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Full Paper

Probability weighted moments and family of non-parametric regression estimators

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Abstract: Regression analysis plays a significant role in statistics by identifying the relationship between variables. However, when the assumptions of ordinary least squares are violated, non-parametric regression becomes a preferable approach. In the field of non-parametric regression, Nadaraya-Watson (NW) kernel regression estimators have gained popularity in recent decades. The adaptive NW kernel regression estimator is considered one of the best and most effective estimators for non-parametric regression due to its varying bandwidth. The effective utilisation of the bandwidth factor is a key aspect of kernel regression. This article proposes the use of probability-weighted moments (PWMs) to enhance the bandwidth factor of the kernel regression estimator. The novelty of this approach lies in replacing traditional measures of central tendency and dispersion with PWMs to introduce a new family of NW kernel regression estimators that are more robust to outliers. Monte Carlo simulation studies are conducted to compare the performance of existing and proposed kernel regression estimators. The simulations use a data set of COVID-19 cases from Africa, highlighting the severity of the current pandemic. The results of the simulations demonstrate that the proposed family of estimators is more robust than existing estimators.

Keywords: probability weighted moments, non-parametric regression estimators, NW kernel estimator, measures of central tendency, percentage relative efficiency

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INTRODUCTION

Before the development of non-parametric statistics, scientists in various fields preferred to use parametric tests for analysis as they were more predictive than non-parametric statistics, especially in regression analysis. This is why science and engineering tend to use parametric statistics because they are predictive. However, parametric statistics are not applicable when ordinary least square assumptions are not fulfilled. In cases where the data are scattered delicately, non-parametric statistics provide a better solution.

The demand for kernel regression estimators is growing rapidly, but it requires more flexible estimators that offer greater accuracy for analysing and inferring results. The method introduced by Nadaraya and Watson [1] in non-parametric regression is primarily based on empirical data with a very large sample size. The descriptive variable's observation is measured by a smoothing parameter. The Nadaraya-Watson (NW) kernel estimator is one of the leading non-parametric regression estimation methods. This estimator provides a precise measurement of observations and the regression curve based on the observed data.

Mathematically, for the given data $\{X_i, Y_i\}_{i=1}^{n}$, consider the regression model $Y = m(x) + \varepsilon$ with observation errors ε_i and unknown regression function m, where the response random variable Y depends on an independent random variable X, and ε is a random variable with mean 0 and variance σ^2. It is widely known that $m(x) = \int \frac{yf(x,y)}{f(x)} dy$ is a conditional mean curve that depends on the joint density function of X and Y $(f(x, y))$ and the marginal function of X $(f(x))$, and it can be estimated by employing the kernel probability density $(K(.))$ for all values of X as

$$\widehat{m}_{\phi 0}(x) = \frac{\sum_{i=1}^{n} y_i K\left(\frac{x-X_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)} \quad . \tag{1}$$

The NW kernel estimators in Eq. (1) depend on a parameter (h) known as the bandwidth. This parameter controls the extent of curve smoothing. This idea is further elaborated by Wand and Jones [2] for a large bandwidth h that can be fixed or varying and estimated as a smooth density function.

The selection of the optimal bandwidth is important. The mean integrated square error is minimised by using the optimal bandwidth, which can be found by integrating the mean square error (MSE). Friedman and Stuetzle [3] suggested a method of finding the parameters of a non-parametric regression model without knowing the bandwidth h. There are different methods for the selection of the bandwidth h described by many authors such as Duin [4], Rudemo [5], Silverman [6], Wand and Jones [2], Härdle et al. [7] and Slaoui [8]. The flexible bandwidth is suggested rather than a fixed bandwidth in the situation of a long-tailed curve.

Abramson [9] suggested a method of inverse square root for selecting the variable kernel density bandwidth, which decreases bias more effectively than the fixed bandwidth estimator. Silverman [6] discussed the assumptions of kernel weights and the properties of the estimator such as mean and variance.

Demir and Toktamis [10] implemented an adaptive NW kernel regression estimator for estimating the regression function. Their simulation studies demonstrated that the NW estimator yielded improved results when using the arithmetic mean instead of the geometric mean to evaluate the local bandwidth factor. Aljuhani and Al-Turk [11] proposed using the range of the probability density function to modify the bandwidth factor h.

