

Full Paper

Exploring soft super metric spaces and their fixed soft points

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Abstract: This paper introduces and examines the concept of soft super metric spaces, which combines the framework of soft set theory with the properties of super metric spaces. We establish fundamental definitions and properties of soft super metric spaces, and present several fixed point theorems for self-soft mappings on complete soft super metric spaces. The main results include theorems establishing the existence and uniqueness of fixed soft points for soft contraction mappings and surjective soft mappings. Additionally, we prove fixed point theorems for weakly compatible mappings under various conditions. These results extend classical fixed point theory to accommodate uncertainty and parameterisation through the soft set framework.

Keywords: soft super metric spaces, fixed point theorems, weakly compatible mappings

INTRODUCTION

Metric spaces form one of the foundational concepts in mathematical analysis, providing a structured way to study distances and their properties. Introduced in the late 19th century, metric spaces have become a central framework for understanding convergence, continuity and fixed point theory. These spaces serve as the foundation for a wide array of mathematical theories and applications including geometry, functional analysis and computer science, making them a versatile and essential concept in mathematics. In parallel, the theory of soft set has also emerged as a powerful tool for handling uncertainty, providing a flexible and comprehensive framework for various mathematical and applied disciplines. Since its introduction by Molodtsov in 1999 [1], the soft set theory has undergone extensive development and has been applied in numerous fields including decision-making, data analysis and optimisation. Building on this foundation, researchers have explored the integration of soft sets with other mathematical structures, giving rise to soft topological spaces [2], soft metric spaces[3] and their generalisations [4-14]. Fixed point theory, which investigates the conditions under which mappings admit invariant points, has also been integrated with soft set theory, driving advances in the analysis of uncertain and parameterised

systems. Recent studies have expanded the scope of fixed point theory within the context of soft sets [15-20]. The concept of fixed soft points was introduced, providing fundamental characterisations linked to infra soft compact spaces [21]. The study of soft b-metric spaces and soft G-metric spaces has led to the development of several fixed point theorems for these new domains [22-26]. Additionally, the investigation of fixed soft points in soft parametric metric spaces, soft S-metric spaces and parametric soft S-metric spaces has been explored by many researchers [27-30]. The concept of soft super-space of soft metric spaces with soft points, extending the soft distance function with examples to support future research and practical applications, has been introduced by Kumawat and Gupta [31].

Super metric spaces, as a generalisation of classical metric spaces, have been widely studied due to their extensive applicability in fixed point theory and analysis [32]. The extension of this concept to the soft set framework paves the way for addressing more complex systems characterised by uncertainty and multi-parameter settings. An investigation was conducted into Meir–Keeler and Ćirić type contractions within the framework of super metric spaces, a generalisation of classical metric spaces [33]. The primary focus was to establish the existence and uniqueness of results for fixed points of such operators in this broader setting. The rational form of contractions within the framework of super metric spaces, an extension of classical metric spaces, has been explored by Karapinar and Fulga [34].

In this context soft super metric spaces offer a natural and versatile generalisation as they combine the structural richness of super metric spaces with the flexibility and adaptability of soft set theory. This synthesis not only extends the scope of classical fixed point theory but also opens new avenues for addressing problems characterised by uncertainty and vagueness, where traditional metric approaches may fall short. To the best of our knowledge, there has been no prior investigation in the literature devoted to the study of soft super metrics. Therefore, the present work constitutes the first attempt in this direction, establishing a foundational framework that can serve as a basis for further theoretical development and practical applications. It is anticipated that this study will stimulate subsequent research on fixed point results and related concepts within soft super metric spaces, thereby enriching both the theory of generalised metric spaces and their interdisciplinary applications. Specifically, this study aims to investigate the properties of soft super metric spaces with a particular focus on the existence and uniqueness of fixed soft points for self-soft mappings defined on complete soft super metric spaces. The concept of a fixed point plays a fundamental role in various branches of mathematics and its applications, making this an important area of research.

PRELIMINARIES

For the sake of this discussion, we consider that X is a non-empty universal set. Similarly, we take E as a non-empty set of all parameters and $SP(X, E)$ as a collection of all soft points of X . Furthermore, $\mathbb{R}(E)^*$ is a collection of non-negative soft real numbers.

Definition 1 [1]. A pair $(\wp, E)$ is deemed a soft set over the universal set on X if and only if the mapping function $\wp: E \rightarrow P(X)$, where $P(X)$ denotes the set of all subsets of X , assigns each element of E a subset of X . In this context a soft set can be conceived as a parametrised family of subsets of X . The element e of E , when paired with the mapping $\wp$, yields a set in X which is an e -approximate element of the given soft set $(\wp, E)$ for each e in E .

Definition 2 [35]. A soft set $(\wp, E)$ is called a null soft set, denoted by Φ , if $\wp(e) = \emptyset$ for all $e \in E$.

Definition 3 [36, 37]. The soft set $(\wp, E)$ is referred to as a soft point, denoted by (x_e, E) , provided that for the element $e \in E$, $\wp(e) = \{x\}$ and $\wp(e') = \emptyset$ for all $e' \in E - \{e\}$. This is briefly denoted by x_e .

