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Full Paper

Neutrosophic Poisson moment exponential distribution: Properties and applications

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Abstract: We propose the neutrosophic Poisson moment exponential distribution (NPMExD) as an extension of the Poisson moment exponential distribution (PMExD) originally developed by Ahsan-ul Haq. We detail the application of neutrosophic logic to the PMExD framework, enhancing its capability to handle uncertainty and indeterminacy. The study explores various statistical and mathematical properties of the NPMExD, including the survival function, moment generating function, hazard rate function, order statistics distribution, cumulative hazard function, index of dispersion, and related measures. Parameter estimation is performed using the maximum likelihood estimation method, followed by a comprehensive simulation study to assess the estimator performance. Finally, the practical efficacy of the proposed distribution is demonstrated through the analysis of two real-world data sets: remission times (in weeks) for 20 leukemia patients and 59 months of actual tax revenue (monthly) data from Egypt. The results indicate that the NPMExD provides a superior fit compared to the neutrosophic discrete Ramos–Louzada distribution for these data sets.

Keywords: neutrosophic, Poisson moment exponential distribution, maximum likelihood estimation, reliability function, hazard rate function

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INTRODUCTION

Traditional statistical methods typically rely on precise numerical values for data collection and analysis. However, in many real-world applications it is often impractical or impossible to obtain exact measurements of statistical parameters. To address this limitation, estimation and approximation techniques are employed. Real-world data frequently involve elements of incompleteness, ambiguity and inconsistency, prompting the development of advanced methodologies such as fuzzy, intuitionistic and neutrosophic distributions. These frameworks extend classical probability theory by incorporating notions of indeterminacy and subjective uncertainty. Such models offer enhanced flexibility and improved accuracy in representing complex systems, making them highly applicable in fields such as artificial intelligence, decision-making, and risk analysis. To overcome the constraints of traditional probability models in dealing with uncertainty and ambiguity, a range of new probability distributions have been introduced. One notable advancement is the shift from classical statistics to neutrosophic statistics [1, 2]. The choice of method for this transition depends on the specific nature of uncertainty present in the data. In recent years substantial progress has been made in leveraging fuzzy logic and neutrosophic theory to effectively model imprecise and uncertain data.

Smarandache [3, 4] introduced a framework for representing logical statements within a three-dimensional neutrosophic space, where each dimension corresponds to the degrees of truth, indeterminacy and falsehood. Building on this concept, several researchers have explored the use of neutrosophic distributions to assess uncertainty in real-world scenarios. Their findings suggest that neutrosophic approaches often outperform traditional statistical methods in capturing and modelling ambiguity and incomplete information. Neutrosophic logic has found widespread application across a variety of fields including decision-making [5], personnel selection and green credit rating systems [6], machine learning, intelligent disease diagnosis, communication services, pattern recognition, social network analysis [7], e-learning systems, physics and more. Within the domain of neutrosophic statistics, researchers have proposed several statistical distributions such as the neutrosophic Poisson and neutrosophic exponential distributions [8]. Additional distributions including the neutrosophic Weibull, neutrosophic uniform and neutrosophic gamma distributions have also been introduced in the literature [9]. A specialised neutrosophic Beta distribution has been developed for modelling interval-valued data [10], while the application of a control chart based on the neutrosophic Maxwell distribution has also been investigated [11].

To address uncertainty and indeterminacy more effectively, several neutrosophic extensions of classical probability distributions have been proposed by incorporating neutrosophic logic. Notable examples include the neutrosophic binomial and neutrosophic normal distributions, which enhance their classical counterparts through the integration of indeterminacy components [12]. The neutrosophic Weibull distribution [13] and the neutrosophic family Weibull distribution are generalisations of the traditional Weibull model, adapted to account for uncertainty within neutrosophic frameworks. Further developments include the neutrosophic Rayleigh distribution, which has been applied in various engineering contexts [14], and the neutrosophic log-logistic distribution [15]. Other contributions to the field include the neutrosophic Kumaraswamy distribution [16], the neutrosophic generalised exponential distribution [17], and the neutrosophic exponentiated inverse Rayleigh distribution [18]. In response to the challenges posed by the COVID-19 pandemic, researchers have also proposed the neutrosophic Burr-III distribution for modelling pandemic-related data [19], as well as the neutrosophic quasi-X Lindley distribution, which has demonstrated applicability in the analysis of COVID-19 data sets [20].

These distributions provide enhanced flexibility in addressing challenges that classical statistical models may fail to capture, particularly in the presence of uncertainty and anomalous data. Furthermore, researchers have extended the application of neutrosophic logic to time series analysis, exploring predictive and modelling techniques such as neutrosophic moving averages, neutrosophic logarithmic models and neutrosophic linear models [21].

The Poisson moment exponential distribution (PMExD) [22] is a discrete probability distribution formed by combining the Poisson and moment exponential distributions to meet the demand for a more adaptable distribution in statistical data analysis with probability mass function (PMF) given by

$$P(x,\beta) = \frac{(x+1)\beta^x}{(\beta+1)^{x+2}}\ , x = 0,1,2 \ldots.,\beta > 0$$

and the corresponding cumulative distribution function (CDF) given by

$$F(x,\beta) = 1 - \frac{\beta^{x+1}\,(\beta+2+x)}{(1+\beta)^{x+2}}.$$

Figure 1 shows the PMF of the PMExD for selected parameter values β=7, 3, 15, 0.6, demonstrating that the distribution can exhibit various shapes including decreasing and increasing-decreasing patterns.

