Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

Maejo International Journal of Science and Technology

e- ISSN 2697-4746
Available online at www.mijst.mju.ac.th

Full Paper

Exact values of fractional dimensions for non-planar symmetric networks

Arooba Fatima [1], Badr S. Alkahtani [2], Muhammad Javaid [1], Sana Shahab [3] and Mohd Anjum [4,*]

[1] Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan
[2] Department of Mathematics, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[3] Department of Business Administration, College of Business Administration, Princess Nourah Bint Abdulrahman University, PO Box 84428, Riyadh 11671, Saudi Arabia
[4] Department of Computer Engineering, Aligarh Muslim University, Aligarh 202002, India
* Corresponding author: Mohd Anjum, e-mail: mohdanjum@zhcet.ac.in

Received: 19 December 2024 / Accepted: 8 August 2025 / Published: 18 August 2025

Abstract: Fractional versions of graph-theoretic invariants have expanded their applicability to diverse fields such as connectivity, scheduling, assignment and operational research. Building on this extension of fractional graph theory, we introduce the local fractional metric dimension (LFMD) of non-planar networks. For a given network G, the local resolving neighbourhood $LR(uw)$ of an edge uw is defined as the set of vertices in G that distinguish between u and w. A function $\rho{:}V(G)\rightarrow[0,1]$ is considered a local resolving function if $\rho(LR(uw))\geq 1$ for every edge uw in G. The LFMD of G is then defined as the minimum value of $\rho(V(G))$ taken over all possible local resolving functions. In this paper we compute the exact values of the LFMD for two non-planar, structurally symmetric network families—generalised gear and generalised helm networks—each constructed with a finite number of levels and characterised by distinct types of vertices. We compare the results for both networks and analyse the impact of pendant vertices in the context of bipartite and non-bipartite structures. Furthermore, we investigate the asymptotic behaviour as the order of the networks approaches infinity and present an emergency exit planning scenario to illustrate the practical significance of our findings.

Keywords: fractional metric dimension, resolving neighbourhood sets, non-planar networks, optimisation problem

---

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

INTRODUCTION

Graph theory has become a foundational tool for analysing complex network systems across disciplines, from biological networks to distributed computing. The core of many network analysis problems is vertex identification—determining unique node positions based on structural attributes [1]—a challenge formally addressed by the metric dimension (MD). Introduced by Slater [2] and Harary and Melter [3], the MD defines the minimal set of reference vertices needed to locate all nodes uniquely via distance vectors. Slater's motivation stemmed from intruder detection in security networks while Harary and Melter approached it from a graph-theoretic perspective on structural uniqueness. Chartrand et al. [4] later characterised extremal MD values for trees and unicyclic graphs. The MD's operational relevance spans various fields including robotic path planning [5] where landmark-based navigation algorithms exhibit $O(n^3)$ complexity for tree networks, and pharmaceutical chemistry [4] where resolving sets aid molecular structure identification. Khuller et al. [5] further demonstrated its practical value in robotic navigation, while Sebő and Tannier [6] applied it to combinatorial optimisation frameworks. Studies on structured graphs further advance the MD: Liu et al. [7] showed constant MD in Toeplitz graphs under certain parameters; Ali et al. [8] proved that three vertices resolve all nodes in Möbius ladders; and Imran et al. [9] computed the exact MD for generalised gear networks. These results underscore the role of symmetry and topology in network resolvability. Extensions such as fault-tolerant MD [10], mixed MD [11] and edge-metric MD [12-14] have since emerged to address robustness and diverse network scenarios.

The transition to the fractional metric dimension (FMD) marks a key theoretical advancement, overcoming limitations of the integer-based formulation. Currie and Ollermann [15] introduced continuous vertex weighting, broadening the resolving-set paradigm for wider optimisation applications. Arumugam and Mathew [16] formalised the FMD by establishing bounds ($1 \leq dim\_f(G) \leq |V(G)|/2$), with paths and complete graphs achieving the extremes. This fractional extension brought three major advancements: (1) enabling probabilistic modelling through vertex weighting [15]; (2) defining FMD behaviour within formal bounds [16]; and (3) expanding its applicability to complex domains including chemical graph theory [17] where fractional weights model probabilistic molecular interactions, and network design [18] where hierarchical products are assessed using FMD metrics. Feng et al. [19] also analysed the FMD in vertex-transitive and Cartesian product graphs, resolving key open problems in the field. Additional studies extended the FMD to Jahangir graphs [20] and comb product graphs [21], highlighting its adaptability in composite networks. Mufti et al. [22] further contributed by applying edge MD to barycentric subdivisions of Cayley graphs, expanding the scope to algebraic structures and symmetric networks. Collectively, these contributions deepen the understanding of both the MD and the FMD in complex and symmetric graph structures.

Javaid et al. [23] advanced the FMD framework by introducing the local fractional metric dimension (LFMD), which focuses on distinguishing adjacent vertices via local resolving function (LRF) $\eta:V(G)\rightarrow[0,1]$, requiring the sum over each edge's local neighbourhood to exceed one. The LFMD is particularly suited to applications involving local interactions, such as fault detection in sensor networks [24] and emergency exit planning [25]. Javaid et al. [26] proved that all connected bipartite graphs satisfy $dim_{lf}(\mathcal{G})$, providing a clean parity-based characterisation, while Aisyah et al. [27] analysed the LFMD in corona product graphs. Recent studies have examined LFMD in structured graphs. For example, Ahmed et al. [28] established sharp bounds for modified prism networks and proved that their LFMD remains bounded as the network size increases due to edge transitivity, and Ali et al. [29] showed convergence behaviours through constructive techniques for rotationally heptagonal symmetrical graphs. Further studies on line networks derived from wheels and prisms have explored LFMD bounds using asymptotic analysis and 3D modelling [30]. Efforts by Zafar et al. [25] on wheel networks and Ali et al. [31] on fault-tolerant designs have started extending LFMD analysis into more complex topologies. Despite these advances, existing research remains largely focused on planar graphs and often provides only bounds rather than exact LFMD values, especially in non-planar hierarchical networks. While the LFMD has been explored in several graph operations including prism-related and corona product structures [24-27], exact computations for structurally rich and non-planar networks remain limited. Javaid et al. [23] and Ali et al. [31] also contributed by characterising the LFMD behaviour in locally dense networks where specific substructures and symmetry give rise to predictable patterns and measurable variations in resolvability, distinguishing LFMD from classical MD.

Recent studies have further expanded the scope of LFMD. Fatima et al. [32] computed exact LFMD values for a non-planar subclass derived from subdivided wheel graphs, demonstrating how fractional techniques capture subtle structural variations. However, their analysis was confined to a single network family and lacked broader consideration of hierarchical structure, symmetry, bipartiteness or asymptotic behaviour. Similarly, Javaid et al. [33] examined generalised sunlet networks providing bounds and selected exact values but without addressing complex topologies or convergence behaviour. Ali et al. [34] analysed LFMD in rotationally symmetric planar graphs using linear programming techniques, offering asymptotic insights but limiting their scope to low-order planar structures without deriving closed-form generalisations.

This study addresses identified research gaps by investigating generalised gear and helm networks, which are two non-planar symmetric families. The objectives are: (1) to derive exact closed-form LFMD expressions; (2) to quantify the impact of pendant vertices and bipartite structures; (3) to analyse asymptotic behaviour as network complexity increases; and (4) to demonstrate practical relevance through an emergency evacuation optimisation scenario. By integrating rigorous theoretical modelling with applied validation, this work offers a comprehensive advancement in both the theory and application of FMDs in complex network systems.

BASIC NOTIONS AND CONSTRUCTION OF NON-PLANAR NETWORKS

Throughout this study, it is assumed that $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a finite, simple (without loops or parallel edges) and connected network, where $V(\mathcal{G})$ and $E(\mathcal{G}) \subseteq V(\mathcal{G}) \times V(\mathcal{G})$ are sets of vertices and edges respectively with cardinalities $|V(\mathcal{G})|$ (order) and $|E(\mathcal{G})|$ (size). For any two vertices $x, y \in V(\mathcal{G})$, the distance $d(x, y)$ is defined as the number of edges along the shortest path between them [35, 36].

