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Full Paper

Ruled surfaces generated by Darboux and instantaneous Pfaff vectors of Salkowski curves in Euclidean 3-space

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Abstract: In the present study ruled surfaces generated by Darboux and instantaneous Pfaff vectors of Salkowski curves in Euclidean 3-space are considered. Some geometric properties of these surfaces such as striction curves, surface normal vectors, tangent and asymptotic planes, distribution parameters, Gaussian and mean curvatures and fundamental forms are investigated. Some results and interpretations of these surfaces are obtained.

Keywords: Salkowski curves, ruled surface, Darboux vector, Gaussian curvature, fundamental forms

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INTRODUCTION

Curves and surfaces are one of the fundamental elements in the field of differential geometry and have important applications in many areas such as mathematical modelling, physics, engineering and computer graphics. In particular, the direction vector defined along a curve and the ruled surfaces constructed by this vector occupy a remarkable place in geometric analysis. In this context, ruled surfaces are defined as surfaces swept by a direction vector moving along a line, and the study of these surfaces offers a wide field of study in both theoretical and applied mathematics. Many authors have conducted various studies on ruled surfaces [1-12].

Salkowski curves are a special class of curves with constant curvature and varying torsion properties [13]. These curves have attracted attention in the geometry literature due to their different characteristics. The defining feature of Salkowski curves is that their curvature remains constant while their torsion varies. This feature distinguishes them from other types of curves and enables their use in the construction of various geometric surfaces. Monterde [14] introduced Frenet frame of Salkowski curves. In another paper we studied ruled surfaces obtained from Frenet vectors of

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Salkowski curves in Euclidean 3-space [15]. Some other papers on Salkowski curves in Euclidean 3-space are available from various sources [16-19].

In addition, one of the frames created at any point of the curve, and the most well-known one, is Frenet frame [20-22]. In the Frenet frame it is considered that the frame rotates at every moment t around a fixed axis. This axis is called Darboux axis, and the unit vector in the direction of this axis is referred to as instantaneous Pfaff vector [23]. Darboux vector is a vector containing the curvature and torsion of a curve and, thank to this vector, the geometric structure of the curve can be analysed in more detail. Ruled surfaces obtained from the Darboux vector defined along a curve offer a rich field of study in terms of both mathematical aesthetics and application [24-26]. These surfaces have important applications in many fields such as differential geometry, elasticity theory and structural engineering.

In this study ruled surfaces generated from Darboux and instantaneous Pfaff vectors of Salkowski curves are discussed. First, the basic properties of Salkowski curves and their Darboux and instantaneous Pfaff vectors are given; then the geometric properties of ruled surfaces generated from these vectors are examined. The aim of the study is to reveal the geometric characteristics of these special surfaces and to shed light on their potential applications.

PRELIMINERIES

For $m \neq \pm\frac{\sqrt{3}}{3}, 0 \in R$ and $n = \frac{m}{\sqrt{m^2+1}}$,

$$\gamma(t) = \frac{n}{4m}\left(\frac{n-1}{1+2n}\sin\left((1+2n)t\right) - \frac{1+n}{1-2n}\sin\left((1-2n)t\right) - 2\sin t,\right.$$
$$\left.\frac{1-n}{1+2n}\cos\left((1+2n)t\right) + \frac{1+n}{1-2n}\cos\left((1-2n)t\right) + 2\cos t, \frac{1}{m}\cos(2nt)\right) \tag{1}$$

is the parametric equation of Salkowski curves in Euclidean 3-space [13] (Figure 1). The interval in which these curves are regular is $\left]-\frac{\pi}{2n}, \frac{\pi}{2n}\right[$; also $\left\|\gamma'(t)\right\| = \frac{n}{m}\cos(nt)$.

The Darboux and instantaneous Pfaff vectors of Salkowski curves are

$$W(t) = \left(\frac{n^2}{m}\sin t, -\frac{n^2}{m}\cos t, \frac{n^2}{m^2}\right) \tag{2}$$

and

$$C(t) = \left(n\sin t, -n\cos t, \frac{n}{m}\right) \tag{3}$$

respectively [17]. The first derivatives of the expressions (1), (2) and (3) with respect to t are respectively as follows:

$$\gamma_t(t) = -\frac{n}{m}\cos(nt)\left(\cos t\cos(nt) + n\sin t\sin(nt), \sin t\cos(nt) - n\sin t\cos(nt), \frac{n}{m}\sin(nt)\right), \tag{4}$$

