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Full Paper

Characterisation of multiplication operator on bicomplex Lorentz spaces with hyperbolic norm

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Abstract: The multiplication operator $M_u f = u \cdot f$ within the bicomplex Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$ is investigated. It is initially established that M_u is $\mathbb{D}$-bounded if and only if the function u is essentially $\mathbb{D}$-bounded. Subsequently, it is proved that the collection of all $\mathbb{D}$-bounded multiplication operators on $\mathbb{BC}$-Lorentz spaces forms a maximal abelian sub-algebra within the Banach algebra of all bounded linear operators on $L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$. Additionally, a necessary and sufficient condition for the compactness of M_u is provided. Finally, by introducing a condition for a multiplication operator to exhibit a closed range, the author identifies some conditions equivalent to M_u being a Fredholm operator.

Keywords: bicomplex numbers, $\mathbb{BC}$-valued functions, hyperbolic norm, $\mathbb{D}$-distribution function, $\mathbb{D}$-rearrangement, multiplication operator, Fredholm operator

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INTRODUCTION

$\mathbb{BC}$-valued functions arise naturally in various mathematical fields including probability theory, mathematical analysis and functional analysis, and understanding their properties is crucial for advancing these areas of study. Functional analysis traditionally deals with vector spaces over a field, such as the complex numbers or the real numbers. However, by considering modules with bicomplex scalars, where the scalars are elements of the bicomplex numbers, a broader framework is introduced. This extension allows for the exploration of new mathematical structures and the investigation of properties beyond the classical setting. One influential work that has contributed to this area is a book by Alpay et al. [1]. It presents notable results, techniques and applications pertaining to the study of modules with bicomplex scalars in the context of functional analysis. These results shed light on the behaviour of modules with bicomplex scalars, reveal connections to other areas of mathematics, and potentially find applications in physics, engineering or other disciplines.

The series of articles mentioned in the references highlight the systematic study of topological bicomplex modules and various fundamental theorems related to them. Here is a breakdown of the articles and their contributions.

Luna-Elizarrarás et al. [2] investigated Hahn-Banach theorem for bicomplex modules and hyperbolic modules. The study of topological bicomplex modules, exploring their topological properties and investigating concepts such as convergence, continuity and compactness in this context was done by Kumar and Saini [3]. Also, fundamental theorems including the principle of uniform boundedness, open mapping theorem, interior mapping theorem and closed graph theorem for bicomplex modules were studied. $\mathbb{BC}$ bounded linear operators and bicomplex functional calculus were examined by Colombo et al. [4].

Saini et al. [5] extended the study of fundamental theorems to the setting of topological bicomplex modules. They delved further into the study of topological hyperbolic modules, topological bicomplex modules, exploring the properties of linear operators, continuity and related topological concepts specific to these settings.

Bicomplex C*-algebras were studied by Kumar et al. [6]. The work covered bicomplex operator algebras, spectral theory and topological properties of C*-algebras defined on bicomplex vector space. Bicomplex linear operators on $\mathbb{BC}$ Hilbert spaces were investigated by Kumar and Singh [7]. They also explored the properties of these operators.

The book authored by Luna-Elizarrarás et al.[8] provides an in-depth exploration of bicomplex analysis and geometry. It covers a wide range of topics including holomorphic functions, integration, differential equations and geometric properties specific to the bicomplex domain. Besides these, bicomplex Lebesgue spaces and some of their geometric and topological properties were defined and studied [9-11]. Bicomplex sequence spaces $l_p(\mathbb{BC})$ were defined and examined with various properties by Değirmen and Sağır [12] and Sağır et al. [13].

These references collectively represent significant contributions to the study of bicomplex modules, functional analysis and related areas. They showcase the exploration of properties, the development of new theorems and the application of functional analysis techniques in the context of bicomplex numbers.