In the adaptive NW kernel estimator, Ali [12] introduced some modifications. He proposed incorporating trimmed mean, median and harmonic mean in the bandwidth factor of the NW estimator instead of arithmetic mean and geometric mean. Based on the results of a simulation study, he concluded that using these measures in the bandwidth factor h of the modified estimator, especially the harmonic mean, improved the estimator's performance. To support his argument, he evaluated the MSE in a simulation study and found that the MSE of the proposed estimators decreased with these innovations. The suggested bandwidth factors by Ali [12] are listed in Table 1.

These kernel regression estimators have been suggested in the last few decades, inspired by the NW kernel regression model. The generalised form of the NW kernel regression function $\hat{m}_{\phi t}(x)$ is given by

$$\hat{m}_{\phi t}(x) = \frac{\sum_{i=1}^{n} \frac{y_i}{d(x_i)} K\left(\frac{x-X_i}{d(x_i)}\right)}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{d(x_i)}\right)} \quad \text{for } t = 1,2,\dots,7 \tag{2}$$

Many authors have extended this idea. The detailed expression of Eq. (2) and its extensions are given below in Table 1, where the local bandwidth factors $d(x_i)$ can be determined by employing different types of λ's.

Table 1. Some family members of $\hat{m}_{\phi t}(x)$

$\hat{m}_{\phi t}(x)$	$d(x_i)$	λ	Author
$\hat{m}_{\phi 1}(x)$	$d(x_1) = \frac{h}{\lambda}$	$\lambda = \frac{1}{\sqrt{\hat{f}_d(x)}}$	Abramson [9]
$\hat{m}_{\phi 2}(x)$	$d(x_2) = h\lambda_{GM}$	$\lambda_{GM} = \left[\frac{\bar{f}(x_i)}{GM}\right]^{-\beta}$	Silverman [6]
$\hat{m}_{\phi 3}(x)$	$d(x_3) = h\lambda_{AM}$	$\lambda_{AM} = \left[\frac{\bar{f}(x_i)}{AM}\right]^{-\beta}$	Demir and Toktamis [10]
$\hat{m}_{\phi 4}(x)$	$d(x_4) = h\lambda_{R}$	$\lambda_{R} = \left[\frac{\bar{f}(x_i)}{R}\right]^{-\beta}$	Aljuhani and Al-Turk [11]
$\hat{m}_{\phi 5}(x)$	$d(x_5) = h\lambda_{Tr}$	$\lambda_{Tr} = \left[\frac{\bar{f}(x_i)}{Tr}\right]^{-\beta}$	Ali [12]
$\hat{m}_{\phi 6}(x)$	$d(x_6) = h\lambda_{MD}$	$\lambda_{MD} = \left[\frac{\bar{f}(x_i)}{MD}\right]^{-\beta}$	
$\hat{m}_{\phi 7}(x)$	$d(x_7) = h\lambda_{HM}$	$\lambda_{HM} = \left[\frac{\bar{f}(x_i)}{HM}\right]^{-\beta}$	

Taking inspiration from Ali [12], we introduce the concept of probability weighted moments (PWMs) to the bandwidth factor of the NW estimator as PWMs are more robust in the presence of outliers.

In most cases the data do not fulfill the conditions of normality when they are collected through experiments or different procedures. In such cases non-parametric kernel regression becomes more effective. The kernel regression estimators (NW and its adaptive version) provide a robust solution in such situations. The NW estimator uses a fixed bandwidth while the adaptive version relies on a varying bandwidth.

In this manuscript some new versions of the NW kernel regression estimator are proposed by utilising PWMs. The NW kernel regression approach is a good choice when ordinary least square assumptions fail. By using these PWMs instead of traditional measures of central tendency and dispersion, the estimator is tuned more efficiently and has proven to be more robust against data with outliers.