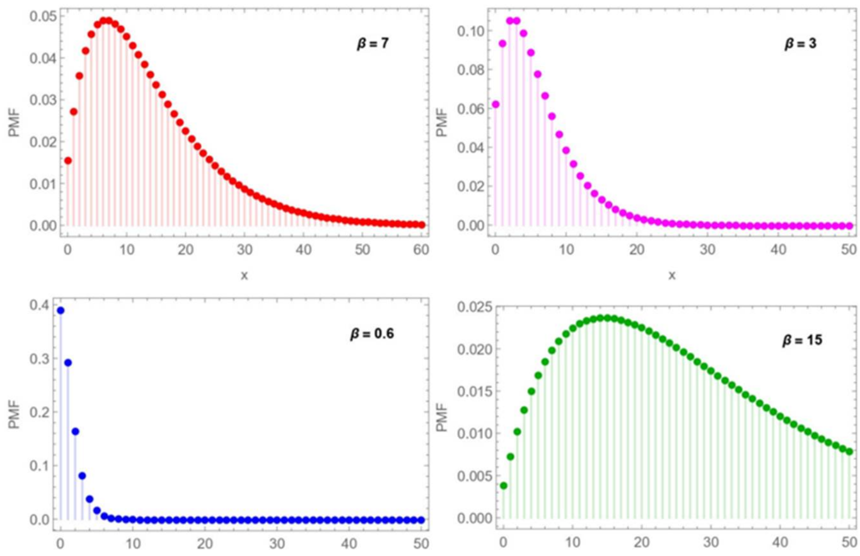

Figure 1. PMF plots of PMExD

This model is well-suited for analysing count data characterised by high variability. Count data models are fundamental in both applied and theoretical domains including healthcare, engineering, insurance, transportation and numerous other fields. They are essential for understanding and predicting the frequency of events in these contexts. In the present study we incorporate neutrosophic logic into the Poisson moment exponential distribution to enhance its capacity to handle uncertainty and indeterminacy inherent in real-world data.

The survival function (SF) of the PMExD is given by

$$SF(x;\beta) = \frac{\beta^{x+1}(\beta+2+x)}{(1+\beta)^{x+2}}.$$

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Figure 2 presents the SF of the PMExD for selected parameter values β = 7, 3, 15, 0.6, revealing that the distribution exhibits a range of decreasing behaviours depending on the parameters.

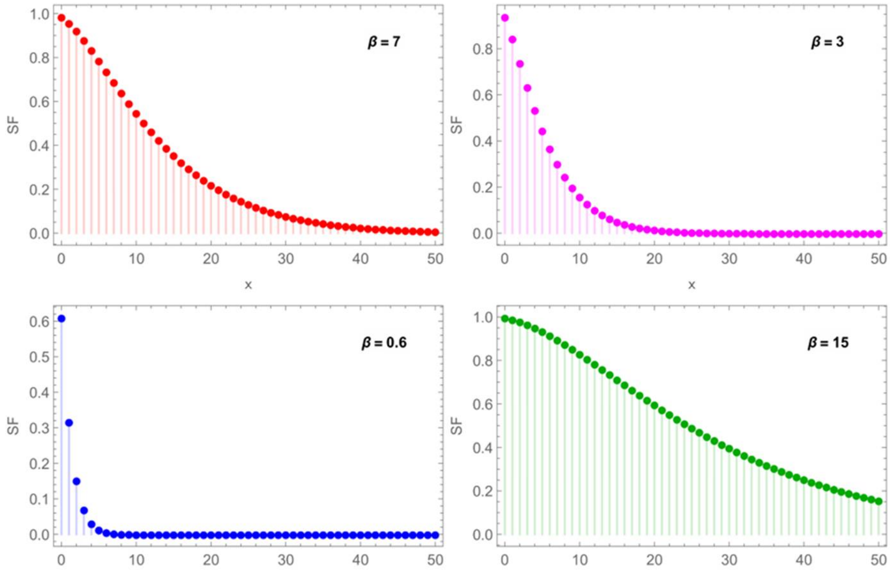

Figure 2. SF plots of PMExD

A summary of the authors' contributions to this study is as follows.

- By adding neutrosophic logic, we develop and expand the PMExD into the neutrosophic Poisson moment exponential distribution (NPMExD).
- We examine and evaluate the NPMExD's mathematical and statistical characteristics such as its hazard rate function, moment generating function, cumulative hazard rate function, index of dispersion, distribution of order statistics, and the survival function.
- We estimate parameters and assess the suggested distribution's performance by applying the maximum likelihood estimation (MLE) technique.
- We perform a simulation study to evaluate the NPMExD's applicability and efficiency compared to alternative distributions.
- We use two real data sets to demonstrate the NPMExD's advantages over the discrete Ramos-Louzada distribution.

NPMExD

The PMF of NPMExD is given by

$$P(x_N,\beta) = P(x-I,\beta) = \frac{(1+(x-I))\beta^{(x-I)}}{(\beta+1)^{(x-I)+2}}\ ,(x-I) = 0,1,2,\dots.,\beta > 0.$$

It is easy to show that the above equation satisfies the conditions of being a PMF, where $P(x_N,\beta)$ is positive and

$$\sum_{x=I}^{\infty}\frac{\big(1+(x-I)\big)\beta^{(x-I)}}{(\beta+1)^{(x-I)+2}} = \frac{1}{(\beta+1)^2}+\frac{2\beta}{(\beta+1)^3}+\frac{3\beta^2}{(\beta+1)^4}+\cdots$$

$$= \frac{1}{(\beta+1)^2}\left[1+\frac{2\beta}{\beta+1}+\frac{3\beta}{(\beta+1)^2}+\cdots\right]$$

$$= \frac{1}{(\beta+1)^2}\left[\frac{(\beta+1)^2}{\beta-\beta+1}\right] = 1.$$

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Figure 3 displays plots of the PMF of the NPMExD for the given values of the distribution parameter β and the indeterminacy factor I. These plots illustrate how changes in β and I affect the shape and behaviour of the distribution. It is noted that for $I = 0.2$, $\beta = 0.5$, the PMF is decreasing, while for other choices considered here the distribution is increasing-decreasing.

