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Full Paper

Quantum Fréchet derivative: A new framework for analysing non-linear operators

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Abstract: This study introduces a derivative concept that integrates the q-derivative and the Fréchet derivative, aiming to overcome the limitations associated with non-linear operators. The theoretical foundation for the newly defined quantum Fréchet derivative is presented, supported by definitions, properties and illustrative examples. In particular, this derivative maintains consistency with classical definitions in finite-dimensional cases, while offering generalisability for applications in infinite-dimensional spaces.

Keywords: q-derivative, Fréchet derivative, non-linear operators, quantum Fréchet derivative

INTRODUCTION

The solution of non-linear operator equations is one of the most actively researched topics in contemporary mathematics. The study of non-linear operators can be facilitated by examining their approximations through local linear operators. Therefore, the investigation of the differential calculus of non-linear operators in normed spaces is of significant importance.

It is known that, for a function $f: A \subseteq \mathbb{R} \to \mathbb{R}$, with x^0 being any non-isolated point in $A°$, the function f is differentiable at x_0 if and only if the following equality holds [1]:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0). \tag{1}$$

The equation does not make sense for functions of the form $f: \mathbb{R}^n \to \mathbb{R}^m$ (or more generally, for operators of the form $f: X \to Y$, where X and Y are Banach spaces). The concept of derivative expressed for real-valued functions of real variables has been generalised to all normed spaces. This extension was carried out by the French mathematician Maurice Fréchet [2]. Therefore, for such transformations, the concept of Fréchet derivative can be utilised. The most significant aspect of Fréchet differentiability is that it can be extended to infinite-dimensional spaces. Consider a transformation $f: A \subseteq X \to Y$, and let $x^0 \in A^\circ$. The necessary and sufficient condition for the function f to be Fréchet differentiable at the point x^0 is the existence of a linear transformation $T: X \to Y$ such that [1]

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0. \tag{2}$$

For a comprehensive survey on Fréchet differentiability, refer to Siddiqi and Nanda [3].

As can be seen, these derivative definitions rely on the concept of a limit. It is known that operators without limits exist. In the field known as limit analysis, by utilising quantum analysis developed by F.H. Jackson [4], quantum derivatives can be employed to obtain the derivatives of operators for which limits cannot be taken.

It should be noted that the quantum derivative converges to the classical derivative as q approaches 1. Generally, q is used within the range $(0,1)$ [5]. For a comprehensive survey on q-calculus, refer to Ernst [6]. Numerous generalisations and variants of the quantum derivative, such as the (p,q)-derivative [7-9], the conformable derivative [10] and the Caputo q-derivative [11,12], have been extensively studied along with their various applications.

The absence of a generally accepted derivative framework appropriate for non-linear operators without computable limits is a significant gap in the literature that this study seeks to fill. To enable wider applicability in normed vector spaces, the goal is to create a new derivative that combines the limit-free nature of the q-derivative with the structural generality of the Fréchet derivative.

In this paper an attempt has been made to address the long-standing issue of being unable to compute the derivative of operators that are non-linear and that do not admit computable limits, using the three derivative definitions found in the literature as a starting point. As known, the Fréchet derivative is a generalisation of the classical derivative. When the Fréchet derivative is taken and the operator is defined in real numbers, the result obtained is the same as that obtained through the classical derivative. Based on this information, in addressing the issue of being unable to compute the derivative of non-linear operators, a solution has been sought by utilising the Fréchet derivative, a generalisation of the classical derivative, instead of the classical derivative. While the Fréchet derivative provides a solution to taking the derivative of non-linear operators, it falls short when dealing with operators without limits [13]. Hence in attempting to address this deficiency in the Fréchet derivative, the quantum derivative is applied to the Fréchet derivative [14-17]. If the anticipated result is achieved, a generalisation of the derivative will be obtained, introducing a novel form of derivative. This research is of significant importance in overcoming this crucial deficiency mentioned in the literature concerning derivatives.