PRELIMINARIES ON $\mathbb{BC}$

Now we give a summary of bicomplex numbers with some basic properties .The set of bicomplex numbers $\mathbb{BC}$ which is a two-dimensional extension of the complex numbers is defined as

$$\mathbb{BC} := \{W = w_1 + jw_2 | \ w_1, w_2 \in \mathbb{C}(i)\}$$

where i and j are imaginary units satisfying $ij = ji$, $i^2 = j^2 = -1$. Here $\mathbb{C}(i)$ is the field of complex numbers with the imaginary unit i. According to the ring structure, for any $Z = z_1 + jz_2, W = w_1 + jw_2$ in $\mathbb{BC}$, the usual addition and multiplication are defined as

$$Z + W = (z_1 + w_1) + j(z_2 + w_2) \text{ and } ZW = (z_1w_1 - z_2w_2) + j(z_2w_1 + z_1w_2).$$

In the sense of module structure, the set $\mathbb{BC}$ is a module in itself. The product of the imaginary units i and j brings out a hyperbolic unit k such that $k^2 = 1$. The product operation of all units i, j and k in the bicomplex numbers is commutative. Specifically, the following relations hold:

$$ij = k, jk = -i \text{ and } ik = -j.$$

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Hyperbolic numbers $\mathbb{D}$ are two-dimensional extension of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\alpha = \beta_1 + k\beta_2$, where β_1 and β_2 are real numbers and k is the hyperbolic unit. For any two hyperbolic numbers $\alpha = \beta_1 + k\beta_2$ and $\gamma = \delta_1 + k\delta_2$, addition and multiplication are defined as follows:

$$\alpha + \gamma = (\beta_1 + \delta_1) + k(\beta_2 + \delta_2) \text{ and } \alpha\gamma = (\beta_1\delta_1 + \beta_2\delta_2) + k(\beta_1\delta_2 + \beta_2\delta_1).$$

The hyperbolic numbers form a ring. Unlike the complex numbers, the hyperbolic numbers do not have a multiplicative inverse for all non-zero elements and they can also be considered a significant subset of the bicomplex numbers $\mathbb{BC}$.

Let $W = w_1 + jw_2 \in \mathbb{BC}$ where $w_1, w_2 \in \mathbb{C}(i)$. By the notation of W with imaginary units i and j, the conjugations are formed for bicomplex numbers as follows: $\overline{W}_1 = \overline{w_1} + j\overline{w_2}$, $\overline{W}_2 = w_1 - jw_2$ and $\overline{W}_3 = \overline{w_1} - j\overline{w_2}$, where $\overline{w_1}$ and $\overline{w_2}$ are the usual complex conjugates of $w_1, w_2 \in \mathbb{C}(i)$ respectively [1, 8, 14].

For any bicomplex number W, the following three moduli: $|W|_i^2 = W \cdot \overline{W}_2 = w_1^2 + w_2^2 \in \mathbb{C}(i)$, $|W|_j^2 = W \cdot \overline{W}_1 = (|w_1|^2 - |w_2|^2) + j\big(2\mathrm{Re}(w_1\overline{w_2})\big) \in \mathbb{C}(j)$ and $|W|_k^2 = W \cdot \overline{W}_3 = (|w_1|^2 + |w_2|^2) + k\big(-2\mathrm{Im}(w_1\overline{w_2})\big) \in \mathbb{D}$, were written [1, 8, 14].

Furthermore, $\mathbb{BC}$ is a normed space with the norm $\|W\|_{\mathbb{BC}} = \sqrt{|w_1|^2 + |w_2|^2}$ for any $W = w_1 + jw_2$ in $\mathbb{BC}$ [2]. According to this, $\|W_1W_2\|_{\mathbb{BC}} \leq \sqrt{2}\|W_1\|_{\mathbb{BC}}\|W_2\|_{\mathbb{BC}}$ for every $W_1, W_2 \in \mathbb{BC}$, and finally $\mathbb{BC}$ is a modified Banach algebra [1, 14].

If the hyperbolic numbers e_1 and e_2 are defined as $e_1 = \frac{1+k}{2}$ and $e_2 = \frac{1-k}{2}$, then it is easy to see that the set $\{e_1, e_2\}$ is a linearly independent set in $\mathbb{C}(i)$-vector space $\mathbb{BC}$. The set $\{e_1, e_2\}$ also satisfies the following properties:

$$e_1^2 = e_1, \ \ e_2^2 = e_2, \ \ \overline{(e_1)}_3 = e_1, \ \ \overline{(e_2)}_3 = e_2, \ \ e_1 + e_2 = 1, \ \ e_1 \cdot e_2 = 0,$$

with $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \sqrt{2}/2$. By using the set $\{e_1, e_2\}$, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear combination of e_1 and e_2 uniquely. That is, $W = w_1 + jw_2$ can be written as

$$W = w_1 + jw_2 = e_1z_1 + e_2z_2, \tag{1}$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$ [1]. Here z_1 and z_2 are elements of $\mathbb{C}(i)$ and the formula in (1) is called the *idempotent representation* of the bicomplex number W.