Definition 4 [37]. A soft real set $(\wp, E)$ is characterised by $\wp: E \rightarrow B(\mathbb{R})$, where $\mathbb{R}$ is the set of all real numbers and $B(\mathbb{R})$ is the collection of all non-empty bounded subsets of $\mathbb{R}$. A soft real set $(\wp, E)$ is a singleton soft set, whereby it is referred to as a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$. Besides, the constant soft real numbers are denoted by $\tilde{r}, \tilde{s}, \tilde{t}$. It is evident that $\tilde{0}(e) = 0$, $\tilde{1}(e) = 1$ for all $e \in E$.

Definition 5 [36, 37]. Two soft points $x_e, y_{e'}$ are considered equal if and only if both the index sets e and e' are identical and corresponding elements x and y are also equal. Consequently, two soft points $x_e, y_{e'}$ are distinct and denoted as $x_e \neq y_{e'}$ if and only if both $x \neq y$ or $e \neq e'$.

Definition 6 [32]. Let Ω be a non-empty set. A function $m: \Omega \times \Omega \rightarrow [0, \infty)$ is called a super metric on Ω if it satisfies the following conditions:

1. If $m(x, y) = 0$, then $x = y$ for all $x, y \in \Omega$;
2. $m(x, y) = m(y, x)$ for all $x, y \in \Omega$;
3. There exists a constant $s \geq 1$ such that for every $y \in \Omega$, there exist distinct sequences $\{x_n\}$ and $\{y_n\} \subset \Omega$ with $\lim_{n \rightarrow \infty} m(x_n, y_n) = 0$, and

$$\limsup_{n \rightarrow \infty} m(y_n, y) \leq s \limsup_{n \rightarrow \infty} m(x_n, y).$$

When these conditions are met, the pair (Ω, m) is referred to as a super metric space.

MAIN RESULTS

The present section is concerned with the study of soft super metric spaces as a generalisation of super metric space. It also considers the existence and uniqueness of fixed soft points of self-soft mappings on a complete soft super metric space.

Definition 7. A mapping $s_m: SP(X, E) \times SP(X, E) \rightarrow \mathbb{R}(E)^*$ is called a soft super metric on $SP(X, E)$ if it satisfies the following axioms:

1. If $s_m(x_e, y_{e'}) = \tilde{0}$, then $x_e = y_{e'}$ for all $x_e, y_{e'} \in SP(X, E)$;
2. $s_m(x_e, y_{e'}) = s_m(y_{e'}, x_e)$ for all $x_e, y_{e'} \in SP(X, E)$;
3. There exists a soft real number $\tilde{s} \geq \tilde{1}$ such that for every $y_{e'} \in SP(X, E)$, there exist distinct soft sequences $\{x_{e_n}^n\}$ and $\{y_{e_n}^n\} \subset SP(X, E)$ with $\lim_{n \rightarrow \infty} s_m(x_{e_n}^n, y_{e_n}^n) = \tilde{0}$, and

$$\limsup_{n \rightarrow \infty} s_m(y_{e_n}^n, y_{e'}) \leq \tilde{s} \limsup_{n \rightarrow \infty} s_m(x_{e_n}^n, y_{e'}).$$

When these conditions are met, the pair $(SP(X, E), s_m)$ is referred to as a soft super metric space.

Example 1. Let $X = [0, \infty)$ and $E = [0, 1]$ be a parameter set. Define the soft super metric $s_m: SP(X, E) \times SP(X, E) \rightarrow \mathbb{R}(E)^*$ by

$$s_m(x_e, y_{e'}) = \begin{cases} \frac{x+y}{1+x+y} + \frac{e+e'}{1+e+e'}, & \text{if } x \neq y, x \neq 0, y \neq 0, e \neq 0, e' \neq 0 \\ \tilde{0}, & \text{if } x_e = y_{e'} \\ \frac{1}{2} \max\{x, y, e, e'\}, & \text{otherwise} \end{cases}.$$

It is evident that conditions 1 and 2 of the soft super metric are satisfied. To verify condition 3, suppose $y_e \in SP(X, E)$ and let $\{x_{e_n}^n\}, \{y_{e'_n}^n\}$ be two distinct soft sequences in $(SP(X, E), s_m)$ such that $\lim_{n \rightarrow \infty} s_m(x_{e_n}^n, y_{e'_n}^n) = \tilde{0}$. Since the soft sequences are distinct, we have

$$s_m(x_{e_n}^n, y_{e'_n}^n) = \frac{x^n + y^n}{1 + x^n + y^n} + \frac{e_n + e'_n}{1 + e_n + e'_n} \rightarrow \tilde{0} \text{ as } n \rightarrow \infty.$$

Therefore, it is evident that we have the following limit equalities:

$$\lim_{n \rightarrow \infty} x^n = \lim_{n \rightarrow \infty} y^n = \lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} e'_n = 0.$$

Then there exists $N > 0$ such that for all $n \geq N$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} s_m(y_{e'_n}^n, y_e) &= \limsup_{n \rightarrow \infty} \left(\frac{y^n + y}{1 + y^n + y} + \frac{e'_n + e}{1 + e'_n + e} \right) \\ &= \frac{y}{1 + y} + \frac{e}{1 + e} \leq s\left(\frac{y}{1 + y} + \frac{e}{1 + e}\right) \\ &= \tilde{s} \limsup_{n \rightarrow \infty} \left(\frac{x^n + y}{1 + x^n + y} + \frac{e_n + e}{1 + e_n + e} \right) \\ &= \tilde{s} \limsup_{n \rightarrow \infty} s_m(x_{e_n}^n, y_e). \end{aligned}$$

This shows that condition 3 of the soft super metric is satisfied.