PRELIMINARIES

In this section we present key definitions of the q-derivative and the Fréchet derivative, along with some of their useful properties.

Definition 1. A function $f: A \subseteq X \to Y$ is said to be Fréchet differentiable at x^0if x^0 is an interior point of A and there exists $T \in \mathcal{L}(X, Y)$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0. \tag{3}$$

This linear function T is called the Fréchet derivative of the function f at x^0 [1]. If the point x^0 is not an interior point of the set A, the linear function T may not be unique.

Definition 2. Let the operator $f: A \subseteq X \to Y$ be Fréchet differentiable at x^0. Then for $x - x^0 = h$,

$$df(x_0\,; h) = Df(x_0)h \tag{4}$$

The expression is called the Fréchet differential of f at x^0 in the direction h [1].

Definition 3. The quantum derivative of a function $f: A \subseteq \mathbb{R} \to \mathbb{R}$, denoted as $D_q f$, is defined in a manner similar to the classical analysis definition, with $x \neq 0$ and $q \in \mathbb{R} - \{1\}$, as follows:

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \tag{5}$$

which is defined as the ratio of two differentials. The quantum derivative of the function f at $x = 0$ is defined to be equivalent to the classical derivative $f'(0)$. According to this, the quantum derivative of a function f is expressed as follows [5]:

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \wedge q \neq 1 \\ f'(x), & x = 0 \vee q = 1 \end{cases}. \tag{6}$$

Definition 4. If f is a transformation, then the quantum differential of f is defined as

$$d_q f(x) = f(x) - f(qx) \tag{7}$$

Specifically, $d_q x = (1-q)x$ [5].

QUANTUM FRÉCHET DERIVATIVE

The classical derivative is inadequate in many functional analysis and applied mathematics applications when non-linear operators or functions whose limits cannot be directly calculated are involved. Despite providing a strong generalisation to infinite-dimensional spaces, the Fréchet derivative still primarily depends on the presence of limits. On the other hand, when classical differentiability fails because there are no limits, quantum calculus, especially the q-derivative, has shown promise. However historically, q-calculus has mostly been used with functions between one-dimensional spaces.

We suggest a new type of derivative, the quantum Fréchet Derivative, which incorporates the limit-free structure of q-derivatives into the general framework of Fréchet differentiability. This definition is motivated by the necessity to bridge these two frameworks, because existence of limit can be too restrictive in higher dimensions. Our construction can potentially broaden the application of quantum calculus to more abstract contexts while also addressing the differentiability of non-linear operators in Banach spaces. Hence it is expected to be a useful tool for analysing complex systems where traditional calculus tools are insufficient, as well as a theoretical generalisation.

In the sequel, we define a generalised form of the derivative by combining the quantum and Fréchet derivatives.

Definition 5. Let X and Y be Banach spaces where $\{e_i: i \in I\}$ represents the Hamel basis of X, and $\{e_j: j \in J\}$ represents the Schauder basis of Y.

Given any transformation $f: A \subseteq X \to Y$ and for $\pi_j: Y \to \mathbb{R}$, we use the representation:

$$f_j = \pi_j \circ f. \tag{8}$$

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Additionally, for any $x \in X$ and $y \in Y$, if we have $x = \sum_{i \in I} x_i e_i$ and $y = \sum_{j \in J} y_j e_j$, we will denote x as $x = (x_i)_{i \in I}$ and y as $y = \left(y_j\right)_{j \in J}$.