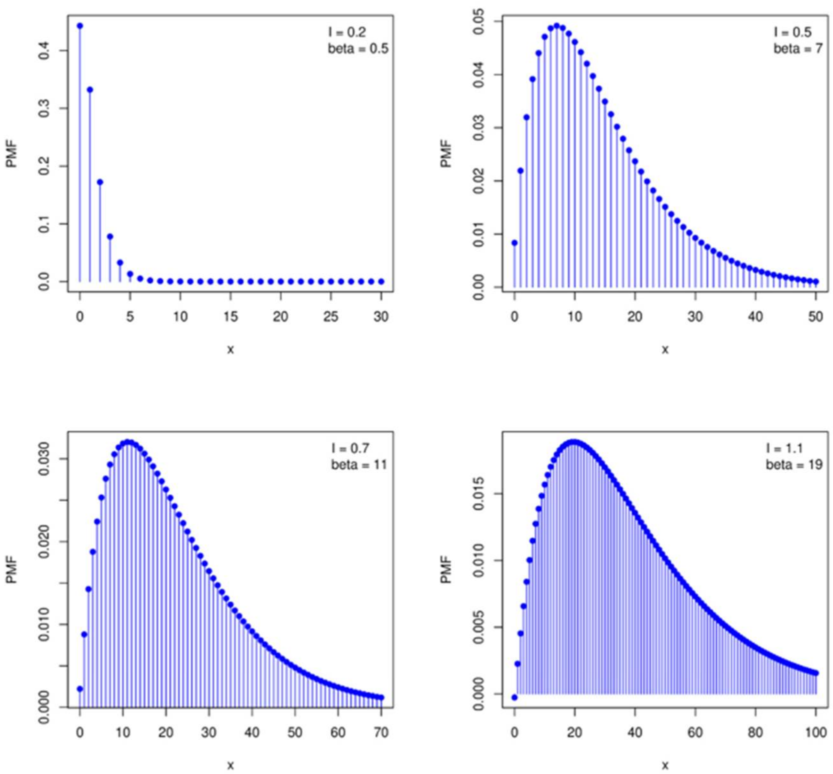

Figure 3. PMF plots of NPMExD

Theorem 1. The CDF of the NPMExD is defined by

$$F(x_N, \beta) = 1 - \frac{(\beta+2+x-I)\beta^{x-I+1}}{(1+\beta)^{(x-I+2)}}.$$

Proof. To prove this theorem, the CDF $F(x)$ is given as

$$F(x_N, \beta) = P(X \leq x) = \sum_{k=I}^{x} P(t_N, \beta) = \sum_{k=I}^{x} \frac{(1+(k-I))\beta^{k-I}}{(\beta+1)^{(k-I)+2}}.$$

For $j = k - I$, the summation starts from 0 when $k = I$. Thus,

$$F(x_N, \beta) = \sum_{j=0}^{x-I} \frac{(1+j)\beta^j}{(\beta+1)^{j+2}} = \sum_{j=0}^{x-I} \frac{\beta^j}{(\beta+1)^{j+2}} + \sum_{j=0}^{x-I} \frac{j\beta^j}{(\beta+1)^{j+2}}.$$

By using $\sum_{j=0}^{x-I} r^j = \frac{1-r^{(x-I)+1}}{1-r}$ and $\sum_{j=0}^{n} jr^j = \frac{r(1-(n+1)r^n+nr^{n+1})}{(1-r)^2}$, where $r = \frac{\beta}{\beta+1}$, we have

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$$F(x_N,\beta)=\frac{1}{\beta+1}\left(1-\left(\frac{\beta}{\beta+1}\right)^{(x-I)+1}\right)$$
$$+\frac{\beta}{\beta+1}\left(1-((x-I)+1)\left(\frac{\beta}{\beta+1}\right)^{(x-I)}+(x-I)\left(\frac{\beta}{\beta+1}\right)^{(x-I)+1}\right)$$
$$=1-\frac{(\beta+2+x-I)\beta^{x-I+1}}{(1+\beta)^{(x-I+2)}}.$$

Figure 4 shows the CDF plots of the NPMExD for different values of β and I.

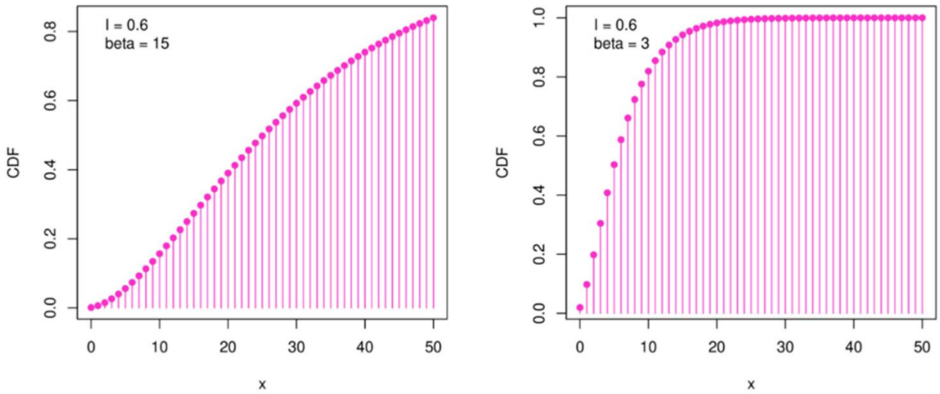

Figure 4. CDF plots of NPMExD

SOME PROPERTIES OF NPMExD

Several statistical characteristics of the NPMExD are described in this section, including the mean, index of dispersion (IOD), variance, moment generating function (MGF), reliability function, cumulative hazard function (CHF), hazard rate function (HRF) and order statistics. These attributes offer important insights into the behaviour and features of the distribution.

Mean, Variance and IOD

The mean of the NPMExD is

$$E(X_N)=\sum_{x=I}^{\infty}x\frac{((x-I)+1)\beta^{(x-I)}}{(\beta+1)^{(x-I+2)}}$$
$$=\frac{I}{(\beta+1)^2}+\frac{2\beta(I+1)}{(\beta+1)^3}+\frac{3\beta^2(I+2)}{(\beta+1)^4}+\cdots=I+2\beta.$$

Similarly, we can find the second moment $E(X^2)=2\beta(3\beta+1)+I\,2+4\beta I,$ and then the variance of the NPMExD is defined as

$$V(X_N)=E\left({X_N}^2\right)-[E(X_N)]^2=2\beta^2+2\beta=2\beta(\beta+1).$$

The IOD is well known as the ratio of the variance to the mean, serves as a metric for assessing whether a given distribution is suitable for the data sets characterised by either under-dispersion or over-dispersion. For the NPMExD the IOD is given by

$$IOD(X_N)=\frac{V(X_N)}{E(X_N)}=\frac{2\beta(\beta+1)}{I+2\beta}.$$

Some values of the mean, variance and IOD of the NPMExD for $I=0.2,0.9$ with $0.1\le\beta\le1.5$ are presented in Table 1. It can be noted that the mean, variance and IOD values are increasing with all values of β for $I = 0.2, 0.9$, while the variance is not affected by the value of I.