For an edge $e = ab \in E(\mathcal{G})$, the local resolving neighbourhood set (LRN) is defined as $LR(e) = \{u \in V(\mathcal{G}) : d(u, a) \neq d(u, b)\}$. A function $\psi: V(\mathcal{G}) \to [0,1]$ is an LRF if $\psi(LR(e)) \geq 1$ for all $e \in E(\mathcal{G})$, where $\psi(LR(e)) = \sum_{v \in LR(e)} \psi(v)$. Moreover, an LRF ψ is called a minimal LRF if there exists a mapping $g: V(\mathcal{G}) \to [0,1]$ such that $g(v) \leq \psi(v)$ for all $v \in V(\mathcal{G})$ and $g(v) \neq \psi(v)$ for at least one $v \in V(\mathcal{G})$ and g is not an LRF of $\mathcal{G}$. Thus, the LFMD of $\mathcal{G}$ is defined as $dim_{lf}(\mathcal{G}) = min\{|\psi|, \psi$ being a minimal LRF of $\mathcal{G}\}$ [23]. The most commonly utilised results in the development of the main findings are outlined below:

Lemma 1 [26]**.** For any integer value $\alpha \geq 2$ and LRNs, i.e. $LR(e)$ of an edge of a connected network $\mathcal{G}$, assume that $|LR(e)| \geq \alpha$. For all $e \in E(\mathcal{G})$ and $|LR(e) \cap W| \geq \alpha$, then

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

$$1 \leq dim_{lf}(\mathcal{G}) \leq \frac{|W|}{\alpha}.$$

Here, W denotes the union of all the LRNs of order α.

Lemma 2 [26]**.** For any connected and bipartite network $\mathcal{G}$, the LFMD is always unity, i.e.

$$dim_{lf}(\mathcal{G}) = 1.$$

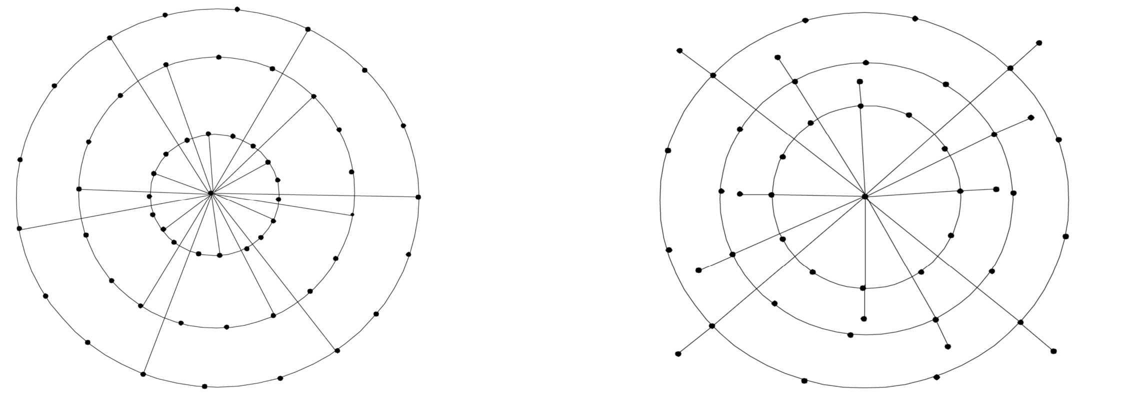
The following present the constructions of generalised n-level gear and generalised p-level helm networks [31].

Generalised N-Level Gear Networks

For an integer $m \geq 3$, each vertex of n copies of the cycle network C_m is joined to form the n-level wheel network. Additionally, it becomes a generalised n-level gear network $J_{m,n}^k$ by addition of $k \geq 1$ vertices to every edge of each cycle. Thus, the vertex and edge sets of $J_{m,n}^k$ are given by $V(\mathrm{J}_{\mathrm{m,n}}^{\mathrm{k}}) = \{a_o, a_i^j, b_{l,j}^i; 1 \leq i,j,l \leq m,n,k\}$ and $E(J_{m,n}^k) = \{a_o a_i^j; 1 \leq i,j \leq m,n\} \cup \{b_{l,j}^i b_{l,j}^{i+1}; 1 \leq i,j,l \leq m,n,k\} \cup \{a_i^j b_{1,j}^i; 1 \leq i,j \leq m,n\} \cup \{a_i^j b_{k,j}^i; 1 \leq i,j \leq m,n\}$ respectively, where $|V(\mathrm{J}_{\mathrm{m,n}}^{\mathrm{k}})| = nm(1+k)+1$. The vertices of $J_{m,n}^k$ can be classified into three types: the central vertex a_o, major vertices a_i^j and minor vertices $b_{l,j}^i$. A generalised n-level gear network $J_{6,3}^2$ is illustrated in Figure 1(a).

Generalised *P*-Level Helm Networks

The generalised p-level helm network $(H_{m,q}^p)$ is formed by adding a pendant vertex to each vertex of every cycle in the generalised n-level gear network [31]. Thus, the vertex set and edge sets of $H_{m,q}^p$ are given by $V(H_{m,q}^p) = \{a_o, a_k^j, b_{l,j}^k, c_k^j; 1 \leq k,j,l \leq m,p,q\}$ and $E(H_{m,q}^p) = \{a_o a_k^j; 1 \leq k,j \leq m,p\} \cup \{b_{l,j}^k b_{l,j}^{k+1}; 1 \leq k,j,l \leq m,p,q\} \cup \{a_k^j b_{1,j}^k; 1 \leq k,j \leq m,p\} \cup \{a_k^j b_{q,j}^k; 1 \leq k,j \leq p\} \cup \{a_k^j c_k^j; 1 \leq k,j \leq m,p\}$ respectively. The order of $H_{m,q}^p$ is $pm(2+q)+1$. A generalised p-level Helm network $H_{4,2}^3$ is illustrated in Figure 1(b).



(a) (b)

Figure 1. Representation of generalised (a) $\boldsymbol{n}$-level gear network $\boldsymbol{J_{6,3}^2}$ and (b) $\boldsymbol{p}$-level helm network $\boldsymbol{H_{4,2}^3}$

EXACT CLOSED-FORM LFMD FOR GENERALISED N-LEVEL GEAR NETWORK

The significant findings on the LFMD of the generalised n-level gear network are included in this section.

Theorem 1. Let $\mathcal{G} \cong J_{m,n}^{0}$ be a generalised n-level gear network. Then

$$dim_{lf}(\mathcal{G}) = \begin{cases} \frac{3n}{2}, & if\ m = 3 \\ \frac{mn}{4}, & if\ m \geq 4 \end{cases}.$$

Proof: Consider the following cases under the assumption that $i \in \{1,2,3\}$ and $1 \leq j \leq n$ throughout the proof:

Case 1. When $m = 3$, then the LRNs of $\mathcal{G}$ are $LR(a_o a_i^j) = V(\mathcal{G}) - \{a_{i+1}^j, a_{i+2}^j\}$ and $LR(a_i^j a_{i+1}^j) = \{a_i^j, a_{i+1}^j\}$. Moreover, $|LR(a_o a_i^j)| = 3n - 1$, $|LR(a_i^j a_{i+1}^j)| = 2$ and $X = \bigcup_{i=1,j=1}^{3,n} LR(a_i^j a_{i+1}^j) = V(\mathcal{G}) - \{a_o\}$, which implies that $|X| = 3n$. For $\alpha > \frac{1}{2}$, define $\psi: V(\mathcal{G}) \to [0,1]$ as

$$\psi(v) = \begin{cases} \lambda, & \text{if } v \in \{a_1^j, a_3^j\} \\ 1 - \lambda, & \text{if } v = a_2^j \\ 0, & \text{if } v = a_o. \end{cases}$$