$$W_t(t) = \left(\frac{n^2}{m}\cos t, \frac{n^2}{m}\sin t, 0\right), \tag{5}$$

and

$$C_t(t) = \left(n\cos t, n\sin t, 0\right). \tag{6}$$

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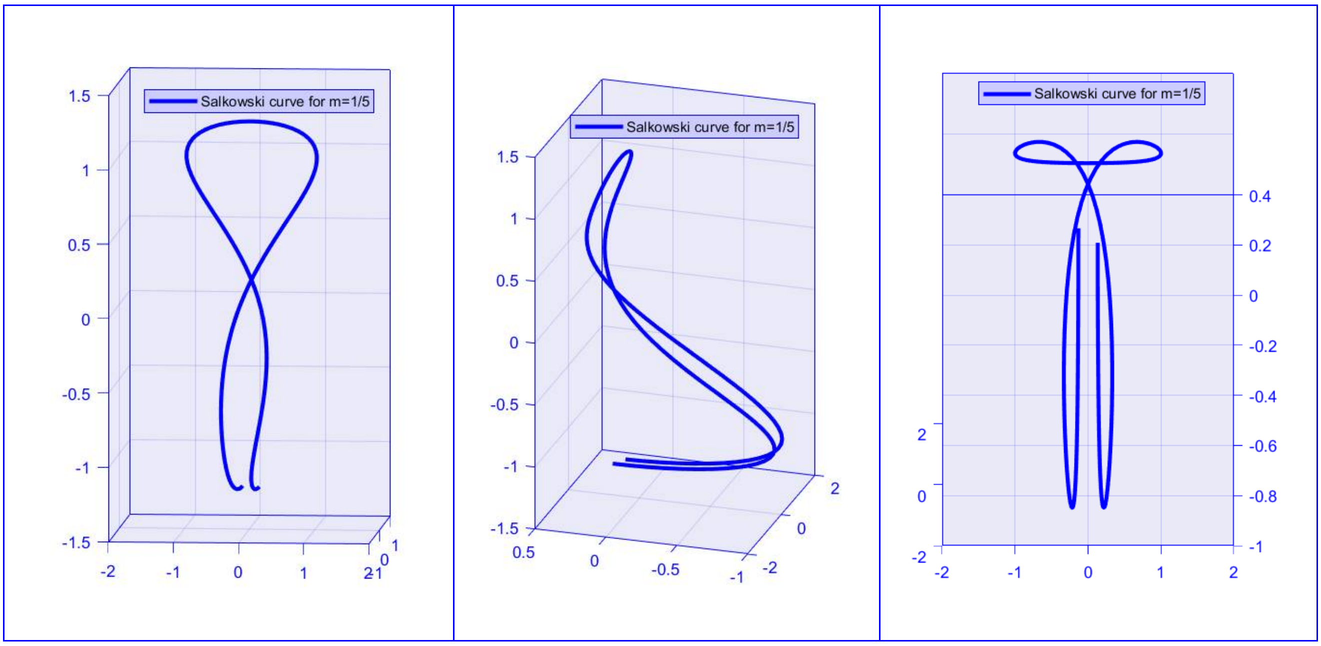

Figure 1. Salkowski curve in Euclidean 3-space $\left(\text{for } m=\frac{1}{5}\right)$ (from left to right: front, left and top views)

RULED SURFACES GENERATED BY DARBOUX VECTOR OF SALKOWSKI CURVES IN EUCLIDEAN 3-SPACE

Throughout this section we denote Salkowski curves by $\gamma(t)$, their Darboux vector by $W(t)$, and ruled surface generated by $W(t)$ along $\gamma(t)$ by $\varphi_W(t, v_W)$.

Theorem 1. $\varphi_W(t, v_W)$ is parameterised as follows (Figure 2):

$$\varphi_W(t,v_W)=\frac{n}{m}\left(\frac{n-1}{4(1+2n)}\sin((1+2n)t)-\frac{n+1}{4(1-2n)}\sin((1-2n)t)-\frac{1}{2}\sin t+nv_W\sin t,\right.$$
$$\frac{1-n}{4(1+2n)}\cos((1+2n)t)+\frac{n+1}{4(1-2n)}\cos((1-2n)t)+\frac{1}{2}\cos t-nv_W\cos t, \tag{7}$$
$$\left.\frac{1}{4m}\cos(2nt)+\frac{nv_W}{m}\right).$$

Proof: According to the definition of ruled surface, we write:

$$\varphi_W(t,v_W)=\gamma(t)+v_W W(t). \tag{8}$$

By using (1) and (2) in (8), we get (7).

Theorem 2. The normal vector $\eta_w(t)$ of $\varphi_W(t, v_W)$ is

$$\eta_w(t)=\left(-\frac{n^3}{m^3}\sin t\cos^2(nt)+\frac{v_W n^4}{m^3}\sin t,\right.$$
$$\left.\frac{n^3}{m^3}\cos t\cos^2(nt)-\frac{v_W n^4}{m^3}\cos t,\frac{n^3}{m^2}\cos^2(nt)-\frac{v_W n^4}{m^2}\right). \tag{9}$$