Table 1. Mean, variance and IOD of NPMExD for I=0.2 and 0.9

	$\boldsymbol{I = 0.2}$			$I = 0.9$		
$\boldsymbol{\beta}$	$E(X_N)$	$V(X_N)$	$IOD(X_N)$	$E(X_N)$	$V(X_N)$	$IOD(X_N)$
0.1	0.4	0.22	0.550	1.1	0.22	0.200
0.2	0.6	0.48	0.800	1.3	0.48	0.369
0.3	0.8	0.78	0.975	1.5	0.78	0.520
0.4	1.0	1.12	1.120	1.7	1.12	0.659
0.5	1.2	1.50	1.250	1.9	1.50	0.789
0.6	1.4	1.92	1.371	2.1	1.92	0.914
0.7	1.6	2.38	1.488	2.3	2.38	1.035
0.8	1.8	2.88	1.600	2.5	2.88	1.152
0.9	2.0	3.42	1.600	2.7	3.42	1.267
1.0	2.2	4.00	1.710	2.9	4.00	1.379
1.1	2.4	4.62	1.925	3.1	4.62	1.490
1.2	2.6	5.28	2.031	3.3	5.28	1.600
1.3	2.8	5.98	2.136	3.5	5.98	1.709
1.4	3.0	6.72	2.240	3.7	6.72	1.816
1.5	3.2	7.50	2.344	3.9	7.50	1.923

MGF

The MGF is a crucial tool in identifying and analysing probability distributions.

Theorem 2. For the NPMExD, the MGF is given by

$$M_{X_N}(t) = M_{X_N}(t)e^{tI} = \frac{e^{tI}}{(1+\beta-e^t\beta)^2}.$$

Proof. The MGF is defined as

$$M_X(t) = \sum_{x=I}^{\infty} e^{tx}\left(\frac{\big(1+(x-I)\big)\beta^{x-I}}{(\beta+1)^{(x-I)+2}}\right).$$

Let $k = x - I$; so $x = k + I$ and $k = 0, 1, 2, \ldots$. Then

$$M_X(t) = \sum_{k=0}^{\infty} e^{t(k+I)}\left(\frac{(1+k)\beta^k}{(\beta+1)^{k+2}}\right) = e^{tI}\left(\frac{1}{(\beta+1)^2}\right)\sum_{k=0}^{\infty}\frac{(1+k)\left(\frac{e^t\beta}{\beta+1}\right)^k}{(\beta+1)^k}$$

$$= e^{tI}\left(\frac{1}{(\beta+1)^2}\right)\sum_{k=0}^{\infty}(1+k)z^k, \qquad z = \frac{e^t\beta}{\beta+1}$$

$$= e^{tI}\left(\frac{1}{(\beta+1)^2}\right)\left(\sum_{k=0}^{\infty} z^k + \sum_{k=0}^{\infty} kz^k\right) = e^{tI}\left(\frac{1}{(\beta+1)^2}\right)\left(\frac{1}{1-z} + \frac{1}{(1-z)^2}\right)$$

$$= e^{tI}\left(\frac{1}{(\beta+1)^2}\right)\left(\frac{1}{(1-z)^2}\right) = e^{tI}\left(\frac{1}{(\beta+1)^2}\right)\left(\frac{1}{\left(1-\frac{e^t\beta}{\beta+1}\right)^2}\right) = \frac{e^{tI}}{(\beta+1-e^t\beta)^2}.$$

The proof is completed by using $\sum_{k=0}^{\infty} z^k = \frac{1}{z-1}$ for $|z| < 1$ and

$$\sum_{k=0}^{\infty} kz^k = z\frac{d}{dz}\left(\sum_{k=0}^{\infty} z^k\right) = z\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{z}{(1-z)^2}\,.$$

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Reliability Function

The probability that a system or component continues to operate or 'survive' after specific amount of time *t* is known as the SF (survival function) or reliability function. The SF is defined as $SF(x) = P(X \geq x)$ and for the NPMExD, it is given by

$$SF(x_N, \beta) = \frac{\beta^{(x-I)}(\beta + 1 + x - I)}{(1+\beta)^{x-I+1}}.$$

For different values of β and I ($\beta = 5, I = 0.2$ and $\beta = 19, I = 0.9$), Figure 5 shows the plots of SF of the NPMExD. The graph illustrates that the reliability function is equal to one at time $t = 0$ and gradually decreases towards zero as time increases. This behaviour is quite reasonable and expected for any item under normal usage conditions. Also, the shape of the curve depends on the value of β and I.