Since for all $e \in \mathcal{G}$, $\psi(LR(e)) \geq 1$, therefore ψ is an LRF of $\mathcal{G}$. Similarly, the function $\phi: V(\mathcal{G}) \to [0,1]$ defined as

$$\phi(v)\text{f(x)} = \begin{cases} \frac{1}{2}, & \text{if } v \in X \\ 0, & \text{if } v \not\in X \end{cases},$$

is also an LRF with $\phi(LR(e)) \geq 1$ for each $e \in \mathcal{G}$. Furthermore, if there exists a mapping $g: V(\mathcal{G}) \to [0,1]$ such that $g(v) \leq \psi(v)$ or $g(v) \leq \phi(v)$ and $g(v) \neq \psi(v)$ $g(v) \neq \phi(v)$ for at least one $v \in V(\mathcal{G})$, then g does not remain as an LRF of $\mathcal{G}$. Consequently, the LRFs ψ and ϕ are minimal LRFs. Now consider: $|\psi| = \sum_{v \in \mathcal{G}} (\psi(v)) = \sum_{i=1,j=1}^{3,n} \psi(a_i^j) = n(1 + \lambda)$ and $|\phi| = \sum_{v \in \mathcal{G}} (\psi(v)) = \sum_{v \in X} (\psi(v)) = |X|\frac{1}{2} = \frac{3n}{2}$ such that $|\phi| < |\psi|$ as $\alpha > \frac{1}{2}$. Thus, ϕ is a minimal LRF of minimum $|\phi|$, which implies that $\frac{3n}{2} \leq dim_{lf}(\mathcal{G})$.

On the other hand, $\mathcal{G}$ has LRs of two types, $LR(a_o a_i^j)$ and $LR(a_i^j a_{i+1}^j)$ with $|LR(a_o a_i^j)| = 3n - 1$ and $|LR(a_i^j a_{i+1}^j)| = 2$ respectively. Additionally, $|LR(a_o a_i^j) \cap X| \geq |LR(a_i^j a_{i+1}^j)| = 2$. Then by Lemma 1, $dim_{lf}(\mathcal{G}) \leq \frac{3n}{2}$. Consequently, $dim_{lf}(\mathcal{G}) = \frac{3n}{2}$.

Case 2. When $m \geq 4$, then the LRNs of $J_{m,n}^{0}$ are of two types: $LR(a_o a_i^j) = V(\mathcal{G}) - \{a_{i-1}^j, a_{i+1}^j\}$ and $LR(a_i^j a_{i+1}^j) = \{a_{i-1}^j, a_i^j, a_{i+1}^j, a_{i+2}^j\}$, with cardinalities $nm - 1$ and 4 respectively. Let us assume that $\alpha_p \in [0,1]$ for $1 \leq p \leq 4$ such that $\sum_{p=1}^{4} \alpha_p = 1$ and $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. Let us define the function $\psi: V(\mathcal{G}) \to [0,1]$ as $\psi(a_0) = 0$ and

$$\psi(a_i^j) = \begin{cases} \alpha_1, & \text{if } i \cong 1(mod4) \\ \alpha_2, & \text{if } i \cong 2(mod4) \\ \alpha_3, & \text{if } i \cong 3(mod4) \\ \alpha_4, & \text{if } i \cong 0(mod4). \end{cases}$$

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

Moreover, if $m \cong 1(mod4)$, then $\psi(a_m^j) = \alpha_4$; if $m \cong 2(mod4)$, then $\psi(a_m^j) = \alpha_4$ and $\psi(a_{m-1}^j) = \alpha_3$; and if $m \cong 3(mod4)$, then $\psi(a_m^j) = \alpha_4$, $\psi(a_{m-1}^j) = \alpha_3$ and $\psi(a_{m-2}^j) = \alpha_2$. Since for each $e \in \mathcal{G}$, $\psi(LR(e)) \geq 1$, therefore, ψ is an LRF of $\mathcal{G}$. Similarly, the function $\phi: V(\mathcal{G}) \to [0,1]$, defined as

$$\phi(v) = \begin{cases} \frac{1}{4}, & \text{if } v \in X \\ 0, & \text{if } v \notin X \end{cases},$$

is also an LRF, where $X = \cup\, LR(a_i^j a_{i+1}^j)$. Furthermore, the LRFs ψ and ϕ are minimal LRFs because, if $g: V(\mathcal{G}) \to [0,1]$ such that $g(v) \leq \psi(v)$ or $g(v) \leq \phi(v)$ and $g(v) \neq \psi(v)$ or $g(v) \neq \phi(v)$ for at least one $v \in V(\mathcal{G})$, then g does not remain as an LRF of $\mathcal{G}$. Now by these minimal LRFs, we have either $|\psi| = \sum_{v \in \mathcal{G}} (\psi(v)) = \frac{mn}{4} \sum_{p=1}^{4} \psi(\alpha_p) = \frac{mn}{4}$ or $|\psi| = \frac{mn}{4} + \beta$ and $|\phi| = \frac{mn}{4}$, where $\beta \in \{\alpha_4, \alpha_4 + \alpha_3, \alpha_4 + \alpha_3 + \alpha_2\}$, which implies that $|\phi| \leq |\psi|$. Thus, $\frac{mn}{4} \leq dim_{lf}(\mathcal{G})$.

Additionally, $|LR(a_o a_i^j)| \geq mn - 1 > 4$, $|LR(a_i^j a_{i+1}^j)| = 4$ and $|LR(a_i^j a_{i+1}^j) \cap Z| \geq 4$, where $Z = \cup\, LR(a_i^j a_{i+1}^j)$. Therefore, by Lemma 1, $dim_{lf}(\mathcal{G}) \leq \frac{mn}{4}$. Thus, $dim_{lf}(\mathcal{G}) = \frac{mn}{4}$. Consequently, from both of the cases, the proof is complete.

Lemma 3. Let $\mathcal{G} \cong J_{m,n}^k$ with $m \geq 4$, $n \geq 1$, $k \geq 2$ and $k \cong 0(mod2)$ be a generalised n-level gear network. Then for each $e \in \mathcal{G}$,

1. $2k + 4 \leq |LR(e)|$,
2. $|LR(e) \cap X| \geq |LR(e^*)|$, where $X = \cup \{LR(e^*): |LR(e^*)| = 2k + 4\}$ for $e^* \in E(\mathcal{G})$.

Proof: 1. There are eight types of LRNs of $\mathcal{G}$:
For $1 \leq i \leq m$, $1 \leq j \leq n$ and $k \geq 2$,

$$LR(a_o a_i^j) = V(\mathcal{G}) - \{b_{\frac{k}{2}+1,j}^i, b_{\frac{k}{2},j}^{i+m-1}\},$$

$$LR(a_i^j b_{1,j}^i) = V(\mathcal{G}) - \left\{b_{\frac{k}{2}+2,j}^i\right\},$$

$$LR(a_i^j b_{k,j}^{i+m-1}) = V(\mathcal{G}) - \left\{b_{\frac{k}{2}-1,j}^{i+m-1}\right\},$$

$$LR\left(b_{\frac{k}{2},j}^i b_{\frac{k}{2}+1,j}^i\right) = \left\{a_i^j, a_{i+1}^j, b_{l,j}^i, b_{z,j}^{i+1}, b_{x,j}^{i+m-1}; 1 \leq l \leq k; 1 \leq z \leq \frac{k}{2} + 1; \frac{k}{2} \leq x \leq k\right\}.$$

For $k \geq 4$,

$$LR\left(b_{\frac{k}{2}-1,j}^i b_{\frac{k}{2},j}^i\right) = V(\mathcal{G}) - \left\{a_{i+1}^j, b_{z,j}^i; 1 \leq z \leq \frac{k}{2}\right\},$$

$$LR\left(b_{\frac{k}{2}+1,j}^i b_{\frac{k}{2}+2,j}^i\right) = V(\mathcal{G}) - \left\{a_i^j, b_{x,j}^i; \frac{k}{2} + 1 \leq x \leq k\right\};$$

and for $k \geq 6$ and $1 \leq h \leq \frac{k-4}{2}$,

$$LR(b_{h,j}^i b_{h+1,j}^i) = V(\mathcal{G}) - \left\{b_{\frac{k}{2}+h+2,j}^i\right\},$$

$$LR\left(b_{\frac{k}{2}+h+1,j}^i b_{\frac{k}{2}+h+2,j}^i\right) = V(\mathcal{G}) - \{b_{h,j}^i\}.$$

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

The cardinalities of all the aforementioned LRNs are listed in Table 1. Accordingly, $|LR(e)| \geq 2k+4 \ \forall e \in E(\mathcal{G})$.