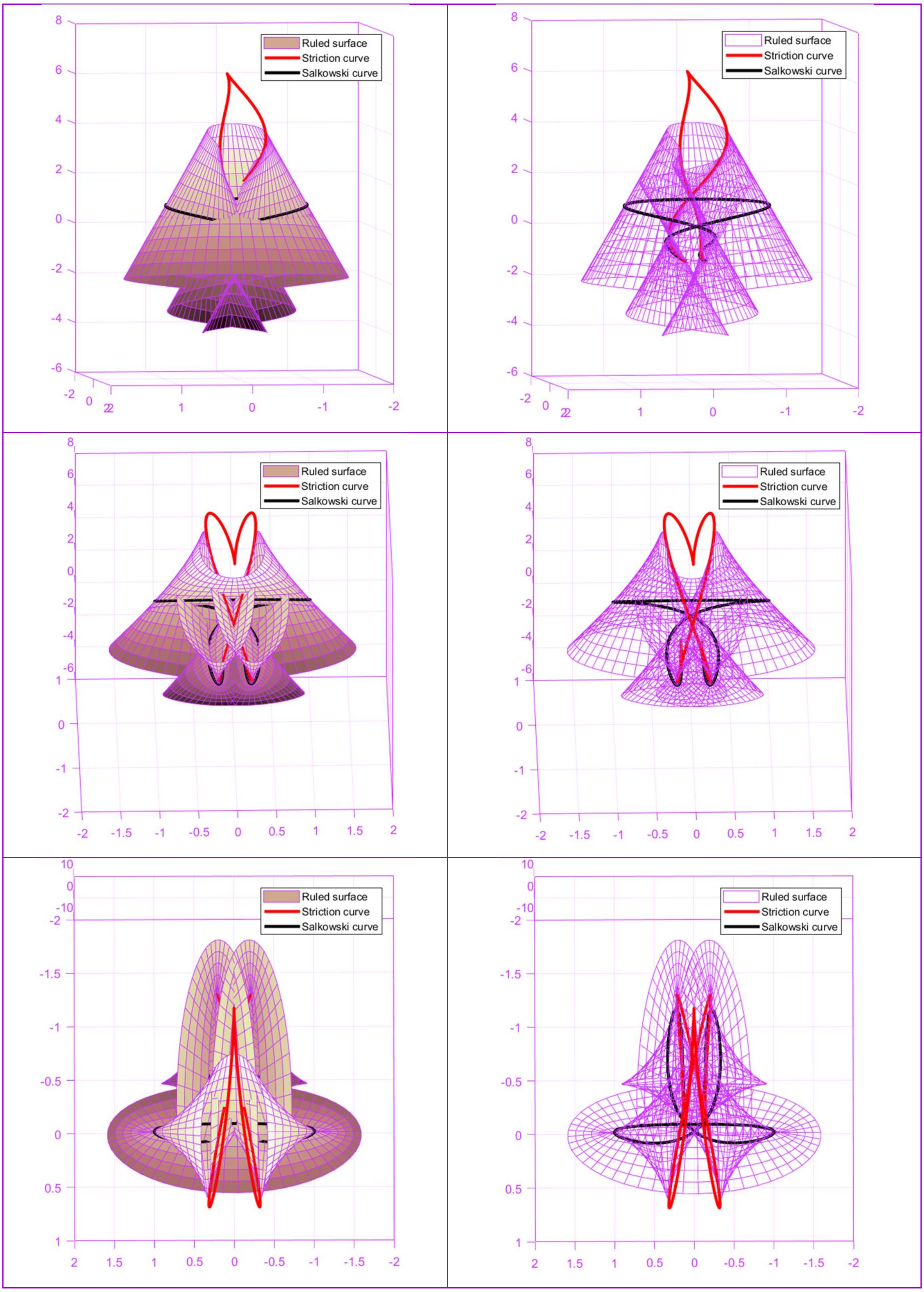

Figure 2. Ruled surface generated by Darboux vector of Salkowski curve (for m=1/5) (top figures: front view; middle figures: top view; bottom figures: bottom view)
(The images on the right are transparent versions of the ones on the left. See striction curve in Theorem 5.)

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Proof: According to the calculation of normal vector of a surface, we write

$$\eta_w(t) = \varphi_{W_t}(t) \wedge \varphi_{W_{v_W}}(t), \tag{10}$$

where $\varphi_{W_t}(t)$ is the derivative of $\varphi_W(t, v_W)$ with respect to t, and $\varphi_{W_{v_W}}(t)$ is the derivative of $\varphi_W(t, v_W)$ with respect to v_W. From (4), (5) and (8), we get

$$\begin{aligned}\varphi_{W_t}(t) = \Big(& -\frac{n}{m}\cos t\cos^2(nt) - \frac{n^2}{m}\sin t\sin(nt)\cos(nt) + \frac{v_W n^2}{m}\cos t, \\ & -\frac{n}{m}\sin t\cos^2(nt) + \frac{n^2}{m}\cos t\cos(nt)\sin(nt) + \frac{v_W n^2}{m}\sin t, \\ & -\frac{n^2}{m^2}\sin(nt)\cos(nt)\Big).\end{aligned} \tag{11}$$

Also from (2) and (8), we get

$$\varphi_{W_{v_W}}(t) = W(t) = \left(\frac{n^2}{m}\sin t, -\frac{n^2}{m}\cos t, \frac{n^2}{m^2}\right). \tag{12}$$

So by using (11) and (12) in (10), we obtain (9).

Theorem 3. The tangent plane to $\varphi_W(t, v_W)$ formed by a fixed point $M = (x, y, z)$ and a variable point $D = (x_0, y_0, z_0)$ is

$$\begin{aligned}&(x - x_0)\left(-\sin t\cos^2(nt) + nv_W \sin t\right) + (y - y_0)\left(\cos t\cos^2(nt) - nv_W \cos t\right) \\ &+ (z - z_0)\left(m\cos^2(nt) - nmv_W\right) = 0.\end{aligned}$$

Proof: The tangent plane of a surface is calculated with the following equation:

$$\langle DM, \eta_W(t)\rangle = 0. \tag{13}$$

From (9) and (13), the proof is completed.