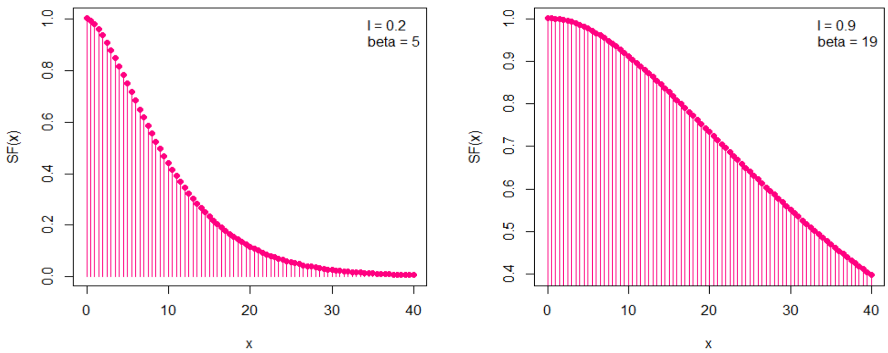

Figure 5. Plots of Sf of NPMExD

HRF

The hazard function evaluates the probability that an object will malfunction or cease to exist after a specific amount of time based on its survival up to that point. The HRF is mathematically defined as the ratio of the PMF to the SF. The HRF of NPMExD is given as

$$HRF(x_N, \beta) = \frac{P(x_N, \beta)}{SF(x_N, \beta)} = \frac{x - I + 1}{(\beta + 1 + x - I)(1+\beta)}.$$

Figure 6 presents plots of the HRF for the NPMExD with varying values of β and I, as $\beta = 5$, $I = 0.2$ and $\beta = 19$, $I = 0.5$. From the graph, we can see that the HRF is increasing, indicating that the items are more likely to fail as they age. This increasing hazard is often associated with the later stages of the item's life cycle. While the graph may seem unrealistic at first glance, it is important to note that increasing hazard rates are commonly observed in many real-world scenarios.

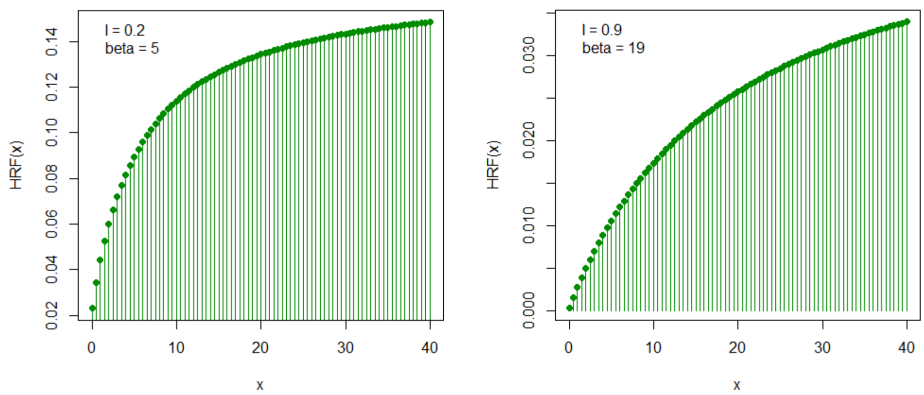

Figure 6. HRF plots of NPMExD

We used the method of Gleser [23] to ascertain the form of the HRF, which is described as follows.

Lemma 1. Based on Gleser [23], let X be a non-negative continuous random variable with twice differentiable PMF $f(x)$, and $\Re(x;\ \beta) = -\frac{f^{/}(x;\beta)}{f(x;\beta)}$.

- Let $\Re^{/}(x;\ \beta) > 0$ ($\Re^{/}(x;\ \beta) < 0$) for all x; then the HRF is increasing (decreasing).
- Suppose that there exists $x_0 > 0$ such that $\Re^{/}(x;\ \beta) < 0, \forall x \in (0, x_0), \Re^{/}(x;\ \beta) = 0$ and $\Re^{/}(x;\ \beta) > 0$, $\forall x \in (x_0, 0)$. Then if $lim_{x\to 0+} f(x) = \infty$, the HRF has a bathtub shape. Now for the NPMExD we have

$$\Re(x;\ \beta) = \frac{\log\beta\,(1 - I + x) + \log(\beta + 1)\,(I - x - 1) + 1}{I - x - 1},$$

and it follows that $\Re^{/}(x;\ \beta) = \frac{1}{(x-I+1)^2} > 0$. Hence the HRF of the NPMExD is increasing.

CHF

The CHF of NPMExD is

$$CHF(x_N, \beta) = -\ln\big(S(x_N, \beta)\big) = -\ln\left(\frac{\beta^{(x-I)}(\beta + 1 + x - I)}{(1+\beta)^{x-I+1}}\right).$$

The CHF plots of the NPMExD in Figure 7 for $\beta = 5, I = 0.2$ and $\beta = 19, I = 0.9$ indicate that the CHF is increasing in both cases.

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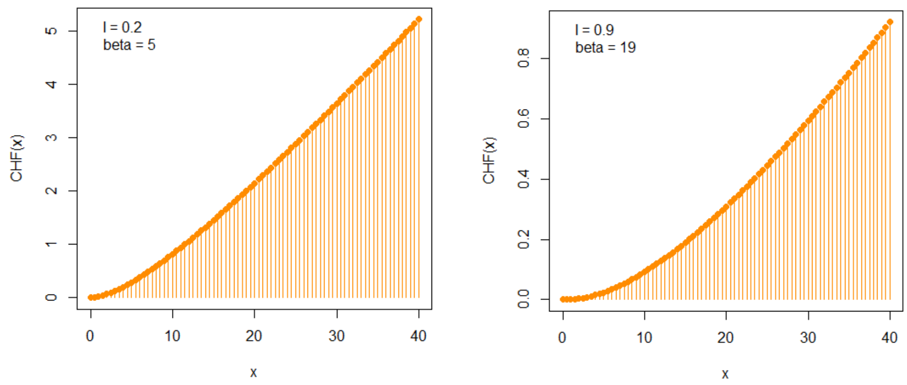