Besides the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$, another norm named ($\mathbb{D}$-valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1z_1 + e_2z_2$ is defined as

$$|W|_k = e_1|z_1| + e_2|z_2|.$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, an idempotent representation can also be written as $\mathbb{D} \subset \mathbb{BC}$. Thus, $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$ can be written as

$$\alpha = e_1\alpha_1 + e_2\alpha_2,$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. If $\beta_1 > 0$ and $\beta_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, then we say that α is a positive hyperbolic number. Thus, the set of non-negative hyperbolic numbers $\mathbb{D}^+ \cup \{0\}$ is defined by

$$\begin{aligned}\mathbb{D}^+ \cup \{0\} &= \{\alpha = \beta_1 + k\beta_2 : \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\} \\ &= \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}.\end{aligned}$$

Now let α and γ be any two elements of $\mathbb{D}$. A partially ordered relation " $\preceq$ " is defined on $\mathbb{D}$ by $\alpha \preceq \gamma \ \Leftrightarrow \gamma - \alpha \in \mathbb{D}^+ \cup \{0\}$ [1, 2]. If the idempotent representations of the hyperbolic

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numbers α and γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \preceq \gamma$ implies that $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha \prec \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$.

Any function f defined on $\mathbb{D}$ is called $\mathbb{D}$-increasing if $f(\alpha) \prec f(\gamma)$, $\mathbb{D}$-decreasing if $f(\alpha) \succ f(\gamma)$, $\mathbb{D}$-non-increasing if $f(\alpha) \succeq f(\gamma)$ and $\mathbb{D}$-non-decreasing if $f(\alpha) \preceq f(\gamma)$, whenever $\alpha \prec \gamma$. More details on hyperbolic numbers $\mathbb{D}$ and partial order "$\preceq$" can be found in the literature [1 (Section 1.5), 8, 14].

Definition 1 [5]. Let A be a subset of $\mathbb{D}$. A is called $\mathbb{D}$-bounded from above if there is a hyperbolic number δ such that $\delta \succeq \alpha$ for all $\alpha \in A$. If $A \subset \mathbb{D}$ is $\mathbb{D}$-bounded from above, then the $\mathbb{D}$-supremum of A is defined as the smallest member of the set of all upper bounds of A.

Similarly, A is called $\mathbb{D}$-bounded from below if there is a hyperbolic number γ such that $\alpha \succeq \gamma$ for all $\alpha \in A$. If $A \subset \mathbb{D}$ is both $\mathbb{D}$-bounded from above and below, it is simply called $\mathbb{D}$-bounded.

Remark 1 [1, Remark 1.5.2]. Let $\mathbb{D} \supset A$ be $\mathbb{D}$-bounded from above and $A_1 := \{\lambda_1 : e_1\lambda_1 + e_2\lambda_2 \in A\}$, $A_2 := \{\lambda_2 : e_1\lambda_1 + e_2\lambda_2 \in A\}$. Then the $sup_{\mathbb{D}}A$ is given by

$$sup_{\mathbb{D}}A := e_1 supA_1 + e_2 supA_2.$$

Similarly, for any $\mathbb{D}$-bounded from below set A, the $\mathbb{D}$-infimum of A is defined as

$$inf_{\mathbb{D}}A = e_1 infA_1 + e_2 infA_2.$$

Remark 2 [1]. A $\mathbb{BC}$-module space or $\mathbb{D}$-module space Y can be decomposed as

$$Y = e_1Y_1 + e_2Y_2, \tag{2}$$

where $Y_1 = e_1Y$ and $Y_2 = e_2Y$ are $\mathbb{R}$-vector or $\mathbb{C}(\mathrm{i})$-vector spaces. The spelling in (2) is called the idempotent decomposition of the space Y.

Definition 2 [15]. Let $\mathfrak{M}$ be a σ-algebra on a set Ω. A bicomplex-valued function $\mu = \mu_1e_1 + \mu_2e_2$ defined on Ω is called a $\mathbb{BC}$-measure on $\mathfrak{M}$ if μ_1, μ_2 are complex measures on $\mathfrak{M}$. In particular, if μ_1, μ_2 are positive measures on $\mathfrak{M}$, i.e. the range of both μ_1, μ_2 being $[0, \infty]$, then μ is called a $\mathbb{D}$-measure on $\mathfrak{M}$, and if μ_1, μ_2 are real measures on $\mathfrak{M}$, i.e. the range of both μ_1, μ_2 being $[0, \infty)$, then μ is called a $\mathbb{D}^+$-measure on $\mathfrak{M}$.