Consider a mapping $f: X \rightarrow Y$, and for each $i \in I$, the partial derivative with respect to the quantum parameter q is defined as follows:

$$\frac{\partial f}{\partial_q u_i} = \frac{f(x + (q-1)x_i e_i) - f(x)}{(q-1)x_i}, (q \neq 1, x_i \neq 0). \tag{9}$$

In this case if we define f_{ji}^q as π_j composed of the partial derivative with respect to the quantum parameter $q \left(\frac{\partial f}{\partial_q u_i}\right)$, then this partial derivative can be expressed as the sum over j in the set J:

$$\frac{\partial f}{\partial_q u_i} = \sum_{j \in J} f_{ji}^q e_j, \tag{10}$$

and the expression $\left(f_{ji}^q\right)_{(i,j) \in I \times J}$ is referred to as the quantum Fréchet derivative of f at the point x. Here, the expression $\frac{\partial f}{\partial_q u_i}$ is also referred to as the partial quantum Fréchet derivative of f at the point x.

It should be noted that if, for every $u \in X$, the operator $T_q(f, x): X \rightarrow Y$ defined as

$$T_q(f, x)(u) = \sum_{j \in J} \sum_{i \in I} f_{ji} u_i e_j, \tag{11}$$

is well-defined, then it is linear.

Accordingly, the quantum Fréchet derivative can be conceptualised as follows:

$$\begin{aligned} \mathcal{F}_q: C_q^1(X, Y) &\longrightarrow \mathcal{L}(X, \mathcal{L}(X, Y)) \\ f &\longmapsto \mathcal{F}_q(f): X \longrightarrow \mathcal{L}(X, Y) \\ & \qquad x \longmapsto \mathcal{F}_q(f)(x): X \longrightarrow Y \\ & \qquad\qquad u \longmapsto \mathcal{F}_q(f)(x)(u) \in Y \end{aligned} \tag{12}$$

where $C_q^1(X, Y)$ represents the set of quantum Fréchet differentiable functions from X to Y. For brevity, we will use the notation $\mathcal{F}_q(f)(x) = T_q(f, x)$.

Let us give some examples of the new definition.

Example 1. Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a function and $f(x_1, x_2) = ({x_1}^3 + x_2, x_1 + {x_2}^2)$ be given. We now compute the quantum Fréchet derivative of the function f at the point (x^1, x^2):

$$\frac{\partial f}{\partial_q u_1} = \frac{f(x + (q-1)x_1 e_1) - f(x)}{(q-1)x_1} = \left((q^2 + q + 1){x_1}^2, 1\right)$$
$$\frac{\partial f}{\partial_q u_2} = \frac{f(x + (q-1)x_2 e_2) - f(x)}{(q-1)x_2} = (1, (q+1)x_2)$$

These expressions yield the following quantum Fréchet derivative:

$$\begin{aligned} T_q(f)(u) &= \begin{pmatrix} \frac{\partial f}{\partial_q u_1} & \frac{\partial f}{\partial_q u_2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \begin{pmatrix} (q^2 + q + 1){x_1}^2 & 1 \\ 1 & (q+1)x_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

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Example 2. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $f(x) = \frac{x^2+1}{3x+2}$ be given. We now compute the quantum Fréchet derivative of f at the point x^0:

$$\frac{\partial f}{\partial_q u} = \frac{f(x + (q-1)xe) - f(x)}{(q-1)x} = \frac{3qx^2 - 3 + 2qx + 2x}{(3x+2)(3qx+2)}$$

This leads to the following expression:

$$T_q(f)(u) = \left(\frac{3qx^2 - 3 + 2qx + 2x}{(3x+2)(3qx+2)}\right)(u).$$

This expression generalises the differentiability properties of operators with linear combinations and states in which the derivatives of these operators are also linear combinations of their respective derivatives.