The PWMs were suggested by Greenwood et al. [13] and were generally used to estimate parameters. According to the classical approach of PWMs ($E[X(1-F)^m]$ or $E[XF^s]$), F is the cumulative distribution function of a random variable X, where the values of m and s should be small, non-negative integers. Usually, they can be either 0, 1 or 2, depending on the parameters being estimated. Meintanis and Ushakov [14] developed a probability-weighted empirical characteristic function for weighted distributions. Caeiro and Mateus [15] defined some new estimators for the parameters of the Pareto distribution based on PWMs. Rasmussen [16] proposed a generalised method of PWMs as an extended class of PWMs with no restrictions on the selection of m and s. The purpose of implementing PWMs in this article is based on the special characteristics of PWMs as outlined below:

- PWMs are less sensitive to outliers in the data set.
- PWMs can effectively summarise and describe the observed data set.
- PWMs are used for non-parametric estimation of the observed data sample.

There are several approaches for estimation as mentioned earlier. However, the significance and novelty of the current paper lie in utilising PWMs in the bandwidth factor of the NW kernel regression function, which is less sensitive to outliers. By using PWMs, a new family of estimators is obtained.

PROPOSED KERNEL REGRESSION ESTIMATORS

PWMs

Outliers can introduce significant bias in regression analysis, particularly affecting bandwidth-dependent estimators. Therefore, estimators are more reliable when they can accommodate the entire data set including outliers, and when their results are not influenced by the presence of outliers. PWMs are more robust in the presence of extreme observations as they are less sensitive to outliers and provide more effective estimates.

Population PWMs are represented by α and β while sample PWMs $\acute{a}_p$ and $\acute{b}_p$ are given below:

$$\acute{a}_p \equiv n^{-1}\sum_{j=1}^{n}\frac{\binom{n-j}{p}x_j}{\binom{n-1}{p}} \text{ for } p = 0,1,\dots,n-1,$$

$$\acute{b}_p \equiv n^{-1}\sum_{j=1}^{n}\frac{\binom{j-1}{p}x_j}{\binom{n-1}{p}} \text{ for } p = 0,1,\dots,n-1,$$

$$\acute{a}_p \equiv \begin{cases} n^{-1}\sum_{j=1}^{n}x_j & \text{for } p=0 \\ n^{-1}\sum_{j=1}^{n}\frac{(n-j)\,x_j}{(n-1)} & \text{for } p=1 \\ n^{-1}\sum_{j=1}^{n}\frac{\binom{n-j}{2}x_j}{\binom{n-1}{2}} & \text{for } p=2 \\ n^{-1}\sum_{j=1}^{n}\frac{\binom{n-j}{3}x_j}{\binom{n-1}{3}} & \text{for } p=3 \end{cases}, \quad \acute{b}_p \equiv \begin{cases} n^{-1}\sum_{j=1}^{n}x_j & \text{for } p=0 \\ n^{-1}\sum_{j=1}^{n}\frac{(j-1)\,x_j}{(n-1)} & \text{for } p=1 \\ n^{-1}\sum_{j=1}^{n}\frac{\binom{j-1}{2}x_j}{\binom{n-1}{2}} & \text{for } p=2 \\ n^{-1}\sum_{j=1}^{n}\frac{\binom{j-1}{3}x_j}{\binom{n-1}{3}} & \text{for } p=3 \end{cases}$$

The presence of outliers in the data is a well-known problem in estimation. Outliers significantly influence the results, making it difficult for researchers to derive valuable conclusions

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from such data. The NW and adaptive NW kernel regression estimators become more appealing in such situations as they play a key role in handling these problems. PWMs are also more effective for outliers. Therefore, incorporating PWMs in the bandwidth factor of the NW kernel regression estimator remarkably improves the performance of the estimator. For more detailed information about robustness and PWMs, interested readers are referred to the Rasmussen [16] and Shahzad et al. [17].