Figure 7. CHF plots of NPMExD

ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n, which follows the NPMExD, and $X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n:n)}$ be the order statistics of the sample. The cumulative distribution function of the r^{th} order statistic is given by

$$F_{(r:n)}(x_N, \beta) = \sum_{r=k}^{n} \binom{n}{r} F^r(x_N, \beta)(1 - F(x_N, \beta))^{n-r}$$

$$= \sum_{r=k}^{n} \binom{n}{r} \left(1 - \frac{(\beta + 2 + (x - I))\beta^{x-I+1}}{(\beta + 1)^{(x-I)+2}}\right)^r \left(\frac{(\beta + 2 + (x - I))\beta^{x-I+1}}{(\beta + 1)^{(x-I)+2}}\right)^{n-r}$$

$$= \frac{1}{(\beta + 1)^{n((x-I)+2)}} \sum_{r=k}^{n} \binom{n}{r} ((\beta + 2 + (x - I))\beta^{x-I+1})^{n-r}$$

and the corresponding PMF is

$$f_{(r:n)}(x_N, \beta) = F_{(r:n)}(x_N, \beta) - F_{(r:n)}(x_N - 1, \beta)$$

$$= \binom{n}{r} (1 - \beta^{x-I}(\beta + 1)^{I-x-1}(\beta - I + x + 1))^k (-\beta^{x-I}(\beta + 1)^{I-x-1}(\beta - I + x + 1))^{n-k}$$

$$\times\ {}_2F_1\left(1, k - n; k + 1; \frac{\beta - \beta^{I-x+1}(\beta + 1)^{x-I} - \left(\frac{\beta}{\beta + 1}\right)^{I-x} - I + x + 1}{\beta - I + x + 1}\right)$$

$$+ (1 - \beta^{-I+x+1}(\beta + 1)^{I-x-2}(\beta - I + x + 2)^k + (\beta^{-I+x+1}(\beta + 1)^{I-x-2}(\beta - I + x + 2)^{n-k}$$

$$\times\ {}_2F_1(1, k - n; k + 1; \Psi)$$

where

$$\Psi = \frac{\beta - (\beta + 1)^{x-I}\beta^{I-x-1} - (\beta + 1)^{x-I}\beta^{I-x+1} - 2\left(\frac{\beta}{\beta + 1}\right)^{I-x} - I + x + 2}{\beta - I + x + 2}$$

and ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{a_k b_k z^k}{k! c_k}$.

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PARAMETER ESTIMATION

In statistical modelling and inference, parameter estimation is fundamental to understanding and quantifying relationships within the data. Among the various estimation techniques, MLE is a cornerstone method, widely recognised for its robustness and effectiveness in estimating the unknown parameters of a statistical model. Here we adopt the MLE approach for estimating the unknown parameter β of the NPMExD. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from the respective distribution with an unknown parameter β. Therefore, the likelihood function is

$$l(x_N, \beta) = \prod_{i=1}^{n} \frac{(x_i - I + 1)\beta^{x_i - I}}{(\beta + 1)^{x_i - I + 2}},$$

and its log is given by

$$\log l(x_N, \beta) = \sum_{i=1}^{n} [\log(x_i - I + 1) + (x_i - I)\log\beta - (x_i - I + 2)\log(\beta + 1)].$$

The first and second partial derivatives of the $\log l(x_N, \beta)$ relative to β are given respectively by

$$\frac{\partial}{\partial\beta}\log l(x_N, \beta) = \sum_{i=1}^{n}\left[\frac{x_i - I}{\beta} - \frac{x_i - I + 2}{\beta + 1}\right],$$

$$\frac{\partial^2}{\partial\beta^2}\log l(x_N, \beta) = \sum_{i=1}^{n}\left[-(x_i - I)\frac{1}{\beta^2} + (x_i - I + 2)\frac{1}{(\beta + 1)^2}\right].$$

By setting the first derivative to zero, $\frac{\partial}{\partial\beta}\log l(x_N, \beta) = 0,$ we get the MLE of β as $\hat{\beta} = \frac{\bar{X} - I}{2}$.

A simulation study is a powerful analytical tool widely used across various domains of statistics. It involves developing a computational model or algorithm to replicate a real-world process or system, enabling the generation of synthetic data under controlled conditions. Simulation provides a flexible framework for analysing complex systems that may be challenging to investigate through purely analytical or empirical methods. By simulating a system's behaviour under diverse scenarios, researchers can gain valuable insights into the dynamics, interactions and emergent properties of the system. In this section we perform a simulation study to evaluate the performance of the MLE for the NPMExD. We generate 1,000 samples of varying sizes, i.e. $n = 25, 50, 75, 100, 125, 150, 200, 250, 300, 350, 400$, for $\beta = 0.2, 0.5, 0.7, 0.9$ from the NPMExD with indeterminacy $I = 0.02$. To assess the properties of the optimal estimator, we use the average bias, mean squared error (MSE) and mean absolute error (MAE) as evaluation metrics and the results are presented in Table 2.

Based on Table 2, we can see that the MSE and MAE values are decreasing as the sample size is increasing for a fixed value of β. For illustration, with $\beta = 0.7$, $n = 25, 400$, the MSE values are 0.07475 and 0,03834 respectively. Also, the MSE and MAE values depend on the parameter value. As an example, for $n = 100$, the MSE values are 0.05266 and 0.05050 for $\beta = 0.2$ and 0.7 respectively.

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Table 2. MLE, bias, MSE and MAE of $\hat{\beta}$ of NPMExD parameter for β = 0.2, 0.5, 0.7, 0.9 and I = 0.02

n	$\beta = 0.2$				$\beta = 0.5$			
	MLE	**Bias**	**MSE**	**MAE**	**MLE**	**Bias**	**MSE**	**MAE**
25	0.42044	0.22044	0.06994	1.17886	0.50532	0.00532	0.02169	0.17996
50	0.39854	0.19854	0.06209	1.08643	0.50541	0.00541	0.02056	0.17991
75	0.38265	0.18265	0.05756	1.02724	0.50769	0.00769	0.01517	0.15135
100	0.37162	0.17162	0.05266	0.96934	0.50294	0.00294	0.01354	0.14511
150	0.37345	0.17345	0.05245	0.96799	0.50103	0.00103	0.01123	0.12971
200	0.37175	0.17175	0.05244	0.96548	0.50057	0.00057	0.00861	0.11528
250	0.37314	0.17314	0.05212	0.96276	0.49960	0.00040	0.00742	0.10932
300	0.36362	0.16362	0.04895	0.91764	0.50108	0.00108	0.00698	0.10289
350	0.37545	0.17545	0.04863	0.91695	0.50746	0.00746	0.00659	0.10051
400	0.36251	0.16251	0.04850	0.91549	0.50354	0.00354	0.00612	0.09739
n	$\beta = 0.7$				$\beta = 0.9$			
25	0.57744	0.12256	0.07475	0.33189	0.65957	0.24043	0.19511	0.43542
50	0.57474	0.12526	0.06075	0.31113	0.67166	0.22834	0.16526	0.40842
75	0.57645	0.12355	0.05479	0.30011	0.71577	0.18423	0.15704	0.39200
100	0.57419	0.12581	0.05050	0.29019	0.72189	0.17812	0.14693	0.37912
150	0.57825	0.12175	0.04547	0.27631	0.72781	0.17219	0.13598	0.36441
200	0.58036	0.11964	0.04549	0.27642	0.72309	0.17691	0.13252	0.35984
250	0.58691	0.11309	0.04173	0.26556	0.72922	0.17078	0.13040	0.35870
300	0.58382	0.11618	0.04079	0.26390	0.74431	0.15569	0.12728	0.35204
350	0.58727	0.11273	0.04034	0.26225	0.75985	0.14015	0.12701	0.35021
400	0.59336	0.10664	0.03834	0.25528	0.74905	0.15095	0.12317	0.34562