Theorem 1. Let $f, g : A \subseteq X \to Y$ be functions where A is a subset of X mapping into Y, and $x^0 \in A$ is a point. For any scalars α, β, the operator $\alpha f + \beta g$ is quantum Fréchet differentiable at the point x^0. In this case the following equality holds:

$$T_q(\alpha f + \beta g)(x_0) = \alpha T_q(f)(x_0) + \beta T_q(g)(x_0) \tag{13}$$

Proof. Let $f: A \subseteq X \to Y$ and $g: A \subseteq X \to Y$ be functions where A is a subset of X mapping to Y space, and $x^0 \in A$ is a point. For any scalars α, β,

$$\begin{aligned} T_q(\alpha f + \beta g) = T_q(\alpha f + \beta g, x_0)(u) &= \sum_{j\in J}\sum_{i\in I}\left(\alpha f_{ji} + \beta g_{ji}\right)u_i e_j \\ &= \sum_{j\in J}\sum_{i\in I}\left(\alpha f_{ji}u_i e_j + \beta g_{ji}u_i e_j\right) \\ &= \sum_{j\in J}\sum_{i\in I}\alpha f_{ji}u_i e_j + \sum_{j\in J}\sum_{i\in I}\beta g_{ji}u_i e_j \\ &= \alpha\sum_{j\in J}\sum_{i\in I} f_{ji}u_i e_j + \beta\sum_{j\in J}\sum_{i\in I} g_{ji}u_i e_j \\ &= \alpha T_q f(x_0) + \beta T_q g(x_0). \blacksquare \end{aligned}$$

Theorem 2. If f is a constant function from a set $A \subseteq X$ into Y, then f is quantum Fréchet differentiable at each interior point x^0 of A, and $T_q(f)(x^0)$ is the zero function from X into Y (so that $T_q(f)$ is constant).

Proof. Let $f: A \subseteq X \to Y$ be a constant function, $p \in Y$ be a fixed point, and for each $x \in A$, let $f(x) = p$. Then

$$\begin{aligned} \frac{\partial f}{\partial_q u_i} &= \frac{f(x + (q-1)x_i e_i) - f(x)}{(q-1)x_i} \\ &= \frac{p - p}{(q-1)x_i} \\ &= \frac{0}{(q-1)x_i} \\ &= 0 \end{aligned}$$

As seen, $T_q(f)(x^0)$ is the zero function from X into Y. $\blacksquare$

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Theorem 3. If f is a linear function from a set $A \subseteq X$ into Y, then f is quantum Fréchet differentiable at each interior point x^0 of A, and $T_q(f)(x^0)$ is equal to $T(f)(x^0)$.

Proof. Let $f: A \subseteq X \to Y$ be a linear function. We now compute its quantum Fréchet derivative:

$$\begin{aligned}\frac{\partial f}{\partial_q u_i} &= \frac{f(x+(q-1)x_i e_i) - f(x)}{(q-1)x_i} \\ &= \frac{f(x) + (q-1)x_i f(e_i) - f(x)}{(q-1)x_i} \\ &= f(e_i).\end{aligned}$$

As can be seen, a real number is obtained. Outputs of Fréchet derivative must be real numbers for each a_{ij}. Here we obtain the derivative of linear function as a real number. Thus, the quantum Fréchet derivative of a linear function is equivalent to the Fréchet derivative. ■

Theorem 4. If f is a polynomial function from a set $A \subseteq X$ into Y, then f is quantum Fréchet differentiable at each interior point x^0 of A, and $T_q(f)(x^0)$ is equal to $D_q(f)(x^0)$.

Proof. Let $f: A \subseteq X \to Y$ be a polynomial function, where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a^2x^2 + a^1x + a^0, n \in \mathbb{N}, a^0, a^1, \ldots, a_n \in \mathbb{R}.$$