Family of NW Kernel Regression Estimators Utilising PWMs

A new family of NW kernel regression estimators is introduced here by incorporating concepts from Ali [12]. The proposed family of NW estimators is defined by modifying the bandwidth factor as follows:

$$d(x_{a0}) = h\lambda_{\acute{g}a} \quad \text{where} \quad \lambda_{\acute{g}a} = \left[\frac{f(x_i)}{\acute{a}_0}\right]^{-\alpha}$$

$$d(x_{a1}) = h\lambda_{\acute{g}b} \quad \text{where} \quad \lambda_{\acute{g}b} = \left[\frac{f(x_i)}{\acute{a}_1}\right]^{-\alpha}$$

$$d(x_{a2}) = h\lambda_{\acute{g}c} \quad \text{where} \quad \lambda_{\acute{g}c} = \left[\frac{f(x_i)}{\acute{a}_2}\right]^{-\alpha}$$

$$d(x_{a3}) = h\lambda_{\acute{g}d} \quad \text{where} \quad \lambda_{\acute{g}d} = \left[\frac{f(x_i)}{\acute{a}_3}\right]^{-\alpha}.$$

Similarly,

$$d(x_{b0}) = h\lambda_{\acute{v}a} \quad \text{where} \quad \lambda_{\acute{v}a} = \left[\frac{f(x_i)}{\acute{b}_0}\right]^{-\beta},$$

$$d(x_{b1}) = h\lambda_{\acute{v}b} \quad \text{where} \quad \lambda_{\acute{v}b} = \left[\frac{f(x_i)}{\acute{b}_1}\right]^{-\beta},$$

$$d(x_{b2}) = h\lambda_{\acute{v}c} \quad \text{where} \quad \lambda_{\acute{v}c} = \left[\frac{f(x_i)}{\acute{b}_2}\right]^{-\beta},$$

$$d(x_{b3}) = h\lambda_{\acute{v}d} \quad \text{where} \quad \lambda_{\acute{v}d} = \left[\frac{f(x_i)}{\acute{b}_3}\right]^{-\beta}.$$

Hence the proposed estimators are expressed in their particular forms in equations (3-10):

$$\widehat{m}_{ga}(x) = \frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{g}a}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}a}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{g}a}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}a}}\right)} \tag{3}$$

$$\widehat{m}_{gb}(x) = \frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{g}b}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}b}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{g}b}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}b}}\right)} \tag{4}$$

$$\widehat{m}_{gc}(x) = \frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{g}c}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}c}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{g}c}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}c}}\right)} \tag{5}$$

$$\widehat{m}_{gd}(x) = \frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{g}d}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}d}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{g}d}}K\left(\frac{x-X_i}{h\lambda_{\acute{g}d}}\right)} \tag{6}$$

$$\widehat{m}_{va}(x) = \frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{v}a}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}a}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{v}a}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}a}}\right)} \tag{7}$$

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$$\widehat{m}_{vb}(x)=\frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{v}b}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}b}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{v}b}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}b}}\right)} \tag{8}$$

$$\widehat{m}_{vc}(x)=\frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{v}c}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}c}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{v}c}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}c}}\right)} \tag{9}$$

$$\widehat{m}_{vd}(x)=\frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{v}d}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}d}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{v}d}}K\left(\frac{x-X_i}{h\lambda_{\acute{v}d}}\right)} \tag{10}$$

Let us write all the proposed estimators of equations (3-10) in the following generalised form:

$$\widehat{m}_{lt}(x)=\frac{\sum_{i=1}^{n}\frac{y_i}{h\lambda_{\acute{l}t}}K\left(\frac{x-X_i}{h\lambda_{\acute{l}t}}\right)}{\sum_{i=1}^{n}\frac{1}{h\lambda_{\acute{l}t}}K\left(\frac{x-X_i}{h\lambda_{\acute{l}t}}\right)};\ \acute{l}=g,v \text{ and } t=a,b,c,d.$$

The bias and variance of $\widehat{m}_{lt}(x)$ are

$$B\left(\widehat{m}_{lt}(x)\right)=\frac{\left(h\lambda_{\acute{l}t}\right)^2}{2}\Omega_u\left(m''_{lt}(x)+\frac{2m'_{lt}(x)g'_{lt}(x)}{g_{lt}(x)}\right)+O\left(h\lambda_{\acute{l}t}\right)^2 \tag{11}$$

$$V\left(\widehat{m}_{lt}(x)\right)=\left(\frac{\sigma^2\sigma_u^2}{g_{lt}(x)}\right)\left(\frac{1}{nh\lambda_{\acute{l}t}}\right)+O\left(\frac{1}{nh\lambda_{\acute{l}t}}\right) \tag{12}$$