2. We have $|LR(b^i_{\frac{k}{2},j} b^i_{\frac{k}{2}+1,j})| \leq LR(e)$ for each $e \in \mathcal{G}$ (refer to Table 1). Let us take $X = \bigcup_{i=1,j=1}^{m,n} LR(b^i_{\frac{k}{2},j} b^i_{\frac{k}{2}+1,j}) = V(\mathcal{G}) - \{a_o\}$ with $|X| = mn(1+k)$. Now

$$\begin{aligned} |LR(e) \cap X| &= |LR(e) \cap (V(\mathcal{G}) - \{a_o\})| \\ &= |LR(e)| \\ aaaaa &\geq (2k+4)aa \ by \ part \ (1) \\ &= |LR(e^*)| \end{aligned}$$

This completes the proof.

Table 1. Cardinalities of LRN sets of n-level gear graph

LR	Cardinality
$LR(b^i_{\frac{k}{2},j} b^i_{\frac{k}{2}+1,j})$	$\alpha = 2k+4$
$LR(a_o a^j_i)$	$\lvert V(\mathcal{G})\rvert - 2 \geq \alpha$
$LR\left(b^i_{\frac{k}{2}-1,j} b^i_{\frac{k}{2},j}\right)$ and $LR(b^i_{\frac{k}{2}+1,j} b^i_{\frac{k}{2}+2,j})$	$\lvert V(\mathcal{G})\rvert - \frac{k+2}{2} \geq \alpha$
$LR(a^j_i b^i_{1,j})$ and $LR(a^j_i b^{i+m-1}_{k,j})$	$\lvert V(\mathcal{G})\rvert - 1 \geq \alpha$
$LR(b^i_{h,j} b^i_{h+1,j})$ and $LR(b^i_{\frac{k}{2}+h+1,j} b^i_{\frac{k}{2}+h+2,j})$	$\lvert V(\mathcal{G})\rvert - 1 \geq \alpha$

Theorem 2. For $m \geq 3$, $k \geq 1$ and $n \geq 1$, if $\mathcal{G} \cong J^k_{m,n}$ is a generalised n-level gear network, then

$$dim_{lf}(\mathcal{G}) = \begin{cases} \dfrac{mn(1+k)}{2k+4}, & \text{if } k \cong 0(mod2) \\ 1, & \text{if } k \cong 1(mod2) \end{cases}.$$

Proof: Let us consider the following cases.

Case 1. When $k \cong 0(mod2)$, let us define a function $g: V(\mathcal{G}) \to [0,1]$ with the conditions of $0 < \alpha \leq \frac{1}{2k+4}$, $1 \leq i,j,l \leq m,n,k-1$ and $i \cong 1(mod2)$ as follows:

$$g(v) = \text{f(x)} = \begin{cases} \dfrac{1}{2k+4} + \alpha, & \text{if } v \in \{a^j_i, b^i_{l,j}\} \\ \dfrac{1}{2k+4} - \alpha, & \text{if } v \in \{a^j_{i+1}, b^{i+1}_{l+1,j}\}. \\ 0, & \text{if } v = a_o \end{cases}$$

Since $|LR(e)| \geq (2k+4)$ for each $e \in E(\mathcal{G})$ (by Lemma 3), therefore $(LR(e)) = \sum_{v \in LR(e)} g(v) \geq \frac{2k+4}{2}(\frac{1}{2k+4} + \alpha + \frac{1}{2k+4} - \alpha) = 1$, which implies that g is an LRF. Now let us define $\psi: V(\mathcal{G}) \to [0,1]$ such that

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

$$\psi(v) = \begin{cases} \dfrac{1}{2k+4}, & \text{if } v \in X \\ 0, & \text{if } v \notin X \end{cases},$$

where $X = \cup \{LR(e): |LR(e)| = 2k + 4a$ for all $e \in \mathcal{G}$. Let us consider $\psi(LR(e)) = \sum_{v \in LR(e)} \psi(v) = \sum_{v \in LR(e) \cap X} \frac{1}{2k+4} = |LR(e) \cap X| \frac{1}{2k+4} = |LR(e)| \frac{1}{2k+4} \geq 1$.

This shows that ψ is an LRF. Now if we have $f: V(\mathcal{G}) \to [0,1]$ such that $f(v) < g(v)$ and $f(v) < \psi(v)\ \forall\ v \in \mathcal{G}$, then $f(LR(e)) < 1$, for $|LR(e)| = 2k + 4 \Rightarrow f$ is not an LRF. Consequently both the LRFs g and ψ are minimal. Now we discuss the following two possibilities.

(i) If m is even, then $|g| = (\frac{mn(1+k)}{2})(\frac{1}{2k+4} + \alpha) + (\frac{mn(1+k)}{2})(\frac{1}{2k+4} - \alpha) = \frac{mn(1+k)}{2k+4} = |\psi|$.

(ii) If m is odd, then $|g| = (\frac{mn(1+k)}{2k+4}) + (\frac{1}{2k+4} + \alpha)n > |\psi|$. Thus,

$$\frac{mn(1+k)}{2k+4} \leq dim_{lf}(\mathcal{G}).$$

Moreover by Lemma 3, for each $e \in \mathcal{G}$, $|LR(e)| \geq 2k + 4$ and $|LR(e) \cap X| \geq 2k + 4$, where $X = \cup \{LR(e): |LR(e)| = 2k + 4\}$ for $e \in E(\mathcal{G})$. Therefore, Lemma 1 shows that

$$dim_{lf}(\mathcal{G}) \leq \frac{|X|}{2k+4} = \frac{mn(1+k)}{2k+4}.$$

Now from above two equations, $dim_{lf}(\mathcal{G}) = \frac{mn(1+k)}{2k+4}$.

Case 2. When $k \cong 1(mod 2)$, then $\mathcal{G}$ is a bipartite network containing no odd-length cycles. Thus by Lemma 2, $dim_{lf}(\mathcal{G}) = 1$.

Consequently, from both of the cases, the proof is complete.

EXACT CLOSED-FORM LFMD FOR GENERALISED *P*-LEVEL HELM NETWORK

The significant findings on the LFMD of the generalised n-level helm network are presented in this section.

Theorem 3. Let us consider that $\mathcal{G} \cong H_{m,0}^{p}$ be a generalised p-level helm network. Then

$$dim_{lf}(\mathcal{G}) = \begin{cases} \dfrac{3p}{2}, & \text{if } m = 3 \\ \dfrac{mp}{4}, & \text{if } m \geq 4. \end{cases}$$

Proof: We consider the following two cases for the proof of above results.

Case 1. When $m = 3$, then the LRNs of $\mathcal{G}$ are $LR(a_o a_k^j) = V(\mathcal{G}) - \{a_{k+1}^j, a_{k+2}^j, c_{k+1}^j, c_{k+2}^j\}$, $LR(a_k^j a_{k+1}^j) = \{a_k^j, a_{k+1}^j, c_k^j, c_{k+1}^j\}$ and $LR(a_k^j c_k^j) = V(\mathcal{G})$, where $1 \leq k \leq 3$ and $1 \leq j \leq p$. In addition, $|LR(a_o a_k^j)| = 6p - 3, |LR(a_k^j a_{k+1}^j)| = 4, |LR(a_k^j c_k^j)| = 6p + 1$ and $Z = \cup_{k=1,j=1}^{3,p} LR(a_k^j a_{k+1}^j) = V(\mathcal{G}) - \{a_o\}$, which implies that $|Z| = 6p$.