$$\begin{aligned}\frac{\partial f}{\partial_q u} &= \frac{a_n q^{\mathrm{n}} x^{\mathrm{n}} + a_{n-1} q^{\mathrm{n}-1} x^{\mathrm{n}-1} + \cdots + a^2 q^2 x^2 + a^1 q x + a^0}{(q-1)x} \\ &\quad - \frac{(a_n q^{\mathrm{n}} x^{\mathrm{n}} + a_{n-1} q^{\mathrm{n}-1} x^{\mathrm{n}-1} + \cdots + a^2 q^2 x^2 + a^1 q x + a^0)}{(q-1)x} \\ &= \frac{a_n x^{\mathrm{n}}(q^{\mathrm{n}} - 1) + a_{n-1} x^{\mathrm{n}-1}(q^{\mathrm{n}-1} - 1) + \cdots + a^2 x^2 (q^2 - 1) + a^1 x (q-1)}{(q-1)x} \\ &= a_n x^{\mathrm{n}-1}(q^{\mathrm{n}-1} + q^{\mathrm{n}-2} + \cdots + 1) + a_{n-1} x^{\mathrm{n}-2}(q^{\mathrm{n}-2} + q^{\mathrm{n}-3} + \cdots + 1) + \cdots + a^2 x (q+1) + a^1 \\ &= D_q(f)\end{aligned}$$

Hence $T_q(f)(x^0)$ is equal to $D_q(f)(x^0)$. ■

Theorem 5. If $f: X \to Y$ is a transformation that is quantum Fréchet differentiable at the point $x \in X$, and $g \in \mathcal{L}(Y, Z)$, then the composition $g \circ f$ is quantum Fréchet differentiable at $x \in X$, and the quantum Fréchet differential of $g \circ f$ at x is given by

$$T_q(g \circ f, x) = g \circ T_q(f, x). \tag{14}$$

Proof. Let I, J and K be pairwise disjoint sets. Consider bases for X, Y and Z given by $\{e_i: i \in I\}, \{e_j: j \in J\}$ and $\{e_k: k \in K\}$ respectively. For $u \in X$,

$$T_q(g \circ f, x)(u) = \sum_{k \in K} \sum_{i \in I} (g \circ f)_{ki}^q u_i e_k$$

and for each $j \in J$ such that g(e_j)=$\sum_{k \in K} g_{kj} e_k$, we have

$$\begin{aligned}\left(g \circ T_q(f, x)\right)(u) &= g\left(\sum_{j \in J} \sum_{i \in I} f_{ji} u_i e_j\right) \\ &= \sum_{j \in J} \sum_{i \in I} f_{ji} u_i g(e_j)\end{aligned}$$

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$$= \sum_{j\in J}\sum_{i\in I} f_{ji}u_i \sum_{k\in K} g_{kj}e_k$$
$$= \sum_{j\in J}\sum_{i\in I}\sum_{k\in K} f_{ji}u_i g_{kj}e_k\,.$$

To establish the desired equality, it suffices to show that for each $i \in I$ and $k \in K$,

$$(g\circ f)_{ki}^{q} = \sum_{j\in J} f_{ji}u_i g_{kj}.$$

Now let us verify this equality. We have

$$\begin{aligned}
(g\circ f)_{ki}^{q} &= \pi_k \circ \frac{\partial(g\circ f)}{\partial_q u} \\
&= \pi_k\left(\frac{(g\circ f)(x+(q-1)x_ie_i)-(g\circ f)(x)}{(q-1)x_i}\right) \\
&= \pi_k\left(g\left(\frac{f(x+(q-1)x_1e_1)-f(x)}{(q-1)x_1}\right)\right) \\
&= \pi_k\left(g\circ\frac{\partial f}{\partial_q u_i}\right) \\
&= \pi_k\left(g\left(\sum_{j\in J} f_{ji}e_j\right)\right) \\
&= \pi_k\left(\sum_{j\in J} f_{ji}g(e_j)\right) \\
&= \pi_k\left(\sum_{j\in J} f_{ji}\sum_{k\in K} g_{kj}e_k\right) \\
&= \sum_{j\in J} f_{ji}g_{kj}.
\end{aligned}$$