It is assumed that $\Omega = (\Omega, \mathfrak{M}, \mu)$ is a σ-finite complete measure space and f_1, f_2 are complex-valued (real-valued) measurable functions on Ω. The function having idempotent decomposition $f = f_1e_1 + f_2e_2$ is called a $\mathbb{BC}$-measurable function and $|f|_k = |f_1|e_1 + |f_2|e_2$ is called a $\mathbb{D}$-valued measurable function on Ω [15].

For any $\mathbb{BC}$-valued measurable function $f = f_1e_1 + f_2e_2$, it is easy to see that $|f|_k = |f_1|e_1 + |f_2|e_2$ is $\mathbb{D}$-valued measurable. Also, for any two $\mathbb{BC}$-valued measurable functions f and g, it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$-measurable functions [15]. More results on $\mathbb{D}$-topology, such as $\mathbb{D}$-limit, $\mathbb{D}$-continuity, $\mathbb{D}$-Cauchy and $\mathbb{D}$-convergence, can be found in the literature [12, 16, 17] and the references therein.

Theorem 1. Let $u = u_1e_1 + u_2e_2, v = v_1e_1 + v_2e_2$ and $u_n = u_1^ne_1 + u_2^ne_2$ be $\mathbb{BC}$-measurable functions and $\lambda \in \mathbb{BC}$. Then:

- Real and imaginary parts of the functions u_1,u_2,v_1 and v_2 are $\mathbb{R}$-valued measurable;
- u_1, u_2, v_1 and v_2 are $\mathbb{C}$-valued measurable;
- $u + v$,$u \cdot v$ and λu are $\mathbb{BC}$-valued measurable;
- $sup_{\mathbb{D}}|u_n|_k$, $inf_{\mathbb{D}}|u_n|_k$, $limsup_{\mathbb{D}}|u_n|_k$, $liminf_{\mathbb{D}}|u_n|_k$ and $lim_{\mathbb{D}}|u_n|_k$ are $\mathbb{D}$-valued measurable, where they are defined.

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Proof. The proof of each item can be done by using the definition of measurable function and similar techniques used in the literature [18, Appendix A].

Definition 3 [15]. Let $(\Omega, \mathfrak{M}, \vartheta)$ be a measure space with $\vartheta = \vartheta_1 e_1 + \vartheta_2 e_2$, $\mathfrak{F}(\Omega, \mathfrak{M})$ indicating the set of all $\mathfrak{M}$-measurable functions on Ω, and $u \in \mathfrak{F}(\Omega, \mathfrak{M})$ be a $\mathbb{BC}$-valued function. Let $E_M = \{x \in \Omega: |u(x)|_k \succ M\}$ for any $M \succeq 0$. If $A = \{M \succ 0: \vartheta(E_M) = 0\} = \{M \in \mathbb{D}^+: |u(x)|_k \preceq M\ \vartheta - a.e.\}$, then the essential $\mathbb{D}$-supremum of u, denoted by $essup_{\mathbb{D}} u$ or $\|u\|_\infty^{\mathbb{D}}$, is defined by $\|u\|_\infty^{\mathbb{D}} = essup_{\mathbb{D}} u = inf_{\mathbb{D}}(A)$.

$\mathbb{D}$ −Distribution and $\mathbb{D}$ −Rearrangement Functions

Now suppose that $(\Omega, \mathfrak{M}, \vartheta)$ is a σ-finite complete $\mathbb{BC}$-measure space and $\mathfrak{F}(\Omega, \mathfrak{M})$ is the set of all $\mathbb{BC}$-measurable, $\mathbb{BC}$-valued functions on Ω.

Definition 4 [19]. Let $u = u_1 e_1 + u_2 e_2$ be an element of $\mathfrak{F}(\Omega, \mathfrak{M})$ and $\vartheta = \vartheta_1 e_1 + \vartheta_2 e_2$ be a $\mathbb{BC}$-measure. Then the $\mathbb{BC}$-distribution function $D_u^{\mathbb{BC}}: \mathbb{D}^+ \cup \{0\} \to \mathbb{D}^+ \cup \{0\}$ of u is given by

$$\begin{aligned} D_u^{\mathbb{BC}}(\lambda) &= D_{u_1}(\lambda_1) e_1 + D_{u_2}(\lambda_2) e_2 \\ &= \vartheta_1\{x \in \Omega: |u_1(x)| > \lambda_1\} e_1 + \vartheta_2\{x \in \Omega: |u_2(x)| > \lambda_2\} e_2 \end{aligned} \tag{3}$$

for all $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \succeq 0$.