APPLICATIONS

Here two data sets are analysed to demonstrate the applicability and flexibility of the newly introduced NPMExD in comparison to established probability distributions. We focus on the neutrosophic discrete Ramos-Louzada distribution (NDRL) [24] for this comparison with the NPMExD when the indeterminacy index $I = 0.01$. The determination of the optimal fit model relies on several model selection criteria including the log-likelihood value (LogLik.), the Kolmogorov-Smirnov (KS) test and information criteria such as Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), and the Bayesian information criterion (BIC). A model is considered superior in fitting the data if it has the minimum values of AIC, BIC, HQIC, CAIC and KS statistics as compared to other competitors. The probability mass function of the NDRL is given by

$$f_Y(y-I) = \begin{cases} \dfrac{(1-\vartheta)^2(1+2\log\vartheta + (y-I)(\log\vartheta)^2)\vartheta^{(y-I)}}{(1+2\log\vartheta)(1-\vartheta)+\vartheta_N\,(\log\vartheta)^2} & , if\ y = I, I+1, I+2, \ldots \\ 0 & ,\quad otherwise \end{cases}$$

Quantile-quantile (Q-Q) and probability-probability (P-P) plots are valuable tools for evaluating the goodness-of-fit of the NPMExD to real data. These plots compare the empirical distribution of the observed data with the theoretical distribution, providing visual insight into the model's suitability. Additionally, the total time on test (TTT) plot is instrumental in model selection, as it reveals the underlying behaviour of failure rates in the data. A straight line on the TTT plot suggests a constant failure rate, a convex shape indicates a decreasing failure rate, while a concave shape suggests an increasing failure rate. A bathtub-shaped curve is characterised by an initial decrease followed by an increase, whereas a concave-convex pattern indicates an inverted bathtub-shaped failure rate.

Data I: The data set used for analysis comprises the remission times measured in weeks for 20 leukemia patients who were randomly assigned to a specific treatment. It is taken from Lawless [25], and the data set is: 1,3,3,6,7,7,10,12,14,15,18,19,22,26,28,29,34,40,48,49. The statistical properties for the data are given in Table 3, which shows that the data are skewed to the right with a standard deviation of 14.7.

Data II: This data set includes the monthly actual tax revenue in Egypt from January 2006 to November 2010. The data were analysed for the five-parameter Lomax distribution [26, 27] for Type II exponentiated half logistic-Gompertz Topp-Leone-G family of distributions. The actual tax revenue values were presented in millions of Egyptian pounds (1000 million) as: 5.90, 20.4, 14.9, 16.2, 17.2, 7.80, 6.10, 9.20, 10.2, 9.60, 13.3, 8.50, 21.6, 18.5, 5.10, 6.70, 17.0, 8.60, 9.70, 39.2, 35.7, 15.7, 9.70, 10.0, 4.10, 36.0, 8.50, 8.00, 9.20, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.50, 10.6, 19.1, 20.5, 7.10, 7.70, 18.1, 16.5, 11.9, 7.0, 8.60, 12.5, 10.3, 11.2, 6.10, 8.40, 11.0, 11.6, 11.9, 5.20, 6.80, 8.90, 7.10, 10.8.

The summary statistics for both data sets presented in Table 3 include mean, standard deviation, median, mean absolute deviation, minimum, maximum, range, skewness, kurtosis and standard error. The P-P and Q-Q plots for the NPMExD based on the leukemia patient's data are illustrated in Figure 8, while Figure 9 presents the TTT and density plots for the NPMExD using the first data of leukemia patients. For the tax revenue data, the P-P and Q-Q plots for the NPMExD are given in Figure 10 and the TTT plot is presented in Figure 11. The results of estimates and goodness of fit are given in Table 4.

Table 3. Summary statistics for remission times of leukemia patients and tax revenue data

Data	*n*	Mean	SD[1]	Median	MAD[2]	Min	Max	Range	Skew[3]	Kur[4]	SE[5]
I	20	19.55	14.7	16.5	14.83	1	49	48	0.60	-0.85	3.29
II	59	13.49	8.05	10.6	5.34	4.1	39.2	35.1	1.57	2.08	1.05

1 = standard deviation, 2 = mean absolute deviation, 3 = skewness, 4 = kurtosis, 5 = standard error

Table 4. MLEs and goodness-of-fit measures for fitted models regarding data sets

	Model	Estimate	LogLik	AIC	BIC	CAIC	HQIC	KS
Data I	NPMExD	$\hat{\beta}$ =0.9725	-35.485	72.97	73.966	74.966	73.164	0.298
	NDRL	$\hat{\vartheta}$ =0.6386	-40.001	82.01	83.010	84.010	82.209	0.377
Data II	NPMExD	$\hat{\beta} = 0.6694$	-85.33	172.66	174.74	175.74	173.47	0.420
	NDRL	$\hat{\vartheta} = 0.5998$	-111.32	224.64	226.71	227.71	225.45	0.447