Theorem 4. The parameter v_W to the striction curve of $\varphi_W(t, v_W)$ is

$$v_W = \frac{1}{n}\cos^2(nt). \tag{14}$$

Proof: v_W is calculated with

$$v_W = -\frac{\langle W(t) \wedge W_t(t), W(t) \wedge \gamma_t(t)\rangle}{\langle W(t) \wedge W_t(t), W(t) \wedge W_t(t)\rangle}. \tag{15}$$

From (2) and (5), we get

$$W(t) \wedge W_t(t) = \left(-\frac{n^4}{m^3}\sin t, \frac{n^4}{m^3}\cos t, \frac{n^4}{m^2}\right). \tag{16}$$

From (2) and (4), we get

$$W(t) \wedge \gamma_t(t) = \left(\frac{n^3}{m^3}\sin t\cos^2(nt), -\frac{n^3}{m^3}\cos t\cos^2(nt), -\frac{n^3}{m^2}\cos^2(nt)\right). \tag{17}$$

Also, from (16) and (17), we have

$$\langle W(t) \wedge W_t(t), W(t) \wedge \gamma_t(t)\rangle = -\frac{n^5}{m^4}\cos^2(nt), \tag{18}$$

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$$\left\langle W(t)\wedge W_t(t), W(t)\wedge W_t(t)\right\rangle = \frac{n^6}{m^4}. \tag{19}$$

By using (18) and (19) in (15), we obtain (14).

Theorem 5. The striction curve $\psi_W(t)$ of $\varphi_W(t,v_W)$ is parameterised as follows (Figure 2):

$$\psi_W(t) = \frac{n}{m}\left(\frac{n-1}{4(1+2n)}\sin((1+2n)t) - \frac{n+1}{4(1-2n)}\sin((1-2n)t) - \frac{1}{2}\sin t + \sin t\cos^2(nt),\right.$$
$$\frac{1-n}{4(1+2n)}\cos((1+2n)t) + \frac{n+1}{4(1-2n)}\cos((1-2n)t) + \frac{1}{2}\cos t - \cos t\cos^2(nt),$$
$$\left.\frac{1}{4m}\cos(2nt) + \frac{1}{m}\cos^2(nt)\right).$$

Proof: We obtain the striction curve $\psi_W(t)$ of $\varphi_W(t,v_W)$ by substituting v_W into (7).

Corollary 1. The striction and base (Salkowski) curves of $\varphi_W(t,v_W)$ never coincide.

Theorem 6. The asymptotic plane to $\varphi_W(t,v_W)$ formed by a variable point $D=(x_0,y_0,z_0)$ and a fixed point $M=(x,\ y,\ z)$ is

$$(x-x_0)\sin t - (y-y_0)\cos t - (z-z_0)m = 0.$$

Proof: From (2) and (5), the normal vector at infinity of $\varphi_W(t,v_W)$ is

$$\mu_W(t) = \left(-\frac{n^4}{m^3}\sin t, \frac{n^4}{m^3}\cos t, \frac{n^4}{m^2}\right). \tag{20}$$

The asymptotic plane of a surface is given by

$$\left\langle DM, \mu_W(t)\right\rangle = 0. \tag{21}$$

From (20) and (21), the proof is completed.

Theorem 7. The distribution parameter $P_W(t)$ of $\varphi_W(t,v_W)$ is

$$P_W(t) = 0.$$

Proof: The distribution parameter of a surface is found by

$$P_W(t) = \frac{\left\langle \gamma_t(t), W(t)\wedge W_t(t)\right\rangle}{\left\|W_t(t)\right\|^2}. \tag{22}$$

From (4) and (16), we get

$$\left\langle \gamma_t(t), W(t)\wedge W_t(t)\right\rangle = 0. \tag{23}$$

So, from (22) and (23), the proof is completed.

Corollary 2. $\varphi_W(t,v_W)$ is a developable surface.

Theorem 8. The Gaussian curvature $K_W(t)$ of $\varphi_W(t,v_W)$ is

$$K_W(t) = 0.$$

Proof: The Gaussian curvature of a surface is found by the following equation:

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$$K_W(t) = -\frac{\langle \gamma_t(t), W(t)\wedge W_t(t)\rangle^2}{\left\| \varphi_t(t)\wedge \varphi_{v_W}(t)\right\|^4}. \tag{24}$$

From (23) and (24), the theorem is proved.

Theorem 9. The first fundamental form I_W of $\varphi_W(t, v_W)$ is

$$I_W = \left(\frac{n^2}{m^2}\cos^2(nt)(1-2nv_W) + \frac{n^4 v_W^2}{m^2}\right)dt^2 - \frac{2n^2}{m^2}\cos(nt)\sin(nt)\,dtdv_W + \frac{n^2}{m^2}dv_W^2.$$

Proof: The first fundamental form of a surface is found by

$$I_W = E_W dt^2 + 2F_W dtdv_W + G_W dv_W^2, \tag{25}$$

where

$$E_W = \langle \varphi_{W_t}(t), \varphi_{W_t}(t)\rangle, \quad F_W = \langle \varphi_{W_t}(t), \varphi_{W_{v_W}}(t)\rangle, \quad G_W = \langle \varphi_{W_{v_W}}(t), \varphi_{W_{v_W}}(t)\rangle. \tag{26}$$

If (11) and (12) are substituted in (26), then we get

$$\begin{cases} E_W = \dfrac{n^2}{m^2}\cos^2(nt)(1-2nv_W) + \dfrac{n^4 v_W^2}{m^2}, \\ F_W = -\dfrac{n^2}{m^2}\cos(nt)\sin(nt), \\ G_W = \dfrac{n^2}{m^2}. \end{cases} \tag{27}$$

By using (27) in (25), the proof is completed.