Definition 8. The soft sequence $\{x_{e_n}^n\} \subset SP(X, E)$ converges to $x_e \in SP(X, E)$ if $\lim_{n \rightarrow \infty} \sup s_m(x_{e_n}^n, x_e) = \tilde{0}$, denoted by $s_m(x_{e_n}^n, x_e) \rightarrow \tilde{0}$ as $n \rightarrow \infty$.

Definition 9. The soft sequence $\{x_{e_n}^n\} \subset SP(X, E)$ is considered a Cauchy soft sequence if $\lim_{n \rightarrow \infty} \sup \{s_m(x_{e_n}^n, x_{e_k}^k) : k > n\} = \tilde{0}$.

Definition 10. soft super metric space $(SP(X, E), s_m)$ is complete if each Cauchy soft sequence is convergent.

Definition 11. Let $(SP(X, E), s_m)$ be a soft super metric space and $(f, \varphi) : (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ be a soft mapping, where $f : X \rightarrow X$, $\varphi : E \rightarrow E$ are two mappings. A soft contraction mapping (f, φ) is characterised by the presence of a soft real number $\tilde{\alpha} \in \mathbb{R}(E)^*$, $\tilde{0} \leq \tilde{\alpha} < \tilde{1}$ such that

$$s_m((f, \varphi)(x_e), (f, \varphi)(y_{e'})) \leq \tilde{\alpha} s_m(x_e, y_{e'})$$

for all $x_e, y_{e'} \in SP(X, E)$.

Theorem 1. Let $(SP(X, E), s_m)$ be a complete soft super metric space and $(f, \varphi) : (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ be a soft contraction mapping. Then there exists a unique fixed soft point $x_e \in SP(X, E)$ such that $(f, \varphi)(x_e) = x_e$.

Proof. Let x_e^0 be an arbitrary soft point in $SP(X, E)$. Let us set

$$x_{e_1}^1 = (f, \varphi)(x_e^0) = (f(x^0))_{\varphi(e)}.$$

Suppose that $x_{e_1}^1 \neq x_e^0$. It follows that $s_m(x_{e_1}^1, x_e^0) > \tilde{0}$. It can therefore be defined without loss of generality that

$$x_{e_{n+1}}^{n+1} = (f, \varphi)(x_{e_n}^n) = (f(x^n))_{\varphi(e_n)},$$

where $x_{e_{n+1}}^{n+1} \neq x_{e_n}^n$, thus ensuring that $s_m(x_{e_{n+1}}^{n+1}, x_{e_n}^n) > \tilde{0}$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} s_m(x_{e_{n+1}}^{n+1}, x_{e_n}^n) &= s_m((f, \varphi)(x_{e_n}^n), (f, \varphi)(x_{e_{n-1}}^{n-1})) \\ &\leq \tilde{\alpha} s_m(x_{e_n}^n, x_{e_{n-1}}^{n-1}) \end{aligned}$$

$$\leq \tilde{\alpha}^2 s_m(x_{e_{n-1}}^{n-1}, x_{e_{n-2}}^{n-2}) \leq \dots \leq \tilde{\alpha}^n s_m(x_{e_1}^1, x_e^0). \quad (1)$$

If we consider the limit from both side of (1), we find that

$$\lim_{n \rightarrow \infty} s_m(x_{e_{n+1}}^{n+1}, x_{e_n}^n) = \tilde{0}.$$

We shall now proceed on the supposition that $m, n \in \mathbb{N}$ and $m > n$. In accordance with the definition of a soft super metric space for $\tilde{s} \geq 1$ and for all $x_{e_{n+2}}^{n+2} \in SP(X, E)$, there exist distinct soft sequences $\{x_{e_{n+1}}^{n+1}\}, \{x_{e_n}^n\}$ with $\lim_{n \rightarrow \infty} s_m(x_{e_{n+1}}^{n+1}, x_{e_n}^n) = \tilde{0}$ such that

$$\limsup_{n \rightarrow \infty} s_m(x_{e_n}^n, x_{e_{n+2}}^{n+2}) \leq \tilde{s} \limsup_{n \rightarrow \infty} s_m(x_{e_{n+1}}^{n+1}, x_{e_{n+2}}^{n+2}).$$