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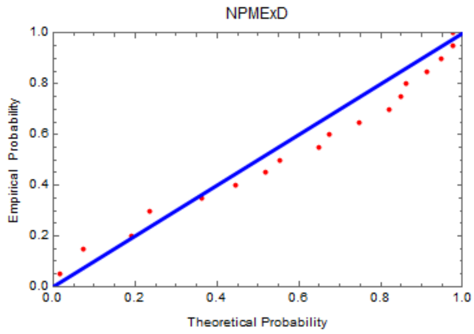
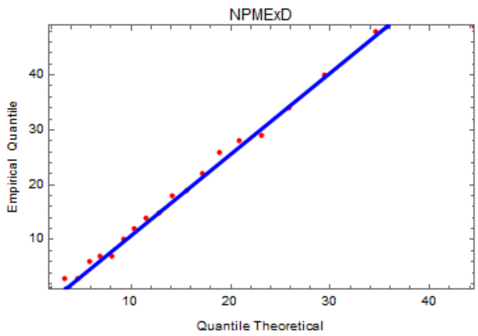
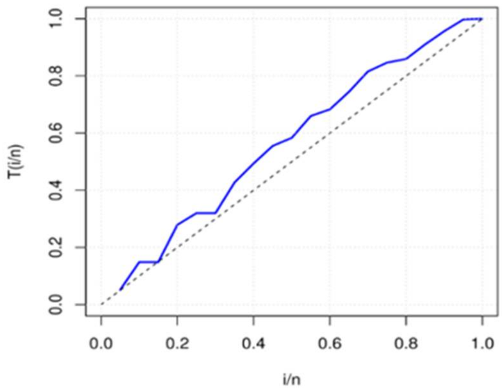
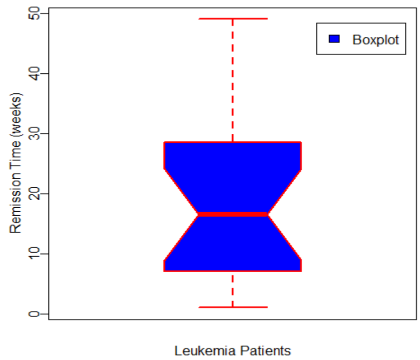




P-P plot Q-Q plot

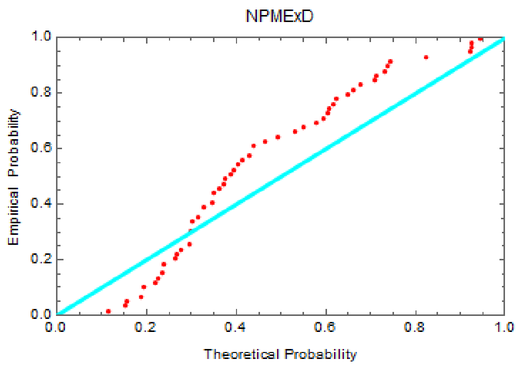
Figure 8. P-P and Q-Q plots of NPMExD based on data I

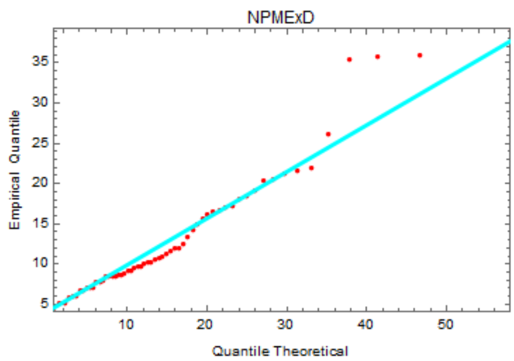




TTT plot Box plot

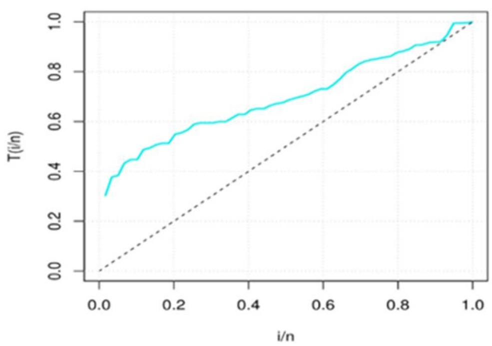
Figure 9. TTT and box plots of NPMExD based on data I

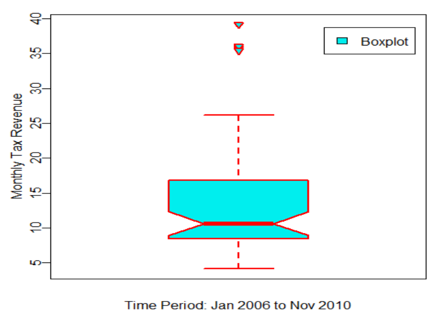




TTT plot Q-Q plot

Figure 10. P-P and Q-Q plots of NPMExD based on data II





TTT plot Box plot

Figure 11. TTT and box plots of NPMExD based on data II

The comparison of the NPMExD and NDRL models suggests that the NPMExD demonstrates superior performance, as evidenced by the smaller values of the AIC, BIC, HQIC, CAIC and KS statistics. This implies that the NPMExD exhibits a better fit to the data and provides a more accurate representation of the underlying phenomena compared to the NDRL model. The reduced AIC, BIC, HQIC and CAIC indicate that the NPMExD model achieves a better goodness of fit, while the smaller KS statistic suggests that it better captures the distributional differences between the observed and predicted values. Thus, based on these metrics, the NPMExD emerges as the preferred choice for modelling the data over the NDRL model.

CONCLUSIONS

The practical applicability and superiority of the proposed NPMExD has been demonstrated through empirical analysis of a real-world data set, where it exhibits enhanced performance compared to the neutrosophic discrete Rayleigh distribution. These findings underscore the improved modelling capabilities and potential utility of the NPMExD. Future research may focus on extending the NPMExD framework to derive novel distributions, with parameter estimation refined through advanced methodologies such as ranked set sampling [28, 29]. Also, the suggested NPMExD can be further modified using the ideas presented by Lathamaheswari et al. [30] and Khan and Gulistan [31].

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