where $g_{lt}(x)$ is the probability distribution function of covariates $X_1, X_2, \ldots, X_n$ and $\Omega_u=\int x^2K(x)dx$. Further, MSE and mean integrated square error (MISE) of $\widehat{m}_{lt}(x)$ are

$$\begin{aligned}MSE\left(\widehat{m}_{lt}(x)\right)&=\left[B\left(\widehat{m}_{lt}(x)\right)\right]^2+V\left(\widehat{m}_{lt}(x)\right),\\&=\frac{\left(h\lambda_{\acute{l}t}\right)^4}{4}\Omega_u^2\left(m''_{lt}(x)+\frac{2m'_{lt}(x)g'_{lt}(x)}{g_{lt}(x)}\right)^2+\left(\frac{\sigma^2\sigma_u^2}{g_{lt}(x)}\right)\left(\frac{1}{nh\lambda_{\acute{l}t}}\right)\\&\quad+O\left(h\lambda_{\acute{l}t}\right)^4+O\left(\frac{1}{nh\lambda_{\acute{l}t}}\right)\end{aligned} \tag{13}$$

$$\begin{aligned}MISE\left(\widehat{m}_{lt}(x)\right)&=\frac{\left(h\lambda_{\acute{l}t}\right)^4}{4}\Omega_u^2\int\left(m''_{lt}(x)+\frac{2m'_{lt}(x)g'_{lt}(x)}{g_{lt}(x)}\right)^2dx\\&\quad+\left(\frac{\sigma^2\sigma_u^2}{nh\lambda_{\acute{l}t}}\right)\int\left(\frac{1}{g_{lt}(x)}\right)dx+O\left(h\lambda_{\acute{l}t}\right)^4+O\left(\frac{1}{nh\lambda_{\acute{l}t}}\right)\end{aligned} \tag{14}$$

SIMULATION STUDY (UTILISATION OF COVID-19 DATA)

There are seven continents in the world. Population-wise, Africa is the second-largest continent, with a population of approximately 1.34 billion people, which accounts for around 17.2% of the total world population. An inspection revealed that a significant portion of the population on this continent were infected with COVID-19. Given the presence of outliers of the data, it is suitable for the proposed NW kernel estimators based on PWMs as described. In analysing the simulation, we used four different variables: total number of deaths, total number of cases per million (1M) population, total number of deaths per million population, and total number of population of the African region.

We obtained secondary data on Africa from a website Worldometer [18] and tailored it for our simulation study. Details of the COVID-19 data set are also available [19-21]. We considered two data sets named population-1 and population-2. In population-1, X represents the total number of cases per million population affected by COVID-19 from February 22, 2020, to August 23, 2020, while the variable Y represents the total number of deaths per million population during the same period. Similarly, in population-2, X indicates the total population of African countries, and Y indicates the total number of deaths due to COVID-19 in African territories during the specified time frame:

Population-1
$X =$ Sum of cases/1M population in Africa
$Y =$ Sum of deaths/1M population in Africa
Population-2
$X =$ Sum of population in Africa
$Y =$ Sum of deaths in Africa

Per cent relative efficiencies (PREs) and MSE are the tools used to assess the performance of estimators in the simulation. The MSE is calculated as shown in Eq. (15), where $\hat{y}_i$ represents estimated values based on all NW versions:

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 \tag{15}$$

The asymptotic MSE (AMSE) is determined by simulating this procedure 1000 times as given in Eq. (16):

$$AMSE = \frac{1}{1000}\sum_{i=1}^{1000} MSE \tag{16}$$

After that, for population-1 and population-2, AMSE-based PREs of the previous versions $\left(\widehat{m}_{\phi t}(x)\right)$ and the proposed version $\left(\widehat{m}_{lt}(x)\right)$ of the estimators with respect to classical estimator $\left(\widehat{m}_{\phi 0}(x)\right)$ are calculated as given in Eq. (17):

$$PRE = \frac{AMSE\left(\widehat{m}_{\phi 0}(x)\right)}{AMSE\left(\widehat{m}_{\phi t}(x) \text{ or } \widehat{m}_{lt}(x)\right)} \times 100 \tag{17}$$

The results of the PRE values related to the two considered populations are listed in Tables 2-5.