Theorem 10. The second fundamental form II_W of $\varphi_W(t, v_W)$ is

$$II_W = -\frac{n^2}{m^2}\left(\cos^2(nt) - nv_W\right)dt^2.$$

Proof: The second fundamental form of a surface is found by

$$II_W = l_W dt^2 + 2m_W dtdv_W + n_W dv_W{}^2, \tag{28}$$

where

$$l_W = \langle \varphi_{W_{tt}}(t), \eta_w(t)\rangle, \quad m_W = \langle \varphi_{W_{tv_W}}(t), \eta_w(t)\rangle, \quad n_W = \langle \varphi_{W_{v_W v_W}}(t), \eta_w(t)\rangle. \tag{29}$$

From (11) and (12), we get

$$\begin{aligned}\varphi_{W_{tt}}(t) = \frac{n}{m}\Big(&\cos(nt)\left(\sin t\cos(nt) + n\cos t\sin(nt)\right) + n^2\sin t\left(\sin^2(nt) - \cos^2(nt)\right) - nv_W\sin t, \\ &-\cos(nt)\left(\cos t\cos(nt) - n\sin t\sin(nt)\right) - n^2\cos t\left(\sin^2(nt) - \cos^2(nt)\right) + nv_W\cos t, \\ &\frac{n^2}{m}\left(\sin^2(nt) - \cos^2(nt)\right)\Big),\end{aligned}$$

$$\varphi_{W_{tv_W}}(t) = \frac{n^2}{m}(\cos t, \sin t, 0)$$

and

$$\varphi_{W_{v_W v_W}}(t) = (0,0,0).$$

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By using these vectors in (29), the following expressions are obtained:

$$\begin{cases} l_W = -\dfrac{n^2}{m^2}\left(\cos^2(nt) - nv_W\right), \\ m_W = 0, \\ n_W = 0. \end{cases} \tag{30}$$

If (30) is substituted in (28), then the proof is completed.

Theorem 11. The third fundamental form III_W of $\varphi_W(t, v_W)$ is

$$III_W = \frac{n^6}{m^6}\left(4m^2\cos^2(nt)\sin^2(nt) + \left(\cos^2(nt) - nv_W\right)^2\right)dt^2 + \frac{4n^6}{m^4}\cos(nt)\sin(nt)\,dt\,dv_W + \frac{n^6}{m^4}dv_W^2.$$

Proof: The third fundamental form III_W of a surface is found by

$$III_W = e_W dt^2 + 2f_W dt dv_W + g_W dv_W{}^2, \tag{31}$$

where

$$e_W = \left\langle \eta_{W_t}(t), \eta_{W_t}(t) \right\rangle, \quad f_W = \left\langle \eta_{W_t}(t), \eta_{W_{v_W}}(t) \right\rangle, \quad g_W = \left\langle \eta_{W_{v_W}}(t), \eta_{W_{v_W}}(t) \right\rangle. \tag{32}$$

From (9), we get

$$\begin{aligned} \eta_{W_t}(t) = \frac{n^3}{m^3}\big(&-\cos t\cos^2(nt) + 2n\sin t\cos(nt)\sin(nt) + nv_W\cos t, \\ &-\sin t\cos^2(nt) - 2n\cos t\cos(nt)\sin(nt) + nv_W\sin t, \\ &-2mn\cos(nt)\sin(nt)\big) \end{aligned}$$

and

$$\eta_{W_{v_W}}(t) = \frac{n^4}{m^3}(\sin t, -\cos t, -m).$$

If these vectors are used in (32), we have

$$\begin{cases} e_W = \dfrac{n^6}{m^6}\left(4m^2\cos^2(nt)\sin^2(nt) + \left(\cos^2(nt) - nv_W\right)^2\right), \\ f_W = \dfrac{4n^6}{m^4}\cos(nt)\sin(nt), \\ g_W = \dfrac{n^6}{m^4}. \end{cases} \tag{33}$$

If (33) is substituted in (31), then the theorem is proved.

Theorem 12. The mean curvature $H_W(t)$ of $\varphi_W(t, v_W)$ is

$$H_W(t) = -\frac{1}{2\left(\cos^2(nt) - nv_W\right)}.$$

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Proof: The mean curvature of a surface is found by the following equation:

$$H_W(t)=\frac{E_W n_W-2F_W m_W+G_W l_W}{2\left(E_W G_W-F_W{}^2\right)}. \tag{34}$$

From (27) and (33), we get

$$E_W n_W-2F_W m_W+G_W l_W=-\frac{n^6}{m^6}\left(\cos^2(nt)-nv_W\right)^2$$

and

$$E_W G_W-F_W{}^2=\left(\frac{n^4}{m^4}\left(\cos^2(nt)-nv_W\right)^2\right)^{3/2}.$$

If the last expressions are substituted in (34), the theorem is proved.

RULED SURFACES GENERATED BY INSTANTANEOUS PFAFF VECTOR OF SALKOWSKI CURVES IN EUCLIDEAN 3-SPACE

Throughout this section, we denote Salkowski curves by $\gamma(t)$, their instantaneous Pfaff vector by $C(t)$, and ruled surface generated by $C(t)$ along $\gamma(t)$ by $\varphi_C(t,v_C)$.