Definition 5 [19]. Let $\lambda \in \mathbb{D}^+ \cup \{0\}$ and u be in $\mathfrak{F}(\Omega, \mathfrak{M})$. The $\mathbb{D}$-decreasing rearrangement of u is the function $u_{\mathbb{BC}}^*: \mathbb{D}^+ \cup \{0\} \to \mathbb{D}^+ \cup \{0\}$ defined by

$$\begin{aligned} u_{\mathbb{BC}}^*(t) &= inf_{\mathbb{D}}\{\alpha \succeq 0: D_u^{\mathbb{BC}}(\alpha) \preceq t\} \\ &= inf\{\alpha_1 \geq 0: D_{u_1}(\alpha_1) \leq t_1\} e_1 + inf\{\alpha_2 \geq 0: D_{u_2}(\alpha_2) \leq t_2\} e_2 \\ &= u_1^*(t_1) e_1 + u_2^*(t_2) e_2, \end{aligned}$$

where $inf_{\mathbb{D}} \emptyset = \infty$.

According to Ghosh and Mondal [15] and Eryilmaz [19], since

$$\|u\|_\infty^{\mathbb{D}} = inf_{\mathbb{D}}\{\alpha \succeq 0: \vartheta\{x \in \Omega: |u(x)|_k \succ \alpha\} = 0\},$$

and $\|u_1\|_\infty, \|u_2\|_\infty \preceq \|u\|_\infty^{\mathbb{D}}$, one can write $\|u\|_\infty^{\mathbb{D}} = \|u_1\|_\infty e_1 + \|u_2\|_\infty e_2$ and so

$$\begin{aligned} u_{\mathbb{BC}}^*(0) &= inf_{\mathbb{D}}\{\alpha \succeq 0: D_u^{\mathbb{BC}}(\alpha) = 0\} \\ &= inf_{\mathbb{D}}\{\alpha \succeq 0: \vartheta_j\{x \in \Omega: |u_j(x)| > \alpha_j\} = 0, j = 1,2\} = \|u\|_\infty^{\mathbb{D}}. \end{aligned} \tag{4}$$

On the other hand, the $\mathbb{D}$-decreasing property of $D_u^{\mathbb{BC}}(\cdot)$ implies that

$$u_{\mathbb{BC}}^*\left(D_u^{\mathbb{BC}}(t)\right) = inf_{\mathbb{D}}\{\alpha \succeq 0: D_u^{\mathbb{BC}}(\alpha) \preceq D_u^{\mathbb{BC}}(t)\} = inf_{\mathbb{D}}\{\alpha \succeq 0: \alpha \succ t\} = t,$$

or

$$\begin{aligned} u_{\mathbb{BC}}^*\left(D_u^{\mathbb{BC}}(t)\right) &= inf\{\alpha_1 \geq 0: D_{u_1}(\alpha_1) \leq D_{u_1}(t_1)\} e_1 + inf\{\alpha_2 \geq 0: D_{u_2}(\alpha_2) \leq D_{u_2}(t_2)\} e_2 \\ &= inf\{\alpha_1 \geq 0: \alpha_1 > t_1\} e_1 + inf\{\alpha_2 \geq 0: \alpha_2 > t_2\} e_2 = t_1 e_1 + t_2 e_2 = t, \end{aligned}$$

and so $u_{\mathbb{BC}}^*(\cdot)$ is the left $\mathbb{D}$-inverse of the distribution function $D_u^{\mathbb{BC}}(\cdot)$. Now let $u_{\mathbb{BC}}^*(t) = \lambda = \lambda_1 e_1 + \lambda_2 e_2 \preceq \infty_{\mathbb{D}}$. Then by Definition 5, there exists a sequence $\lambda_n = \lambda_n^{(1)} e_1 + \lambda_n^{(2)} e_2$ in $\mathbb{D}^+$ such that $\lambda_n^{(1)} \downarrow \lambda_1$, $\lambda_n^{(2)} \downarrow \lambda_2$, $D_{u_1}\left(\lambda_n^{(1)}\right) \leq t_1$ and $D_{u_2}\left(\lambda_n^{(2)}\right) \leq t_2$. By using the techniques used in the continuation of Castillo and Rafeiro [18, Definition 4.4] and the right continuity of the usual distribution function, we get