Since $\lim_{n \rightarrow \infty} s_m(x_{e_{n+1}}^{n+1}, x_{e_n}^n) = \tilde{0}$, we have $\lim_{n \rightarrow \infty} \sup s_m(x_{e_n}^n, x_{e_{n+2}}^{n+2}) = \tilde{0}$. This reasoning may be extended to cover all $\tilde{s} \geq 1$, and for all $x_{e_{n+3}}^{n+3} \in SP(X, E)$, there exist distinct soft sequences $\{x_{e_{n+2}}^{n+2}\}, \{x_{e_n}^n\}$ with $\lim_{n \rightarrow \infty} s_m(x_{e_{n+2}}^{n+2}, x_{e_n}^n) = \tilde{0}$ such that

$$\limsup_{n \rightarrow \infty} s_m(x_{e_n}^n, x_{e_{n+3}}^{n+3}) \leq \tilde{s} \limsup_{n \rightarrow \infty} s_m(x_{e_{n+2}}^{n+2}, x_{e_{n+3}}^{n+3}).$$

Therefore, we can conclude that

$$\limsup_{n \rightarrow \infty} \{s_m(x_{e_n}^n, x_{e_m}^m) : m > n\} = \tilde{0}.$$

This means that $\{x_{e_n}^n\} \subset SP(X, E)$ is a Cauchy soft sequence. The completeness of $(SP(X, E), s_m)$ implies the existence of a soft point $x_e^* \in SP(X, E)$ for which $\lim_{n \rightarrow \infty} s_m(x_{e_n}^n, x_e^*) = \tilde{0}$ is valid. We contend that x_e^* represents the fixed soft point of (f, φ) . Suppose that $s_m(x_e^*, (f, \varphi)(x_e^*)) > \tilde{0}$. It is observed that

$$s_m(x_{e_{n+1}}^{n+1}, (f, \varphi)(x_e^*)) = s_m((f, \varphi)(x_{e_n}^n), (f, \varphi)(x_e^*)) \leq \tilde{\alpha} s_m(x_{e_n}^n, x_e^*). \quad (2)$$

Hence $\lim_{n \rightarrow \infty} s_m(x_{e_{n+1}}^{n+1}, (f, \varphi)(x_e^*)) = \tilde{0}$. If there exists a natural number k such that for all $n > k$, $x_{e_{k+1}}^{k+1} = x_e^*$, then (2) implies that $s_m(x_e^*, (f, \varphi)(x_e^*)) = \tilde{0}$. Therefore, we can conclude that x_e^* is a fixed soft point of the (f, φ) .

Now if $y_{e'}^*$ is another fixed soft point of (f, φ) , then

$$s_m(x_e^*, y_{e'}^*) = s_m((f, \varphi)(x_e^*), (f, \varphi)(y_{e'}^*)) \leq \tilde{\alpha} s_m(x_e^*, y_{e'}^*).$$

Hence for $\tilde{0} \leq \tilde{\alpha} < \tilde{1}$ we have $s_m(x_e^*, y_{e'}^*) = \tilde{0}$, which implies that $x_e^* = y_{e'}^*$. Consequently, the fixed soft point of (f, φ) is a unique entity.

Example 2. Let $X = [2, 3]$, $E = \left[0, \frac{1}{2}\right]$ and define $s_m: SP(X, E) \times SP(X, E) \rightarrow \mathbb{R}(E)^*$ as follows:

$$s_m(x_e, y_{e'}) = \begin{cases} xy + ee' & \text{if } x_e \neq y_{e'} \\ \tilde{0} & \text{if } x_e = y_{e'} \end{cases}.$$

Let $\{x_{e_n}^n\}$ and $\{y_{e'_n}^n\}$ be two distinct soft sequences such that $s_m(x_{e_n}^n, y_{e'_n}^n) \rightarrow \tilde{0}$ as $n \rightarrow \infty$. Since the sequences are distinct, it follows that

$$s_m(x_{e_n}^n, y_{e'_n}^n) = x^n y^n + e_n e'_n \Rightarrow x^n y^n \rightarrow 0 \text{ and } e_n e'_n \rightarrow 0.$$

There are two possible scenarios:

- 1) $x^n \rightarrow u$, $y^n \rightarrow 0$ and $e_n \rightarrow 0$, $e'_n \rightarrow e'$
- 2) $x^n \rightarrow u$, $y^n \rightarrow 0$ and $e_n \rightarrow e$, $e'_n \rightarrow 0$

Case 1. For any $y_a \in SP(X, E)$, we observe:

$$\limsup_{n \rightarrow \infty} s_m(y_{e'_n}^n, y_a) = \limsup_{n \rightarrow \infty} (y^n y + e'_n a) = e' a \leq \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \sup s_m(x_{e_n}^n, y_a) = \lim_{n \rightarrow \infty} \sup (x^n y + e_n a) = uy \geq 2.$$

Thus, we conclude:

$$\lim_{n \rightarrow \infty} \sup s_m(y_{e_n}^n, y_a) \leq \tilde{s} \lim_{n \rightarrow \infty} \sup s_m(x_{e_n}^n, y_a).$$