Theorem 13. $\varphi_C(t,v_C)$ is parameterised as follows (Figure 3):

$$\begin{aligned}\varphi_C(t,v_C)=\frac{n}{m}\Bigg(&\frac{n-1}{4(1+2n)}\sin((1+2n)t)-\frac{n+1}{4(1-2n)}\sin((1-2n)t)-\frac{1}{2}\sin t+mv_C\sin t,\\ &\frac{1-n}{4(1+2n)}\cos((1+2n)t)+\frac{n+1}{4(1-2n)}\cos((1-2n)t)+\frac{1}{2}\cos t-mv_C\cos t,\\ &\frac{1}{4m}\cos(2nt)+v_C\Bigg).\end{aligned} \tag{35}$$

Proof: According to the definition of ruled surface, we write:

$$\varphi_C(t,v_C)=\gamma(t)+v_C C(t). \tag{36}$$

By using (1) and (3) in (36), we get (35).

Theorem 14. The normal vector $\eta_C(t)$ of $\varphi_C(t,v_C)$ is

$$\begin{aligned}\eta_C(t)=\frac{n^2}{m^2}\big(&-\sin t\cos^2(nt)+mv_C\sin t,\cos t\cos^2(nt)-mv_C\cos t,\\ &m\cos^2(nt)-m^2v_C\big).\end{aligned} \tag{37}$$

Proof: From (4), (6) and (36), we get

$$\begin{aligned}\varphi_{C_t}(t)=\frac{n}{m}\Big(&-\cos t\cos^2(nt)-n\sin t\sin(nt)\cos(nt)+mv_C\cos t,\\ &-\sin t\cos^2(nt)+n\cos t\cos(nt)\sin(nt)+mv_C\sin t,\\ &-\frac{n}{m}\sin(nt)\cos(nt)\Big),\end{aligned} \tag{38}$$

and from (3) and (36), we get

$$\varphi_{C_{v_C}}(t)=C(t)=\left(n\sin t,-n\cos t,\frac{n}{m}\right). \tag{39}$$

So by considering the expression (10) and using (38) and (39), we obtain (37).

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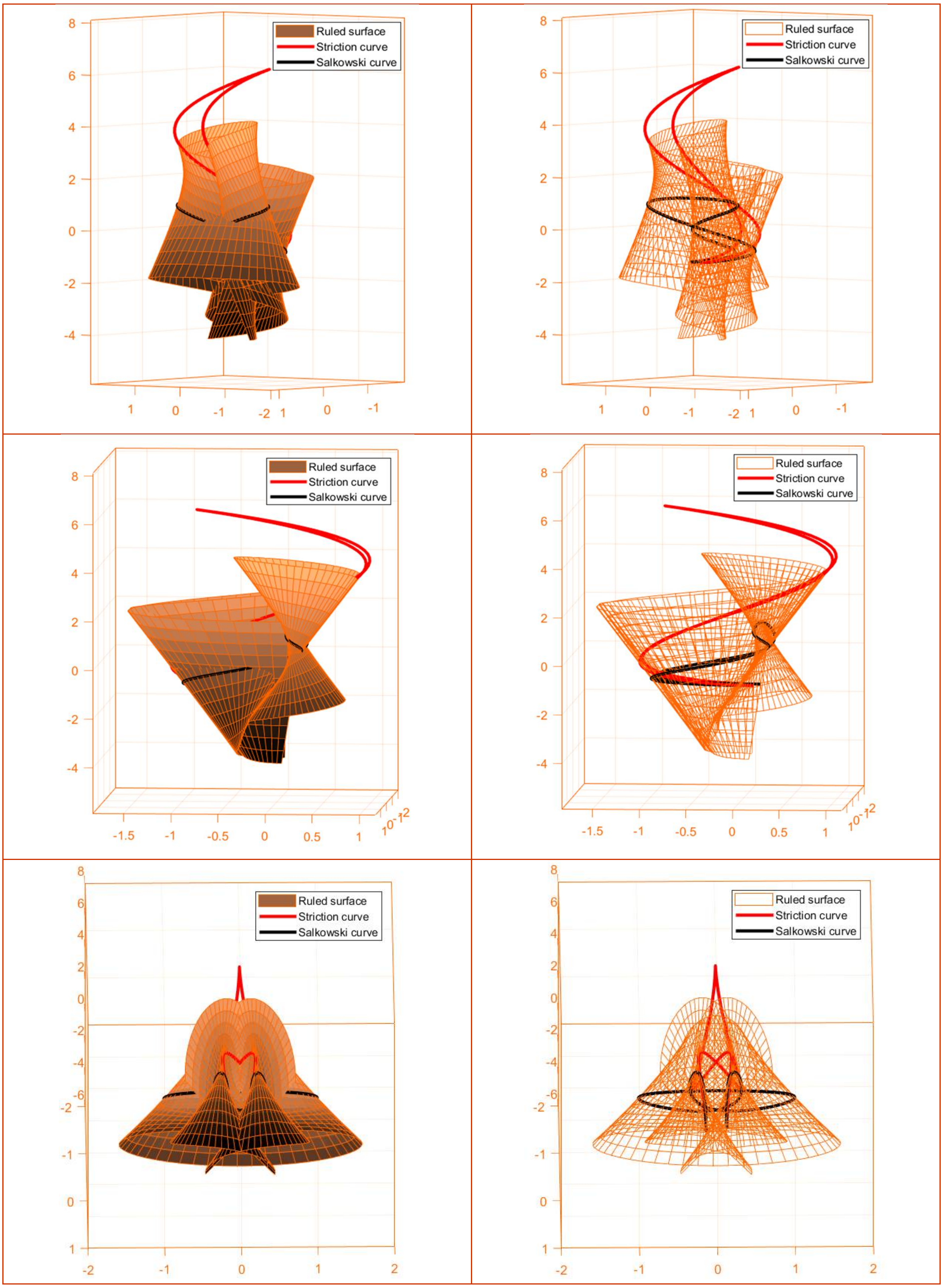

Figure 3. Ruled surfaces generated by instantaneous Pfaff vector of Salkowski curve (for m=1/5) (top figures: right view; middle figures: left view: bottom figures: bottom view)
(The images on the right are transparent versions of the ones on the left. See striction curve in Theorem 17.)