Table 2. PREs for population-1 for Alpha (α)

$\widehat{m}$	$h = 0.5$	$h = 0.75$	$h = 1$
$\widehat{m}_{\phi 0}(x)$	100	100	100
$\widehat{m}_{\phi 1}(x)$	103.3158	103.6362	103.5463
$\widehat{m}_{\phi 2}(x)$	103.4131	103.7533	103.3734
$\widehat{m}_{\phi 3}(x)$	103.9193	103.7531	103.6731
$\widehat{m}_{\phi 4}(x)$	103.1135	103.6534	103.7739
$\widehat{m}_{\phi 5}(x)$	103.5492	103.5901	103.6102
$\widehat{m}_{\phi 6}(x)$	103.4134	103.4536	103.2736
$\widehat{m}_{\phi 7}(x)$	103.1232	103.9540	103.4373
$\widehat{m}_{ga}(x)$	116.6133	116.9534	116.1735
$\widehat{m}_{gb}(x)$	116.3742	116.4195	116.2425
$\widehat{m}_{gc}(x)$	116.2690	116.2143	116.8374
$\widehat{m}_{gd}(x)$	116.3691	116.1142	116.9375

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Table 3. PREs for population-1 for Beta (β)

$\widehat{m}$	$h = 0.5$	$h = 0.75$	$h = 1$
$\widehat{m}_{\phi 0}(x)$	100	100	100
$\widehat{m}_{\phi 1}(x)$	104.2158	104.6362	104.4462
$\widehat{m}_{\phi 2}(x)$	104.3131	104.8544	104.2733
$\widehat{m}_{\phi 3}(x)$	104.8139	104.9536	104.6821
$\widehat{m}_{\phi 4}(x)$	104.2135	104.6535	104.7739
$\widehat{m}_{\phi 5}(x)$	104.4492	104.2902	104.6132
$\widehat{m}_{\phi 6}(x)$	104.4143	104.1533	104.2699
$\widehat{m}_{\phi 7}(x)$	104.5736	104.7321	104.4801
$\widehat{m}_{va}(x)$	118.5133	118.9535	118.3734
$\widehat{m}_{vb}(x)$	118.6742	118.3195	118.3575
$\widehat{m}_{vc}(x)$	118.1690	118.2144	118.8291
$\widehat{m}_{vd}(x)$	118.0691	118.1143	118.8560

Table 4. PREs for population-2 for Alpha (α)

$\widehat{m}$	$h = 0.5$	$h = 0.75$	$h = 1$
$\widehat{m}_{\phi 0}(x)$	100	100	100
$\widehat{m}_{\phi 1}(x)$	107.3322	107.3328	107.3331
$\widehat{m}_{\phi 2}(x)$	107.3330	107.3342	107.3348
$\widehat{m}_{\phi 3}(x)$	107.3326	107.3339	107.3345
$\widehat{m}_{\phi 4}(x)$	107.3310	107.3322	107.3328
$\widehat{m}_{\phi 5}(x)$	107.3316	107.3329	107.3335
$\widehat{m}_{\phi 6}(x)$	107.3320	107.3332	107.3330
$\widehat{m}_{\phi 7}(x)$	107.2123	107.2135	107.2141
$\widehat{m}_{ga}(x)$	520.3548	520.6964	520.4999
$\widehat{m}_{gb}(x)$	520.1504	520.5152	520.9187
$\widehat{m}_{gc}(x)$	520.4505	520.6806	520.4842
$\widehat{m}_{gd}(x)$	520.5392	520.4578	520.5613

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Table 5. PREs for population-2 for Beta (β)

$\hat{m}$	$h = 0.5$	$h = 0.75$	$h = 1$
$\hat{m}_{\phi 0}(x)$	100	100	100
$\hat{m}_{\phi 1}(x)$	108.4221	108.4229	108.4232
$\hat{m}_{\phi 2}(x)$	108.4230	108.4243	108.4249
$\hat{m}_{\phi 3}(x)$	108.4225	108.4238	108.4248
$\hat{m}_{\phi 4}(x)$	108.4211	108.4221	108.4227
$\hat{m}_{\phi 5}(x)$	108.4214	108.4228	108.4234
$\hat{m}_{\phi 6}(x)$	108.4221	108.4231	108.4237
$\hat{m}_{\phi 7}(x)$	108.4223	108.4234	108.4242
$\hat{m}_{va}(x)$	535.4547	535.4965	535.4998
$\hat{m}_{vb}(x)$	535.4503	535.5153	535.5187
$\hat{m}_{vc}(x)$	535.4504	535.4805	535.4843
$\hat{m}_{vd}(x)$	535.4391	535.4577	535.4614