Case 2.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup s_m(y_{e_n}^n, y_a) &= \tilde{0} \\ \lim_{n \rightarrow \infty} \sup s_m(x_{e_n}^n, y_a) &= uy + ea \Rightarrow \lim_{n \rightarrow \infty} \sup s_m(y_{e_n}^n, y_a) \leq \tilde{s} \lim_{n \rightarrow \infty} \sup s_m(x_{e_n}^n, y_a), \end{aligned}$$

which implies that $(SP(X, E), s_m)$ is a soft super metric space. Now consider a soft mapping $(f, \varphi): (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ defined as

$$f(x) = \begin{cases} 2, & \text{if } x \neq 3 \\ \frac{3}{2}, & \text{if } x = 3 \end{cases} \text{ and } \varphi(e) = \frac{1}{2}e, \forall e \in \left[0, \frac{1}{2}\right].$$

Let $s = \frac{9}{4}, \alpha = \frac{1}{2}$, and take $x \neq 3, y = 3$. Then

$$s_m((f, \varphi)(x_e), (f, \varphi)(y_{e'})) = 3 \leq \frac{1}{2}(3x + ee') = \tilde{\alpha}s_m(x_e, y_{e'}).$$

Therefore, $(f, \varphi): (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ is a soft contraction mapping on the soft super metric space. Consequently, by invoking Theorem 1, it can be concluded that the mapping admits at least one soft fixed point, specifically $x_e = 2_0$.

Theorem 2. Let $(SP(X, E), s_m)$ be a complete soft super metric space and let $(f, \varphi): (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ be a surjective soft mapping. Suppose there exists a soft real number $\tilde{\alpha} > \tilde{1}$ such that

$$s_m((f, \varphi)(x_e), (f, \varphi)(y_{e'})) \geq \tilde{\alpha}s_m(x_e, y_{e'})$$

for all $x_e, y_{e'} \in SP(X, E)$. Under this condition, the mapping (f, φ) admits a unique fixed soft point.

Proof. Let x_e^0 be an arbitrary soft point in $SP(X, E)$. Since (f, φ) is a surjective soft mapping, it follows that $(f, \varphi)^{-1}(x_e^0) \neq \Phi$. Therefore, we select soft point $x_{e_1}^1$ from $(f, \varphi)^{-1}(x_e^0)$, which implies that $(f, \varphi)(x_{e_1}^1) = x_e^0$. Next, consider $(f, \varphi)^{-1}(x_{e_1}^1) \neq \Phi$. Hence there exists a soft point $x_{e_2}^2 \in (f, \varphi)^{-1}(x_{e_1}^1)$ such that $(f, \varphi)(x_{e_2}^2) = x_{e_1}^1$. By continuing this process iteratively, we construct a soft sequence of soft points satisfying the following relation:

$$x_{e_n}^n = (f, \varphi)(x_{e_{n+1}}^{n+1}), n = 0, 1, 2, \dots$$

If $x_e^0 = x_{e_1}^1$, then x_e^0 is a fixed soft point and the proof is completed. Otherwise, if $x_e^0 \neq x_{e_1}^1$, we have $s_m(x_e^0, x_{e_1}^1) \geq \tilde{0}$. Without loss of generality, we assume that $x_{e_n}^n \neq x_{e_{n+1}}^{n+1}$. Consequently, for every $n = 0, 1, 2, \dots$ the inequality $s_m(x_{e_n}^n, x_{e_{n+1}}^{n+1}) \geq \tilde{0}$ holds. Thus, we have

$$\begin{aligned} s_m(x_{e_n}^n, x_{e_{n+1}}^{n+1}) &= s_m((f, \varphi)(x_{e_{n+1}}^{n+1}), (f, \varphi)(x_{e_{n+2}}^{n+2})) \\ &\geq \tilde{\alpha}s_m(x_{e_{n+1}}^{n+1}, x_{e_{n+2}}^{n+2}). \end{aligned}$$

Consequently,

$$\begin{aligned} s_m(x_{e_{n+1}}^{n+1}, x_{e_{n+2}}^{n+2}) &\leq \frac{1}{\tilde{\alpha}}s_m(x_{e_n}^n, x_{e_{n+1}}^{n+1}) \\ &\leq \frac{1}{\tilde{\alpha}^2}s_m(x_{e_{n-1}}^{n-1}, x_{e_n}^n) \\ &\dots \\ &\leq \frac{1}{\tilde{\alpha}^{n+1}}s_m(x_e^0, x_{e_1}^1). \end{aligned}$$

Then we conclude that $\{x_{e_n}^n\}$ is a Cauchy soft sequence. Since $(SP(X, E), s_m)$ is a complete soft super metric space, there exists a soft point $z_{e_0} \in SP(X, E)$ such that $s_m(x_{e_n}^n, z_{e_0}) \rightarrow \tilde{0}$ as $n \rightarrow \infty$. Therefore, we have

$$s_m(x_{e_{n-1}}^{n-1}, (f, \varphi)(z_{e_0})) = s_m((f, \varphi)(x_{e_n}^n), (f, \varphi)(z_{e_0})) \leq \frac{1}{\tilde{\alpha}} s_m(x_{e_n}^n, z_{e_0}).$$