Theorem 15. The tangent plane to $\varphi_C(t, v_C)$ formed by a fixed point $M = (x, y, z)$ and a variable point $D = (x_0, y_0, z_0)$ is

$$(x - x_0)\left(-\sin t \cos^2(nt) + mv_C \sin t\right) + (y - y_0)\left(\cos t \cos^2(nt) - mv_C \cos t\right)$$
$$+ (z - z_0)\left(m \cos^2(nt) - m^2 v_C\right) = 0.$$

Proof: By considering the expressions (13) and (37), the proof is completed.

Theorem 16. The parameter v_C of the striction curve of $\varphi_C(t, v_C)$ is

$$v_C = \frac{1}{m}\cos^2(nt). \tag{40}$$

Proof: From (3) and (6), we have

$$C(t) \wedge C_t(t) = \left(-\frac{n^2}{m}\sin t, \frac{n^2}{m}\cos t, n^2\right), \tag{41}$$

and from (3) and (4), we get

$$C(t) \wedge \gamma_t(t) = \left(\frac{n^2}{m^2}\sin t \cos^2(nt), -\frac{n^2}{m^2}\cos t \cos^2(nt), -\frac{n^2}{m}\cos^2(nt)\right). \tag{42}$$

Also, from (41) and (42), we obtain

$$\left\langle C(t) \wedge C_t(t), C(t) \wedge \gamma_t(t)\right\rangle = -\frac{n^2}{m}\cos^2(nt), \tag{43}$$

$$\left\langle C(t) \wedge C_t(t), C(t) \wedge C_t(t)\right\rangle = n^2. \tag{44}$$

If (13), (43) and (44) are considered, then we have (40).

Theorem 17. The striction curve $\psi_C(t)$ of $\varphi_C(t, v_C)$ is parameterised as follows (Figure 3):

$$\psi_C(t) = \frac{n}{m}\left(\frac{n-1}{4(1+2n)}\sin((1+2n)t) - \frac{n+1}{4(1-2n)}\sin((1-2n)t) - \frac{1}{2}\sin t + \sin t \cos^2(nt),\right.$$
$$\frac{1-n}{4(1+2n)}\cos((1+2n)t) + \frac{n+1}{4(1-2n)}\cos((1-2n)t) + \frac{1}{2}\cos t - \cos t \cos^2(nt),$$
$$\left.\frac{1}{4m}\cos(2nt) + \frac{1}{m}\cos^2(nt)\right).$$

Proof: We obtain the striction curve $\psi_C(t)$ of $\varphi_C(t, v_C)$ by substituting v_C into (36).

Corollary 3. The striction and base (Salkowski) curves of $\varphi_C(t, v_C)$ never coincide.

Theorem 18. The asymptotic plane to $\varphi_C(t, v_C)$ formed by a variable point $D = (x_0, y_0, z_0)$ and a fixed point $M = (x, y, z)$ is

$$(x - x_0)\sin t - (y - y_0)\cos t - (z - z_0)m = 0.$$

Proof: From (2) and (5), the normal vector at infinity of $\varphi_C(t, v_C)$ is

$$\mu_C(t) = \left(-\frac{n^2}{m}\sin t, \frac{n^2}{m}\cos t, n^2\right). \tag{45}$$

From (21) and (45), the proof is completed.

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Theorem 19. The distribution parameter $P_C(t)$ of $\varphi_C(t,v_C)$ is
$P_C(t)=0.$

Proof: From (5) and (41), we get
$$\langle \gamma_t(t), C(t)\wedge C_t(t)\rangle = 0. \tag{46}$$
So, by considering (22) and (46), the proof is completed.

Corollary 4. $\varphi_C(t,v_C)$ is a developable surface.

Theorem 20. The Gaussian curvature $K_C(t)$ of $\varphi_C(t,v_C)$ is
$K_C(t)=0$.

Proof: If (24) and (46) are considered, the proof is completed.

Theorem 21. The first fundamental form I_C of $\varphi_C(t,v_C)$ is
$$I_C=\left(\frac{n^2}{m^2}\cos^2(nt)(1-2v_C)+n^2v_C^2\right)dt^2+dtdv_C-\frac{n}{m}\cos(nt)\sin(nt)dv_C^2.$$

Proof: The first fundamental form of a surface is found by
$$I_C=E_Cdt^2+2F_Cdtdv_C+G_Cdv_C^2, \tag{47}$$
where
$$E_C=\langle \varphi_{C_t}(t),\varphi_{C_t}(t)\rangle,\quad F_C=\langle \varphi_{C_t}(t),\varphi_{C_{v_C}}(t)\rangle,\quad G_C=\langle \varphi_{C_{v_C}}(t),\varphi_{C_{v_C}}(t)\rangle. \tag{48}$$
If (38) and (39) are substituted in (48), then we get
$$\begin{cases} E_C=\dfrac{n^2}{m^2}\cos^2(nt)(1-2v_C)+n^2v_C^2, \\ F_C=-\dfrac{n}{m}\cos(nt)\sin(nt), \\ G_C=1. \end{cases} \tag{49}$$
By using (49) in (47), the proof is completed.