Thus, the equality holds. ■

Let $f:\mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{m}$ be a transformation where $x = (x^1, \dots, x_n) \in \mathbb{R}^{\mathrm{n}}$ is a point, and $f(x) \in \mathbb{R}^{m}$. For $T_q:\mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{m}$ as a linear transformation and $q \in \mathbb{R} - \{1\}$, the transformation $T_q(f)(u)$ is defined as

$$T_q(f)(u) = \begin{pmatrix} \frac{\partial f_1}{\partial_q u_1} & \cdots & \frac{\partial f_1}{\partial_q u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial_q u_1} & \cdots & \frac{\partial f_m}{\partial_q u_n} \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \tag{15}$$

$$\frac{\partial f}{\partial_q u_i} = \frac{f(x+(q-1)x_ie_i)-f(x)}{(q-1)x_i} \tag{16}$$

The transformation $T_q(f)$ is referred to as the quantum Fréchet derivative of the transformation f. When the limit is taken as $q \to 1$, it is equivalent to the classical Fréchet derivative. Now let us see this with an example.

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Example 3. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^2$. We now compute the quantum Fréchet derivative of f at the point $x = x^1$. The quantum Fréchet derivative, denoted as $Tq(f)$, is given by

$$Tq(f) = \frac{\partial f}{\partial_q u_1} = \frac{f(x + (q-1)x_1 e_1) - f(x)}{(q-1)x_1}.$$

The derivative for the function $f(x) = 3x^2$ at $x = x^1$ is calculated:

$$\frac{\partial f}{\partial_q u_1} = 3(q+1)x_1.$$

Taking the limit as $q \to 1$, we get

$$\lim_{q \to 1} \frac{f(x + (q-1)x^1 e^1) - f(x)}{(q-1)x^1} = 6x^1.$$

Next, we consider the Fréchet derivative of the same function. The Fréchet derivative is a linear operator T such that:

$$\lim_{x \to x_0} \frac{\|f(x) - f(x^0) - T(x - x^0)\|}{\|x - x^0\|} = 0.$$

For our function $f(x) = 3x^2$, we find that the linear operator $T(x)$ is given by $T(x) = 6x^0 x$. This satisfies the condition for the Fréchet derivative.

We also observe that for a constant function $f(x) = c$, the quantum Fréchet derivative yields $\frac{\partial f}{\partial_q u}$, consistent with the classical derivative. In conclusion, the quantum Fréchet derivative, in the limit as $q \to 1$, converges to the classical Fréchet derivative.

As seen in the example above, the quantum Fréchet derivative generalises the classical Fréchet derivative. In the limit as $q \to 1$, the proposed derivative recovers the classical form, thereby ensuring compatibility with existing methods. However, unlike classical derivatives, it remains well-defined even in cases where limits do not exist, thus broadening its applicability.

Theorem 6. Let $f: X \to Y$ be a function between Banach spaces, where $\{e_i : i \in I\}$ is a normal Hamel basis of the Banach space X, and let T be the Fréchet derivative of f at $x \in X$. Then $\lim_{q \to 1} \frac{\partial f}{\partial_q u_i}(x) = T(e_i)$.

Proof. By the definition of the Fréchet derivative,

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - T(h)\|}{\|h\|} = 0,$$

that is, $f(x+h) = f(x) + T(h) + r(h)$, where $\|r(h)\| = o(\|h\|)$ with the little o notation. Now consider the vector $k = (q-1)x_i e_i$. $\|k\| = \|(q-1)x_i e_i\| = |(q-1)x_i|\|e_i\|$, and thus $\|k\| \to 0$ as $q \to 1$. This means that $f(x+k) = f(x) + T(k) + r(k)$ and $\frac{\|r(k)\|}{\|k\|} \to 0$. It can be observed that

$$f(x + (q-1)x_i e_i) = f(x) + T\big((q-1)x_i e_i\big) + r\big((q-1)x_i e_i\big),$$

and

$$\begin{aligned} \frac{\partial f}{\partial_q u_i}(x) &= \frac{f(x + (q-1)x_i e_i) - f(x)}{(q-1)x_i} \\ &= \frac{T\big((q-1)x_i e_i\big) + r\big((q-1)x_i e_i\big)}{(q-1)x_i} \end{aligned}$$