For $\alpha > \frac{1}{4}$, $\beta > \frac{1}{4}$ and $1 \leq j \leq p$, we define a mapping $\psi: V(\mathcal{G}) \to [0,1]$ as

$$\psi(v) = \begin{cases} \alpha, & \text{if } v \in \{a_1^j, a_3^j\} \\ 1 - \alpha, & \text{if } v = a_2^j \\ 0, & \text{if } v = a_o \\ \beta, & \text{if } v \in \{c_1^j, c_3^j\} \\ 1 - \beta, & \text{if } v = c_2^j \end{cases}.$$

Since $\forall\ e \in \mathcal{G}$, $\psi(LR(e)) \geq 1$, therefore ψ is an LRF of $\mathcal{G}$. Similarly, the function $\phi: V(\mathcal{G}) \to [0,1]$, defined as

$$\phi(v) = \begin{cases} \frac{1}{4}, & \text{if } v \in Z \\ 0, & \text{if } v \notin Z \end{cases},$$

is also an LRF with $\phi(LR(e)) \geq 1$ for each $e \in \mathcal{G}$. Additionally, if $g: V(\mathcal{G}) \to [0,1]$ such that $g(v) \leq \psi(v)$ or $g(v) \leq \phi(v)$ and $g(v) \neq \psi(v)$ or $g(v) \neq \phi(v)$ for at least one $v \in V(\mathcal{G})$, then g does not remain as an LRF of $\mathcal{G}$. Consequently, the LRFs ψ and ϕ are minimal. Now let us consider $|\psi| = \sum_{v\in\mathcal{G}} (\psi(v)) = \sum_{k=1,j=1}^{3,p} \psi(a_k^j) + \sum_{k=1,j=1}^{3,p} \psi(c_k^j) = p(2+\alpha+\beta)$ and $|\phi| = \sum_{v\in\mathcal{G}} (\psi(v)) = \sum_{v\in Z} (\psi(v)) = |Z|\frac{1}{4} = \frac{3p}{2}$ such that $|\phi| < |\psi|$ as $\alpha > \frac{1}{4}$.
Thus, ϕ is a minimal LRF of minimum $|\phi|$, which implies that $\frac{3p}{2} \leq dim_{lf}(\mathcal{G})$.

On the other hand, $\mathcal{G}$ has LRs of three types, i.e. $LR(a_o a_k^j)$, $LR(a_k^j a_{k+1}^j)$ and $LR(a_k^j c_k^j)$ with $|LR(a_o a_k^j)| = 6p-3$, $|LR(a_k^j a_{k+1}^j)| = 4$ and $|LR(a_k^j c_k^j)| = 6p+1$ respectively. Additionally, $|LR(a_o a_k^j) \cap Z| \geq |LR(a_k^j a_{k+1}^j)| = 4$ and $|LR(a_k^j c_k^j) \cap Z| \geq |LR(a_k^j a_{k+1}^j)| = 4$. Then by Lemma 1, $dim_{lf}(\mathcal{G}) \leq \frac{3p}{2}$. Consequently, $dim_{lf}(\mathcal{G}) = \frac{3p}{2}$.

Case 2. When $m \geq 4$, then the LRNs of $H_{m,0}^p$ are of three types, i.e. $LR(a_o a_k^j) = V(\mathcal{G}) - \{a_{k+1}^j, a_{k+m-1}^j, c_{k+1}^j, c_{k+m-1}^j\}$, $LR(a_k^j a_{k+1}^j) = \{a_k^j, a_{k+1}^j, a_{k+2}^j, a_{k+m-1}^j, c_k^j, c_{k+1}^j, c_{k+2}^j, c_{k+m-1}^j\}$ and $LR(a_k^j c_k^j) = V(\mathcal{G})$ with $|LR(a_o a_k^j)| = 2mp-3$, $|LR(a_k^j a_{k+1}^j)| = 8$ and $|LR(a_k^j c_k^j)| = 2mp+1$, where $1 \leq k \leq m$ and $1 \leq j \leq p$.

Let us assume that $\alpha_p, \beta_p \in [0,1]$ for $1 \leq p \leq 4$ such that $\sum_{p=1}^4 \alpha_p = \frac{1}{2}$, $\sum_{p=1}^4 \beta_p = \frac{1}{2}$ and $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ &, $\beta_1 < \beta_2 < \beta_3 < \beta_4$. For $1 \leq j \leq p$, we define a mapping $\psi: V(\mathcal{G}) \to [0,1]$ as

$$\psi(a_0) = 0, \psi(a_k^j) = \begin{cases} \alpha_1, & \text{if } k \cong 1(mod4) \\ \alpha_2, & \text{if } k \cong 2(mod4) \\ \alpha_3, & \text{if } k \cong 3(mod4) \\ \alpha_4, & \text{if } k \cong 0(mod4) \end{cases} \text{ and } \psi(c_k^j) = \begin{cases} \beta_1, & \text{if } k \cong 1(mod4) \\ \beta_2, & \text{if } k \cong 2(mod4) \\ \beta_3, & \text{if } k \cong 3(mod4). \\ \beta_4, & \text{if } k \cong 0(mod4) \end{cases}$$

Moreover, if $m \cong 1(mod4)$, then $\psi(a_m^j) = \alpha_4$; if $m \cong 2(mod4)$, then $(a_m^j) = \alpha_4$ and $\psi(a_{m-1}^j) = \alpha_3$; and if $m \cong 3(mod4)$, then $\psi(a_m^j) = \alpha_4$, $\psi(a_{m-1}^j) = \alpha_3$ and $\psi(a_{m-2}^j) = \alpha_2$. Since for each $e \in \mathcal{G}$, $\psi(LR(e)) \geq 1$, therefore ψ is an LRF of $\mathcal{G}$. Similarly, $\phi: V(\mathcal{G}) \to [0,1]$, defined as

$$\phi(v) = \begin{cases} \frac{1}{8}, & \text{if } v \in Z \\ 0, & \text{if } v \notin Z \end{cases},$$

is also an LRF with $\phi(LR(e)) \geq 1\ \forall\ e \in \mathcal{G}$, where $Z = \cup\, LR(a_k^j a_{k+1}^j)$ for $1 \leq k, j \leq m, p$.

Furthermore, the LRFs ψ and ϕ are minimal LRFs because, if $g: V(\mathcal{G}) \to [0,1]$ such that $g(v) \leq \psi(v)$ or $g(v) \leq \phi(v)$ and $g(v) \neq \psi(v)$ or $g(v) \neq \phi(v)$ for at least one $v \in V(\mathcal{G})$, then g does not remain as an LRF of $\mathcal{G}$. Now by these minimal LRFs, we have either $|\psi| = \sum_{v\in\mathcal{G}} (\psi(v)) = \frac{mn}{4}[\sum_{p=1}^4 \psi(\alpha_p) + \sum_{p=1}^4 \psi(\beta_p)] = \frac{mn}{4}$ or $|\psi| = \frac{mn}{4} + \gamma_1 + \gamma_2$ and $|\phi| = \frac{mn}{4}$,

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

where $\gamma_1 \in \{\alpha_4, \alpha_4 + \alpha_3, \alpha_4 + \alpha_3 + \alpha_2\}$ and $\gamma_2 \in \{\beta_4, \beta_4 + \beta_3, \beta_4 + \beta_3 + \beta_2\}$, which implies that $|\phi| \leq |\psi|$. Thus, $\frac{mp}{4} \leq dim_{lf}(\mathcal{G})$.

Additionally, $|LR(a_o a_k^j)| = 2mp - 3 > 8$, $|LR(a_k^j a_{k+1}^j)| = 8$ and $|LR(a_k^j c_k^j)| = 2mp + 1 \geq 8$, $|LR(a_k^j a_{k+1}^j) \cap Z| \geq 8$, where $Z = \cup\, LR(a_k^j a_{k+1}^j)$ for $1 \leq k, j \leq m, p$. Therefore by Lemma 1, $dim_{lf}(\mathcal{G}) \leq \frac{mn}{4}$. Thus, $dim_{lf}(\mathcal{G}) = \frac{mn}{4}$. Consequently, from both the cases, the proof is complete.