Theorem 22. The second fundamental form II_C of $\varphi_C(t,v_C)$ is
$$II_C=-\frac{n^2}{m^2}\left(\cos^2(nt)-mv_C\right)dt^2.$$

Proof: The second fundamental form of a surface is found by
$$II_C=l_Cdt^2+2m_Cdtdv_C+n_Cdv_C{}^2, \tag{50}$$
where
$$l_C=\langle \varphi_{C_{tt}}(t),\eta_C(t)\rangle,\quad m_C=\langle \varphi_{C_{tv_C}}(t),\eta_C(t)\rangle,\quad n_C=\langle \varphi_{C_{v_Cv_C}}(t),\eta_C(t)\rangle. \tag{51}$$
From (38) and (39), we get

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$$\varphi_{C_{tt}}(t)=\Big(\frac{n}{m}\big(\cos(nt)\big(\sin t\cos(nt)+n\cos t\sin(nt)\big)+n^2\sin t\big(\sin^2(nt)-\cos^2(nt)\big)\big)-nv_C\sin t,$$
$$-\frac{n}{m}\big(\cos(nt)\big(\cos t\cos(nt)-n\sin t\sin(nt)\big)-n^2\cos t\big(\sin^2(nt)-\cos^2(nt)\big)\big)+nv_C\cos t,$$
$$\frac{n^3}{m^2}\big(\sin^2(nt)-\cos^2(nt)\big)\Big),$$

$$\varphi_{C_{tv_C}}(t)=n(\cos t,\sin t,0)$$

and

$$\varphi_{C_{v_Cv_C}}(t)=(0,0,0).$$

By using these vectors in (51), the following expressions are got:

$$\begin{cases} l_C=-\dfrac{n^2}{m^2}\left(\cos^2(nt)-mv_C\right), \\ m_C=0, \\ n_C=0. \end{cases} \tag{52}$$

By using (52) in (50), the proof is completed.

Theorem 23. The third fundamental form III_C of $\varphi_C(t,v_C)$ is

$$III_C=\frac{n^4}{m^4}\left(4m^2\cos^2(nt)\sin^2(nt)+\left(\cos^2(nt)-mv_C\right)^2\right)dt^2$$
$$-\frac{2n^4}{m^3}\left(\cos^2(nt)-2mn\cos(nt)\sin(nt)-mv_C\right)dtdv_C+n^2dv_C^2.$$

Proof: The third fundamental form of a surface is found by

$$III_C=e_Cdt^2+2f_Cdtdv_C+g_Cdv_C^{\,2}, \tag{53}$$

where

$$e_C=\left\langle\eta_{C_t}(t),\eta_{C_t}(t)\right\rangle,\quad f_C=\left\langle\eta_{C_t}(t),\eta_{C_{v_C}}(t)\right\rangle,\quad g_C=\left\langle\eta_{C_{v_C}}(t),\eta_{C_{v_C}}(t)\right\rangle. \tag{54}$$

From (37), we get

$$\eta_{C_t}(t)=\frac{n^2}{m^2}\big(-\cos t\cos^2(nt)+2n\sin t\cos(nt)\sin(nt)+mv_C\cos t,$$
$$-\sin t\cos^2(nt)-2n\cos t\cos(nt)\sin(nt)+mv_C\sin t,$$
$$-2mn\cos(nt)\sin(nt)\big)$$

and

$$\eta_{C_{v_C}}(t)=\frac{n^2}{m}(\sin t,-\cos t,-m).$$

If these vectors are used in (54), we have

$$\begin{cases} e_C = \frac{n^4}{m^4}\left(4m^2\cos^2(nt)\sin^2(nt) + \left(\cos^2(nt) - mv_C\right)^2\right), \\ f_C = -\frac{2n^4}{m^3}\left(\cos^2(nt) - 2mn\cos(nt)\sin(nt) - mv_C\right), \\ g_C = n^2. \end{cases} \tag{55}$$

Bu using (55) in (53), the proof is completed.

Theorem 24. The mean curvature $H_C(t)$ of $\varphi_C(t, v_C)$ is

$$H_C(t) = -\frac{1}{2\left(\cos^2(nt) - mv_C\right)}.$$

Proof: From (49) and (55), we get

$$E_C n_C - 2F_C m_C + G_C l_C = -\frac{n^3}{m}\left(\frac{1}{m}\cos^2(nt) - v_C\right)^2$$

and

$$E_C G_C - F_C^{\ 2} = n^3\left(\frac{1}{m}\cos^2(nt) - v_C\right)^3.$$

If the last two expressions are considered in (34), the proof is completed.

CONCLUSIONS

In the present paper we have introduced ruled surfaces generated on Salkowski curves in Euclidean 3-space by their Darboux and instantaneous Pfaff vectors. We have studied some geometric properties of these surfaces such as fundamental forms and Gaussian and mean curvatures and concluded that they are developable. Similar studies can be done with anti-Salkowski curves or Salkowski curves in the Minkowski 3-space.

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