Discussion

The proposed and previous versions of the NW kernel estimators are compared in terms of the PRE values. According to the simulated results in Tables 2-5, it is clear that the PREs for the proposed estimators are higher than those of the existing estimators. Therefore, the increase in the value of PRE means a decrease in the value of the MSE. These results are evidence that the proposed estimators perform significantly well. The proposed estimators demonstrate a clear convergence in performance. However, Tables 2 and 5 with $h = 0.5$, as well as Tables 2, 3 and 4 with $h = 0.75$, show that the first estimator is preferred in such cases.

Further, except for the classical NW estimator, i.e. $\hat{m}_{\phi 0}(x)$ from Tables 2, 3 and 4, we found that the performance of all estimators has been affected (generally enhanced) by increasing the value of the bandwidth h. It is worth noting that the performance of all estimators in population-2 for Beta (Table 5) is completely enhanced by increasing the value of h from 0.5 to 0.75 to 1, where the best performance of the estimators is recorded at $h = 1$. In this context the lower and upper values of PRE for two populations for Alpha and Beta associated with the proposed and existing estimators are identified from Tables 2-5 and expressed as an interval shown in Table 6. From this Table, it is noted that all proposed estimators $\hat{m}_{lt}(x)$ achieve excellent performance compared to all the existing estimators, and the values of PRE associated with Beta are always higher than those corresponding to Alpha.

Finally, the PRE values associated with the estimators can be stated as follows:

$$PREs\big(\hat{m}_{lt}(x)\big) > PREs\left(\hat{m}_{\phi t}(x)\right) > PREs\left(\hat{m}_{\phi 0}(x)\right).$$

Hence the proposed family of NW kernel estimators significantly outperforms $\hat{m}_{\phi t}(x)$ and is strongly recommended for use.

Table 6. Lower and upper PREs for population-1 and population-2

Population	$\hat{m}$	h	For Alpha (α)	For Beta (β)
1	$\hat{m}_{lt}(x)$	0.5	[116.2690-116.6133]	[118.0691-118.6742]
		0.75	[116.1142-116.9534]	[118.1143-118.9535]
		1	[116.1735-116.9375]	[118.3575-118.8560]
	$\hat{m}_{\phi t}(x)$	0.5	[103.1135-103.9193]	[104.2135-104.8139]
		0.75	[103.4536-103.9540]	[104.1533-104.9536]
		1	[103.2736-103.7739]	[104.2699-104.7739]
2	$\hat{m}_{lt}(x)$	0.5	[520.1504-520.5392]	[535.4391-535.4547]
		0.75	[520.4578-520.6964]	[535.4577-535.5153]
		1	[520.4842-520.9187]	[535.4614-535.5187]
	$\hat{m}_{\phi t}(x)$	0.5	[107.2123-107.3330]	[108.4211-108.4230]
		0.75	[107.2135-107.3342]	[108.4221-108.4243]
		1	[107.2141-107.3348]	[108.4227-108.4249]

CONCLUSIONS

In this work it is concluded that implementing PWMs in the bandwidth factor of the NW kernel function provides a new family of NW kernel regression estimators that significantly boost the performance of the estimators in the presence of outliers. As PWMs are not as influenced by outliers and are comparatively less sensitive than traditional measures of central tendency and dispersion, they perform better. Simulation results illustrate that the PREs of the proposed estimators are significantly greater than those of existing estimators. Additionally, the increase in the value of PRE of the proposed estimators means a decrease in the value of the MSE, indicating the robustness of the proposed family of kernel estimators in the presence of outliers. The obtained method is computationally tractable, more consistent and comprehensive, making it more useful and applicable for such conditions. As concrete directions for future studies, this work can be adapted for multivariate regression or time series data and extended in light of Ali et al. [22].

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