Lemma 4. Let $\mathcal{G} \cong H_{m,q}^p$ with $m \geq 3$, $p \geq 1$, $q \geq 2$ and $k \cong 0(mod2)$ be a generalised p-level helm network. Then for each $e \in \mathcal{G}$,

1. $2q + 6 \leq |LR(e)|$,
2. $|LR(e) \cap Z| \geq |LR(e^*)|$, where $Z = \cup \{LR(e^*): |LR(e^*)| = 2q + 6\}$ for $e^* \in E(\mathcal{G})$.

Proof: 1. There are nine types of LRNs of $\mathcal{G}$:

For $1 \leq K \leq m$, $1 \leq j \leq P$ & $q \geq 2$,

$$LR\left(a_o a_k^j\right) = V(\mathcal{G}) - \left\{b_{\frac{q}{2}+1,j}^i, b_{\frac{q}{2},j}^{k+m-1}\right\},$$

$$LR\left(a_k^j b_{q-2t-3,j}^k\right) = V(\mathcal{G}) - \left\{b_{\frac{q}{2}+2,j}^k\right\},$$

$$LR\left(a_k^j b_{q,j}^{k+m-1}\right) = V(\mathcal{G}) - \left\{b_{\frac{q}{2}-1,j}^{k+m-1}\right\},$$

$$LR\left(a_k^j c_k^j\right) = V(\mathcal{G})$$

$$LR(b_{\frac{q}{2},j}^k b_{\frac{q}{2}+1,j}^k) = \{a_k^j, a_{k+1}^j, b_{l,j}^k, b_{z,j}^{k+1}, b_{x,j}^{k+m-1}, c_k^j, c_{k+1}^j; 1 \leq l \leq q; 1 \leq z \leq \tfrac{q}{2} + 1; \tfrac{q}{2} \leq x \leq q\}.$$

For $q \geq 4$,

$$LR(b_{\frac{q}{2}-1,j}^k b_{\frac{q}{2},j}^k) = V(\mathcal{G}) - \{a_{k+1}^j, b_{z,j}^k, c_{k+1}^j; 1 \leq z \leq \tfrac{q}{2}\},$$

$$LR\left(b_{\frac{q}{2}+1,j}^k b_{\frac{q}{2}+2,j}^k\right) = V(\mathcal{G}) - \{a_k^j, b_{x,j}^k, c_k^j; \tfrac{q}{2} + 1 \leq x \leq q\};$$

and for $q \geq 6$ and $1 \leq h \leq \frac{q-4}{2}$,

$$LR\left(b_{h,j}^k b_{h+1,j}^k\right) = V(\mathcal{G}) - \{b_{\frac{q}{2}+h+2,j}^k\},$$

$$LR\left(b_{\frac{q}{2}+h+1,j}^k b_{\frac{q}{2}+h+2,j}^k\right) = V(\mathcal{G}) - \{b_{h,j}^k\}.$$

The cardinalities of all the aforesaid LRNs are listed in Table 2. According to this, $|LR(e)| \geq 2q + 6\ \forall\ e \in E(\mathcal{G})$.

2. We have $|LR(b_{\frac{q}{2},j}^k b_{\frac{q}{2}+1,j}^k)| \leq LR(e)$ for each $e \in \mathcal{G}$ (refer to Table 2). Let us consider $Z = \cup_{k=1,j=1}^{m,p} LR(b_{\frac{q}{2},j}^k b_{\frac{q}{2}+1,j}^k)) = V(\mathcal{G}) - \{a_o\}$ with $|Z| = mp(q + 2)$. Now

$$\begin{aligned} |LR(e) \cap Z| &= |LR(e) \cap (V(\mathcal{G}) - \{a_o\})| \\ &= |LR(e)| \\ aaaaa &\geq (2q + 6) aa\, by\ (1) \\ &= |LR(e^*)|. \end{aligned}$$

Hence the proof is complete.

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

Table 2. Cardinalities of LRN sets of p-level helm graph

LR	Cardinality
$LR(b^k_{\frac{q}{2},j}b^k_{\frac{q}{2}+1,j})$	$\alpha = 2q+6$
$LR(a_o a^j_k)$	$\lvert V(\mathcal{G})\rvert - 2 \geq \alpha$
$LR(b^k_{\frac{q}{2}-1,j}b^k_{\frac{q}{2},j})$, $LR(b^k_{\frac{q}{2}+1,j}b^k_{\frac{q}{2}+2,j})$	$\lvert V(\mathcal{G})\rvert - (\frac{q+2}{2}) \geq \alpha$
$LR(a^j_k b^k_{q-2t-3,j})$, $LR(a^j_k b^{k+m-1}_{q,j})$	$\lvert V(\mathcal{G})\rvert - 1 \geq \alpha$
$LR(b^k_{h,j}b^k_{h+1,j})$, $LR(b^k_{\frac{q}{2}+h+1,j}b^k_{\frac{q}{2}+h+2,j})$	$\lvert V(\mathcal{G})\rvert - 1 \geq \alpha$
$LR(a^j_k c^j_k)$	$V(\mathcal{G})$

Theorem 4. For $m \geq 3$, $q \geq 1$ and $p \geq 1$, if $\mathcal{G} \cong H^p_{m,q}$ is a generalised p-level helm network, then

$$dim_{lf}(\mathcal{G}) = \begin{cases} \dfrac{mp(q+2)}{2q+6}, & \text{if } q \cong 0(mod2) \\ 1, & \text{if } q \cong 1(mod2). \end{cases}$$

Proof: Let us consider the following cases.

Case 1. When $q \cong 0(mod2)$, then we define $g: V(\mathcal{G}) \to [0,1]$ with the conditions of $0 < \alpha \leq \frac{1}{2q+6}$, $1 \leq k, j, l \leq m, p, q-1$ and $k \cong 1(mod2)$ as follows:

$$g(v) = \begin{cases} \dfrac{1}{2q+6} + \alpha, & \text{if } v \in \{a^j_k, b^k_{l,j}\} \\ \dfrac{1}{2q+6} - \alpha, & \text{if } v \in \{a^j_{k+1}, b^{k+1}_{l+1,j}\} \\ 0, & \text{if } v = a_o \end{cases}.$$

Since $|LR(e)| \geq (2q+6)$ for each $e \in E(\mathcal{G})$ (by Lemma 4), therefore $g(LR(e)) = \sum_{v \in LR(e)} g(v) \geq (q+3)(\frac{1}{2q+6} + \alpha + \frac{1}{2q+6} - \alpha) = 1$, which implies that g is an LRF. Now we define another mapping $\psi: V(\mathcal{G}) \to [0,1]$ such that

$$\psi(v) = \begin{cases} \dfrac{1}{2q+6}, & \text{if } v \in Z \\ 0, & \text{if } v \notin Z \end{cases},$$

where $Z = \cup \{LR(e): |LR(e)| = 2q + 6a \text{ and } ae \in \mathcal{G}\}$. Consider $\psi(LR(e)) = \sum_{v \in LR(e)} \psi(v) = \sum_{v \in LR(e) \cap Z} \frac{1}{2q+6} = |LR(e) \cap Z| \frac{1}{2q+6} = |LR(e)| \frac{1}{2q+6} \geq 1$. This shows that ψ is an LRF. Let us consider that if $f: V(\mathcal{G}) \to [0,1]$ such that $f(v) < g(v)$ and $f(v) < \psi(v)$ for each $v \in \mathcal{G}$, then $f(LR(e)) < 1$, for $|LR(e)| = 2q+6 \Rightarrow f$ is not an LRF. Consequently, both the LRFs g and ψ are minimal LRFs.

Now we discuss the following two possibilities.

(i) If m is even, then $|g| = (\frac{mp(q+2)}{2})(\frac{1}{2q+6} + \alpha) + (\frac{mp(q+2)}{2})(\frac{1}{2q+6} - \alpha) = \frac{mp(q+2)}{2q+6} = |\psi|$.