So $s_m(z_{e_0}, (f, \varphi)(z_{e_0})) = \tilde{0}$, which implies that $(f, \varphi)(z_{e_0}) = z_{e_0}$. Now suppose $y_{e'_0}$ is another distinct fixed soft point of (f, φ) , i.e. $(f, \varphi)(y_{e'_0}) = y_{e'_0}$. We then have

$$\begin{aligned} s_m(y_{e'_0}, z_{e_0}) &\leq \frac{1}{\tilde{\alpha}} s_m((f, \varphi)(y_{e'_0}), (f, \varphi)(z_{e_0})) \\ &= \frac{1}{\tilde{\alpha}} s_m(y_{e'_0}, z_{e_0}). \end{aligned}$$

Since $\tilde{\alpha} > \tilde{1}$, this implies

$$s_m(y_{e'_0}, z_{e_0}) = \tilde{0}.$$

Therefore, we conclude that $y_{e'_0} = z_{e_0}$.

Definition 12. Let (f, φ) and (g, ψ) be two self soft mappings on $SP(X, E)$. The two mappings (f, φ) and (g, ψ) are said to be weakly compatible, provided that there exists a soft point $x_e \in SP(X, E)$ satisfying the following conditions:

$$\begin{aligned} (f, \varphi)(x_e) &= (g, \psi)(x_e), \\ (f, \varphi)((g, \psi)(x_e)) &= (g, \psi)((f, \varphi)(x_e)). \end{aligned}$$

Theorem 3. Let $(SP(X, E), s_m)$ be a complete soft super metric space. Consider two weakly compatible mappings $(f, \varphi), (g, \psi): (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ and assume that

$$(g, \psi)(SP(X, E)) \subset (f, \varphi)(SP(X, E)).$$

If the following inequality holds:

$$s_m((f, \varphi)(x_e), (f, \varphi)(y_{e'})) \leq \tilde{\alpha} s_m((g, \psi)(x_e), (g, \psi)(y_{e'})), \quad (3)$$

$0 \leq \tilde{\alpha} < \tilde{1}$, for all $x_e, y_{e'} \in SP(X, E)$, and if either $(f, \varphi)(SP(X, E))$ or $(g, \psi)(SP(X, E))$ is complete, then (f, φ) and (g, ψ) have a unique common fixed soft point in $SP(X, E)$.

Proof. Let $x_{e_0}^0$ be an arbitrary soft point in $SP(X, E)$. Since $(g, \psi)(SP(X, E)) \subset (f, \varphi)(SP(X, E))$, a soft point $x_{e_1}^1$ is selected such that $y_{e'_1}^1 = (f, \varphi)(x_{e_0}^0) = (g, \psi)(x_{e_1}^1)$. In general, the choice of $x_{e_{n+1}}^{n+1}$ is made such that

$$y_{e'_n}^n = (f, \varphi)(x_{e_n}^n) = (g, \psi)(x_{e_{n+1}}^{n+1})$$

for all $n = 0, 1, 2, \dots$. Subsequently, from condition (3), the following can be deduced:

$$\begin{aligned} s_m(y_{e'_n}^n, y_{e'_{n+1}}^{n+1}) &= s_m((f, \varphi)(x_{e_n}^n), (f, \varphi)(x_{e_{n+1}}^{n+1})) \\ &\leq \tilde{\alpha} s_m((g, \psi)(x_{e_n}^n), (g, \psi)(x_{e_{n+1}}^{n+1})) \\ &= \tilde{\alpha} s_m(y_{e'_{n-1}}^{n-1}, y_{e'_n}^n) \\ &\dots \\ &\leq \tilde{\alpha}^n s_m(y_{e'_0}^0, y_{e'_1}^1). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we obtain

$$s_m(y_{e'_n}^n, y_{e'_{n+1}}^{n+1}) = \tilde{0}.$$

It is evident that $\{y_{e_n}^n\}$ is a Cauchy soft sequence. Furthermore, it is evident from the hypothesis of Theorem 1 that the soft sequence converges to a soft point $z_{e_0} \in SP(X, E)$, i.e.

$$\lim_{n \rightarrow \infty} y_{e_n}^n = \lim_{n \rightarrow \infty} (f, \varphi)(x_{e_n}^n) = \lim_{n \rightarrow \infty} (g, \psi)(x_{e_{n+1}}^{n+1}) = z_{e_0}.$$

Since one of the soft images $(f, \varphi)(SP(X, E))$ or $(g, \psi)(SP(X, E))$ is the complete subspace of $SP(X, E)$ and the relation $(g, \psi)(SP(X, E)) \subset (f, \varphi)(SP(X, E))$ holds, there exists an element $\rho_e^* \in SP(X, E)$ such that $(g, \psi)(\rho_e^*) = z_{e_0}$. It can be concluded from (3) that

$$\begin{aligned} s_m((f, \varphi)(\rho_e^*), (f, \varphi)(x_{e_n}^n)) &\leq \tilde{\alpha} s_m((g, \psi)(\rho_e^*), (g, \psi)(x_{e_n}^n)) \\ &= \tilde{\alpha} s_m(z_{e_0}, y_{e_{n-1}}^{n-1}). \end{aligned}$$

