q^{\mathbb{k}}(\mathbb{BC})$ of bounded, convergent, null and q-absolutely summable bicomplex sequences by using hyperbolic valued norm $|.|_{\mathbb{k}}$ and examining their geometric properties. Since the theory of double sequences is extremely active and has extensive applications, we believe that our newly obtained results will be used by many researchers for further work and applications to other related areas.

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