(ii) If m is odd, then $|g| = (\frac{mp(q+2)}{2q+6}) + (\frac{1}{2q+6} + \alpha)p > |\psi|$. Thus,

$$\frac{mp(q+2)}{2q+6} \leq dim_{lf}(\mathcal{G}).$$

Moreover, by Lemma 4, $\forall\ e \in \mathcal{G}$, $|LR(e)| \geq 2q+6$ and $|LR(e) \cap Z| \geq 2q+6$, where $Z = \cup \{LR(e): |LR(e)| = 2q+6\}$ for $e \in E(\mathcal{G})$. Therefore Lemma 1 shows that

$$dim_{lf}(\mathcal{G}) \leq \frac{|Z|}{2k+4} = \frac{mp(q+2)}{2q+6}.$$

Now from the above two equations, $dim_{lf}(\mathcal{G}) = \frac{mp(q+2)}{2q+6}$.

Case 2. When $q \cong 1(mod2)$, then $\mathcal{G}$ is a bipartite network with no cycle of odd length. Thus, by Lemma 2, $dim_{lf}(\mathcal{G}) = 1$.

Consequently, from both of the cases, the proof is complete.

APPLICATION: EMERGENCY EVACUATION OPTIMISATION

In this section we present an application of the obtained results. To illustrate these findings, we consider an emergency escape route within a building. We focus on one floor that includes stores, restrooms, food corners, juice shops, ice cream stalls, lifts and escalators as depicted in Figures 3 and 4. This floor layout corresponds to the network $J^4_{6,2}$. From Theorem 3, the LFMD of $J^4_{6,2}$ is calculated as $\frac{60}{12} = 5$, which indicates that complete evacuation in the event of an emergency will take approximately five minutes.

This study demonstrates how computing the LFMD can greatly enhance the design and optimisation of network structures such as emergency exit plans. The findings provide valuable insights for improving network efficiency and guide decision-making in practical applications, particularly in safety and communication systems.

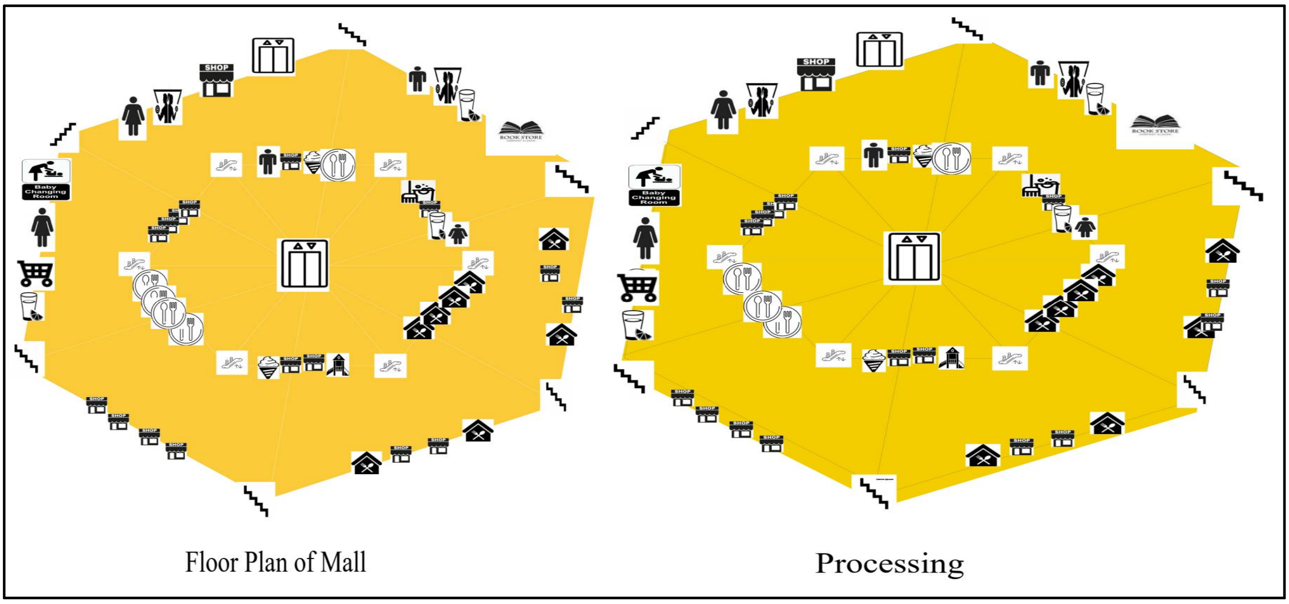
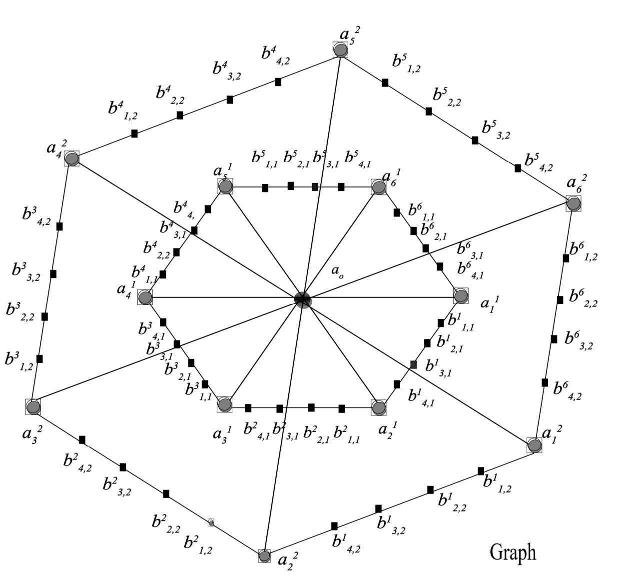


Figure 3. Emergency exit plan model (Left: simple model and Right: processing model with links (edges))

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195



(a)

Shops	$b^2_{2,1}$, $b^2_{3,1}$, $b^4_{1,1}$, $b^4_{4,1}$, $b^5_{2,1}$, $b^1_{2,2}$, $b^1_{3,2}$, $b^6_{4,2}$, $b^2_{1,2}$, $b^2_{4,2}$, $b^2_{3,2}$, $b^6_{2,2}$, $b^6_{3,2}$
Stairs	a^2_1, a^2_2, a^2_3, a^2_4, a^2_5, a^2_6
Lift	a_0, $b^4_{4,2}$
Food corners	$b^1_{1,1}$, $b^1_{2,1}$, $b^1_{3,1}$, $b^1_{4,1}$, $b^1_{1,2}$, $b^1_{4,2}$, $b^6_{1,2}$, $b^6_{4,2}$
Washrooms	$b^5_{1,1}$, $b^6_{4,1}$, $b^3_{3,2}$, $b^4_{1,2}$, $b^5_{1,2}$, $b^3_{4,2}$
Cleaning room	$b^6_{1,1}$
Juice corner	$b^6_{3,1}$, $b^3_{1,2}$, $b^5_{3,2}$
Playing area	$b^2_{1,1}$
Grocery store	$b^3_{2,2}$
Restaurants	$b^3_{1,1}$, $b^3_{2,1}$, $b^3_{3,1}$, $b^3_{4,1}$, $b^5_{4,1}$
Book store	$b^5_{4,2}$
Gym	$b^5_{2,2}$, $b^4_{2,2}$
Ice-cream corner	$b^2_{4,1}$, $b^5_{3,1}$
Escalators	a^1_1, a^1_2, a^1_3, a^1_4, a^1_5, a^1_6

(b)

Figure 4. (a) Emergency exit plan network (converted from Figure 3); (b) Corresponding vertices-labelling

CONCLUSIONS

In this work LFMDs of two non-planar networks have been computed in the form of exact values rather than their lower and upper bounds. This study is summarised as follows:

- Tables 1 and 2 show the cardinalities of LRNs of generalised n-level gear and p-level helm networks.
- In the case of bipartite networks, both the non-planar networks have the same LFMD which is exactly 1 with no impact from the addition of pendant vertices.
- In the case of non-bipartite networks, by taking the same number of levels n, the same number of vertices adjacent to the central vertex at each level (m), and the same subdivided vertices on each cycle edge k, we have $dim_{lf}(H^n_{m,k}) - dim_{lf}(J^k_{m,n}) = \frac{mn}{2(k+2)(k+3)} > 0$, which implies that $dim_{lf}(H^n_{m,k}) > dim_{lf}(J^k_{m,n})$. This shows that the addition of pendant vertices enhances the LFMD for the helm network.
- For $k \to \infty$, the asymptotic behaviour shows that both the networks have the same LFMD. Thus, the numerical value of this error approaches 0 as k approaches infinity.