In the limit as $n \rightarrow \infty$, we obtain the following result:

$$(f, \varphi)(\rho_e^*) = z_{e_0}.$$

Since (f, φ) and (g, ψ) are two weakly compatible mappings, it follows that

$$(f, \varphi)((g, \psi)(\rho_e^*)) = (g, \psi)((f, \varphi)(\rho_e^*)).$$

Hence we have

$$(f, \varphi)(z_{e_0}) = (g, \psi)(z_{e_0}).$$

Next, we demonstrate that z_{e_0} is a common fixed soft point of (f, φ) and (g, ψ) . According to (3), it follows that

$$\begin{aligned} s_m((f, \varphi)(z_{e_0}), (f, \varphi)(x_{e_n}^n)) &\leq \tilde{\alpha} s_m((g, \psi)(z_{e_0}), (g, \psi)(x_{e_n}^n)) \\ s_m((f, \varphi)(z_{e_0}), z_{e_0}) &\leq \tilde{\alpha} s_m((g, \psi)(z_{e_0}), z_{e_0}). \end{aligned}$$

As $\tilde{\alpha} < \tilde{1}$, this leads to $(f, \varphi)(z_{e_0}) = z_{e_0}$. Consequently, we have $(f, \varphi)(z_{e_0}) = z_{e_0} = (g, \psi)(z_{e_0})$, establishing that z_{e_0} is a common fixed soft point of (f, φ) and (g, ψ) . The uniqueness of the common fixed soft point is readily obtained from Theorem 1.

Theorem 4. Let $(SP(X, E), s_m)$ be a complete soft super metric space, and consider two weakly compatible mappings $(f, \varphi), (g, \psi): (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$. Suppose that

$$(g, \psi)(SP(X, E)) \subset (f, \varphi)(SP(X, E)).$$

If the following inequality holds for all $x_e, y_{e'} \in SP(X, E)$ and for some $\tilde{0} \leq \tilde{\alpha} < \tilde{1}$, i.e.

$$s_m((f, \varphi)(x_e), (f, \varphi)(y_{e'})) \leq \tilde{\alpha} s_m((g, \psi)(x_e), (g, \psi)(y_{e'})), \quad (4)$$

and if either $(f, \varphi)(SP(X, E))$ or $(g, \psi)(SP(X, E))$ is a closed subset, then (f, φ) and (g, ψ) have a unique common fixed soft point in $SP(X, E)$.

Proof. Let $x_{e_0}^0$ be an arbitrary soft point in $SP(X, E)$. Since $(g, \psi)(SP(X, E)) \subset (f, \varphi)(SP(X, E))$, a soft point $x_{e_1}^1$ is selected such that $y_{e_1}^1 = (f, \varphi)(x_{e_0}^0) = (g, \psi)(x_{e_1}^1)$. In general, the choice of $x_{e_{n+1}}^{n+1}$ is made such that

$$y_{e_n}^n = (f, \varphi)(x_{e_n}^n) = (g, \psi)(x_{e_{n+1}}^{n+1})$$

for all $n = 0, 1, 2, \dots$. Subsequently, from the conditions of Theorem 3, we conclude that $\{y_{e_n}^n\}$ is a Cauchy soft sequence. Moreover, by the hypothesis, $(g, \psi)(SP(X, E))$ is a closed subset. It is noteworthy that $\{y_{e_n}^n\}$ is contained in $(g, \psi)(SP(X, E))$ and it has a soft limit point $z_{e_0} \in SP(X, E)$. In

the case of letting $\rho_e^* = (g, \psi)^{-1}(z_{e_0})$, it can be demonstrated that $(g, \psi)(\rho_e^*) = z_{e_0}$ for some $\rho_e^* \in SP(X, E)$. Following from (4), we derive

$$\begin{aligned} s_m \left((f, \varphi)(\rho_e^*), (f, \varphi)(x_{e_n}^n) \right) &\leq \tilde{\alpha} s_m \left((g, \psi)(\rho_e^*), (g, \psi)(x_{e_n}^n) \right) \\ &= \tilde{\alpha} s_m \left(z_{e_0}, y_{e_{n-1}}^{n-1} \right). \end{aligned}$$

In the limit as $n \rightarrow \infty$, we obtain the following result:

$$(f, \varphi)(\rho_e^*) = z_{e_0}.$$

Since (f, φ) and (g, ψ) are two weakly compatible mappings, it can be shown that z_{e_0} is a unique common fixed soft point of (f, φ) and (g, ψ) , following a proof similar to Theorem 3.

Theorem 5. Let $(SP(X, E), s_m)$ be a complete soft super metric space, and let $(f, \varphi): (SP(X, E), s_m) \rightarrow (SP(X, E), s_m)$ be a soft contraction mapping. Suppose there exists a soft mapping

$$\tilde{\gamma}: \mathbb{R}(E)^* \rightarrow \mathbb{R}(E)^*$$

satisfying the following conditions:

- (i) For any $\tilde{r} > \tilde{0}$, $\tilde{\gamma}(\tilde{r}) < \tilde{r}$ holds.
- (ii) $\tilde{\gamma}(\tilde{r}) = \tilde{0}$ if and only if $\tilde{r} = \tilde{0}$.