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$$= T(e_i) + \frac{r\big((q-1)x_i e_i\big)}{(q-1)x_i}$$

by the linearity of T. Since

$$\left\|\frac{r\big((q-1)x_i e_i\big)}{(q-1)x_i}\right\| = \frac{\|r(k)\|}{|(q-1)x_i|\|e_i\|} = \frac{\|r(k)\|}{\|k\|} \to 0$$

as $q \to 1$, we have $\lim\limits_{q\to 1} \frac{\partial f}{\partial_q u_i}(x) = T(e_i)$. ■

Since all values of a linear transformation defined on a vector space can be constructed solely based on its behaviour on the basis vectors, the theorem above demonstrates that, in the limit case as $q \to 1$, the quantum Fréchet derivative uniquely describes the operator corresponding to the classical Fréchet derivative.

Theorem 7. For $f: \mathbb{R}^n \to \mathbb{R}^m$, let $T_q(f, x)$ be the quantum Fréchet derivative of the operator f. For all $i, j \in \{1, \dots, n\}$, the following holds for the derivatives $T_q(f, x_i)$ and $T_q\big(f, x_j\big)$:

$$T_q\big(T_q(f, x_i), x_j\big) = T_q\big(T_q(f, x_j), x_i\big). \tag{17}$$

Proof. For all $i, j \in \{1, \dots, n\}$ where $i \neq j$, the quantum partial derivative with respect to $u_i$is given by

$$\frac{\partial f}{\partial_q u_i} = \frac{f(x + (q-1)x_i e_i) - f(x)}{(q-1)x_i} = \frac{f(x^1, \dots, qx_i, \dots, x_n) - f(x^1, \dots, x_n)}{(q-1)x_i}.$$

On the other hand,

$$\frac{\partial f}{\partial_q u_j}\left(\frac{\partial f}{\partial_q u_i}\right) = \frac{\big[f\big(x^1, \dots, qx_i, \dots, qx_j, \dots, x_n\big) - f\big(x^1, \dots, qx_j, \dots, x_n\big)\big]}{(q-1)^{2x_i x_j}}$$
$$- \frac{f(x_1, \dots, qx_i, \dots, x_n) - f(x_1, , \dots, x_n)}{(q-1)^{2x_i x_j}}.$$

Similarly,

$$\frac{\partial f}{\partial_q u_j} = \frac{f\big(x + (q-1)x_j e_j\big) - f(x)}{(q-1)x_j} = \frac{f(x_1, \dots, qx_j, \dots, x_n) - f(x_1, \dots, x_n)}{(q-1)x_j}$$

and

$$\frac{\partial f}{\partial_q u_i}\left(\frac{\partial f}{\partial_q u_j}\right) = \frac{\big[f\big(x^1, \dots, qx_i, \dots, qx_j, \dots, x_n\big) - f(x^1, \dots, qx_i, \dots, x_n)\big]}{(q-1)^2 x_i x_j}$$
$$- \frac{f\big(x_1, , \dots, qx_j, \dots, x_n\big) - f(x_1, , \dots, x_n)}{(q-1)^2 x_j x_i}.$$

Since these derivatives are equal for each component, the equality $T_q\big(T_q(f, x_i), x_j\big) = T_q\big(T_q(f, x_j), x_i\big)$ holds. ■

CONCLUSIONS

In order to address non-linear operators, this paper has presented the quantum Fréchet derivative which is not based on limits., a novel generalisation that gets around the drawbacks of both classical and Fréchet derivatives. Important theoretical characteristics have been established, and examples have shown that it is consistent with established techniques in classical situations.

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Determining an integral counterpart for this derivative and looking into its numerical applications in the solution of operator equations in functional analysis or non-linear differential equations can be the main goals of future research.

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