This study's outcomes advance the theoretical understanding of fractional dimensions in symmetric networks while providing practical insights for network design and optimisation. The findings are particularly relevant to applications in communication networks, transportation systems and facility layout planning, where efficient node distinguishability is crucial.

ACKNOWLEDGMENTS

This research was supported by the Princess Nourah bint Abdulrahman University Researchers Supporting Project, PNURSP2025R259, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

REFERENCES

1. J. L. Gross and J. J. Yellen, "Graph Theory and Its Applications", 2nd Edn., *CRC Press*, Boca Raton, **2005**.
2. P. J. Slater, "Leaves of trees", *Congr. Numer.*, **1975**, *14*, 549-559.
3. F. Harary and R. A. Melter, "On the metric dimension of a graph", *Ars Comb.*, **1976**, *2*, 191-195.
4. G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph", *Discrete. Appl. Math.*, **2000**, *105*, 99-113.
5. S. Khuller, B. Raghavachari and A. Rosenfeld, "Landmarks in graphs", *Discrete Appl. Math.*, **1996**, *70*, 217-229.
6. A. Sebő and E. Tannier, "On metric generators of graphs", *Math. Oper. Res.*, **2024**, *29*, 383-393.
7. J. B. Liu, M. F. Nadeem, H. M. A. Siddiqui and W. Nazir, "Computing metric dimension of certain families of toeplitz graphs", *IEEE Access*, **2019**, *7*, 126734-126741.
8. M. Ali, G. Ali, M. Imran, A. Q. Baig and M. K. Shafiq, "On the metric dimension of Möbius ladders", *ARS Comb.*, **2012**, *105*, 403-410.
9. S. Imran, M. K. Siddiqui, M. Imran, M. Hussain, H. M. Bilal, I. Z. Cheema, A. Tabraiz and Z. Saleem, "Computing the metric dimension of gear graphs", *Symmetry*, **2018**, *10*, Art.no.209.
10. H. Raza, S. Hayat and X. F. Pan, "On the fault-tolerant metric dimension of certain interconnection networks", *J. Appl. Math. Comput.*, **2019**, *60*, 517-535.
11. A. N. A. Koam, A. Ahmad, S. Husain and M. Azeem, "Mixed metric dimension of hollow coronoid structure", *Ain Shams Eng. J.*, **2023**, *14*, Art.no.102000.
12. A. N. A. Koam, A. Ahmad, M. S. Alatawi, M. Azeem and M. F. Nadeem, "Metric basis of four-dimensional Klein bottle", *Comput. Model. Eng. Sci.*, **2023**, *136*, 3011-3024.
13. S. Abbas, Z. Raza, N. Siddiqui, F. Khan and T. Whangbo, "Edge metric dimension of honeycomb and hexagonal networks for IoT", *Comput. Mater. Contin.*, **2021**, *71*, 2683-2695.
14. Z. Raza and M. S. Bataineh, "The comparative analysis of metric and edge metric dimension of some subdivisions of the wheel graph", *Asian-Eur. J. Math.*, **2021**, *14*, Art.no.2150062.
15. J. Currie and O. R. Oellerman, "The metric dimension and metric independence of a graph", *J. Comb. Math. Comb. Comput.*, **2001**, *39*, 157-167.
16. S. Arumugam, V. Mathew and J. Shen, "On fractional metric dimension of graphs", *Discrete Math. Algor. Appl.*, **2013**, *5*, Art.no.1350037.
17. M. Raza, M. Javaid and N. Saleem, "Fractional metric dimension of metal-organic frameworks", *Main Group Metal Chem.*, **2001**, *44*, 92-102.
18. M. Feng and K. Wang, "On the metric dimension and fractional metric dimension of the hierarchical product of graphs", *Appl. Anal. Discrete Math.*, **2013**, *7*, 302-313.
19. M. Feng, B. Lv and K. Wang, "On the fractional metric dimension of graphs", *Discrete Appl. Math.*, **2014**, *170*, 55-63.
20. J. B. Liu, A. Kashif, T. Rashid and M. Javaid, "Fractional metric dimension of generalized Jahangir graph", *Math.*, **2019**, *7*, Art.no.100.
21. S. W. Saputro, A. Semaničová-Feňovčíková, M. Bača and M. Lascsáková, "On fractional metric dimension of comb product graphs", *Stat. Optim. Inform. Comput.*, **2018**, *6*, 150-158.

Maejo Int. J. Sci. Technol. **2025**, *19*(02), 181-195

22. Z. S. Mufti, M. F. Nadeem, A. Ahmad and Z. Ahmad, "Computation of edge metric dimension of barycentric subdivision of Cayley graphs", *Italalian J. Pure Appl. Math.*, **2020**, *44*, 714-722.
23. I. Javaid, H. Benish and M. Murtaza, "The fractional local metric dimension of graphs", *Contrib. Discrete Math.*, **2024**, *19*, 163-177.
24. M. Javaid, H. Zafar, Q. Zhu and A. M. Alanazi, "Improved lower bound of LFMD with applications of prism-related networks", *Math. Problems Eng.*, **2021**, *2021*, Art.no.9950310.
25. H. Zafar, M. Javaid and M. A. Ashebo, "Distance-based fractional dimension of certain wheel networks", *J. Math.*, **2024**, *2024*, Art.no.870335.
26. M. Javaid, M. Raza, P. Kumam and J.-B. Liu, "Sharp bounds of local fractional metric dimensions of connected networks", *IEEE Access*, **2020**, *8*, 172329-172342.
27. S. Aisyah, M. I. Utoyo and L. Susilowati, "On the local fractional metric dimension of corona product graphs", *IOP Conf. Ser. Earth Environ. Sci.*, **2019**, *243*, Art.no.012043.
28. A. Alamer, H. Zafar and M. Javaid, "Study of modified prism networks via fractional metric dimension", *AIMS Math.*, **2023**, *8*, 10864-10886.
29. S. Ali, R. Ismail, F. J. H. Campena, H. Karamti and M. U. Ghani, "On rotationally symmetrical planar networks and their local fractional metric dimension", *Symmetry*, **2023**, *15*, Art.no.530.
30. R. Ismail, M. Javaid and H. Zafar, "Metric-based fractional dimension of rotationally-symmetric line networks", *Symmetry*, **2023**, *15*, Art.no.1069.
31. I. Ali, M. Javaid and S. A. Fufa, "Studies of multilevel networks via fault-tolerant metric dimensions", *IEEE Access*, **2022**, *10*, 88226-88233.
32. A. Fatima, M. Javaid and M. A. Ashebo, "Metric dimension of nonplanar networks by fractional technique with application", *IEEE Access*, **2024**, *12*, 195918-195925.
33. M. Javaid, H. Zafar and E. Bonyah, "Fractional metric dimension of generalized sunlet networks", *J. Math.*, **2021**, *2021*, Art.no.4101869.
34. S. Ali, R. M. Falcón and M. K. Mahmood, "Local fractional metric dimension of rotationally symmetric planar graphs arisen from planar chorded cycles", *Rend. Mat. Appl.*, **2023**, *44*, 159-179.
35. M. Kairanbay and H. M. Jani, "A review and evaluations of shortest path algorithms", *Int. J. Sci. Technol. Res.*, **2013**, *2*, 99-104.
36. D. B. West, "Introduction to Graph Theory", 2nd Edn., Prentice Hall, Upper Saddle River, **2001**, pp.82-83