Furthermore, assume that the mapping (f, φ) satisfies the contractive condition

$$s_m \left((f, \varphi)(x_e), (f, \varphi)(y_{e'}) \right) \leq \tilde{\gamma}(s_m(x_e, y_{e'})) \quad (5)$$

for all $x_e, y_{e'} \in SP(X, E)$. Under these assumptions, the mapping (f, φ) admits a unique fixed soft point in $SP(X, E)$.

Proof. Let x_e^0 be an arbitrary soft point in $SP(X, E)$. Consider the soft sequence $\{x_{e_n}^n\}$ in $SP(X, E)$ defined recursively by

$$x_{e_{n+1}}^{n+1} = (f, \varphi)(x_{e_n}^n)$$

for all $n = 0, 1, 2, \dots$ and define the soft sequence $\{\tilde{\alpha}_n\}$ by $\tilde{\alpha}_n = s_m(x_{e_n}^n, x_{e_{n-1}}^{n-1})$. From the contractive condition (5), it follows that

$$\begin{aligned} \tilde{\alpha}_n &= s_m(x_{e_n}^n, x_{e_{n-1}}^{n-1}) = s_m \left((f, \varphi)(x_{e_{n-1}}^{n-1}), (f, \varphi)(x_{e_{n-2}}^{n-2}) \right) \\ &\leq \tilde{\gamma} \left(s_m(x_{e_{n-1}}^{n-1}, x_{e_{n-2}}^{n-2}) \right) = \tilde{\gamma}(\tilde{\alpha}_{n-1}). \end{aligned}$$

By assumption (i), the following strict inequality holds:

$$\tilde{\alpha}_n \leq \tilde{\gamma}(\tilde{\alpha}_{n-1}) < \tilde{\alpha}_{n-1}.$$

This establishes that $\{\tilde{\alpha}_n\}$ is a strictly decreasing soft sequence of soft real numbers, converging to some soft point $\tilde{\beta} \geq \tilde{0}$ in $\mathbb{R}(E)^*$, i.e.

$$\lim_{n \rightarrow \infty} s_m(x_{e_n}^n, x_{e_{n-1}}^{n-1}) = \tilde{\beta}.$$

Asserting $\tilde{\beta} = \tilde{0}$, if $\tilde{\beta} > \tilde{0}$, then it follows that $\tilde{\beta} \leq \tilde{\gamma}(\tilde{\beta}) < \tilde{\beta}$, a circumstance which constitutes a contradiction; therefore $\tilde{\beta} = \tilde{0}$. Consequently, we obtain $\lim_{n \rightarrow \infty} s_m(x_{e_n}^n, x_{e_{n-1}}^{n-1}) = \tilde{0}$. By Theorem 1, it follows that the soft sequence $\{x_{e_n}^n\}$ in $SP(X, E)$ is Cauchy. Given that $(SP(X, E), s_m)$ is a complete soft super metric space, there exists a soft point $x_e^* \in SP(X, E)$. Furthermore, from (5), we have

$$s_m \left(x_{e_n}^n, (f, \varphi)(x_e^*) \right) \leq \tilde{\gamma} \left(s_m(x_{e_n}^n, x_e^*) \right).$$

Taking the limit as $n \rightarrow \infty$, we obtain

$$(f, \varphi)(x_e^*) = x_e^*.$$

Thus, x_e^* is a fixed soft point of (f, φ) . Now suppose that $y_{e'}^*$ is another fixed soft point of (f, φ) , then we have

$$\begin{aligned} s_m(x_e^*, y_{e'}^*) &= s_m((f, \varphi)(x_e^*), (f, \varphi)(y_{e'}^*)) \\ &\leq \tilde{\gamma}(s_m(x_e^*, y_{e'}^*)) < s_m(x_e^*, y_{e'}^*). \end{aligned}$$

This is a contradiction. Hence we conclude that $x_e^* = y_{e'}^*$, proving the uniqueness of the fixed soft point. Therefore, the mapping (f, φ) has a unique fixed soft point in $SP(X, E)$.

CONCLUSIONS

We have introduced and investigated the concept of soft super metric spaces, establishing a novel framework that combines the generality of super metric spaces with the flexibility of soft set theory. Our contributions include the formal definition and characterisation of soft super metric spaces, extending the classical notion of super metric spaces to accommodate parameterised uncertainty through the soft set framework. We have also developed several fixed point theorems for self-soft mappings on complete soft super metric spaces including a fundamental fixed point theorem for soft contraction mappings, an existence theorem for surjective soft mappings with specific metric conditions, and fixed point results for weakly compatible mappings under various completeness conditions. These theoretical results significantly extend the existing literature on fixed point theory in soft metric spaces and provide a robust foundation for analysing mappings in parameterised metric structures.

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