$$\begin{aligned} D_u^{\mathbb{BC}}\left(u_{\mathbb{BC}}^*(t)\right) &= D_u^{\mathbb{BC}}(\lambda) = D_{u_1}(\lambda_1) e_1 + D_{u_2}(\lambda_2) e_2 \\ &= \left(\lim D_{u_1}\left(\lambda_n^{(1)}\right)\right) e_1 + \left(\lim D_{u_2}\left(\lambda_n^{(2)}\right)\right) e_2 \preceq t_1 e_1 + t_2 e_2 = t. \end{aligned}$$

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Therefore,

$$u^*_{\mathbb{BC}}\left(D_u^{\mathbb{BC}}(\alpha)\right) \preceq \alpha \text{ and } D_u^{\mathbb{BC}}\left(u^*_{\mathbb{BC}}(t)\right) \preceq t. \tag{5}$$

Definition 6 [19]. The function $u^{**}_{\mathbb{BC}}: \mathbb{D}^+ \to \mathbb{D}^+ \cup \{0\}$ is defined as

$$u^{**}_{\mathbb{BC}}(t) = \left(\frac{1}{t_1}\int_0^{t_1} u_1^*(s)ds\right)e_1 + \left(\frac{1}{t_2}\int_0^{t_2} u_2^*(s)ds\right)e_2 = u_1^{**}(t_1)e_1 + u_2^{**}(t_2)e_2,$$

where $t = t_1e_1 + t_2e_2$ and $u^*_{\mathbb{BC}} = u_1^*e_1 + u_2^*e_2$. This function $u^{**}_{\mathbb{BC}}(\cdot)$ is called the $\mathbb{D}$-maximal function of u since it is the $\mathbb{D}$-largest of all $\mathbb{D}$-average values over $u^*_{\mathbb{BC}}$.

Remark 3. Even if the value of $u^{**}_{\mathbb{BC}}(t)$ at $t = 0$ is not included in the definition above, the $\mathbb{D}$-limit as t_1, t_2 approach zero from the right for $t = t_1e_1 + t_2e_2$ is defined for all rearrangements. In fact,

$$\begin{aligned}\lim_{t_1,t_2\to 0^+}{}_{\mathbb{D}}\, u^{**}_{\mathbb{BC}}(t) &= \lim_{t_1,t_2\to 0^+}{}_{\mathbb{D}}\,(u_1^{**}(t_1)e_1 + u_2^{**}(t_2)e_2) = \lim_{t_1\to 0^+} u_1^{**}(t_1)e_1 + \lim_{t_2\to 0^+} u_2^{**}(t_2)e_2 \\ &= u_1^*(0)e_1 + u_2^*(0)e_2 = u^*_{\mathbb{BC}}(0) = \|u\|_\infty^{\mathbb{D}},\end{aligned}$$

where the last equality is from (4).

Theorem 2 [18, Theorem 4.17]. Suppose that $(X, \mathcal{A}, \mu)$ is a non-atomic measure space and let $\mathfrak{F}(X, \mathcal{A})$ denote the set of all complex-value $\mathcal{A}$-measurable functions on X. Then

$$sup\left\{\int_E |f(x)|d\mu(x) : m(E) = t\right\} = \int_0^t f^*(s)ds.$$

Theorem 3 [19]. Let $u = u_1e_1 + u_2e_2$, $v = v_1e_1 + v_2e_2$ be two elements of $\mathfrak{F}(\Omega, \mathfrak{M})$ and $\vartheta = \vartheta_1e_1 + \vartheta_2e_2$ be a $\mathbb{BC}$-measure with resonant measures ϑ_1 and ϑ_2. Then

$$(u+v)^{**}_{\mathbb{BC}}(t) \preceq u^{**}_{\mathbb{BC}}(t) + v^{**}_{\mathbb{BC}}(t)$$

for all $t \in \mathbb{D}^+$.

Definition 7. Let $\vartheta = \vartheta_1e_1 + \vartheta_2e_2$ be a $\mathbb{BC}$-measure, $(\Omega, \mathfrak{M}, \vartheta)$ be a σ-finite complete $\mathbb{BC}$-measurable space and $\mathfrak{F}(\Omega, \mathfrak{M})$ be the set of all measurable $\mathbb{BC}$-valued functions on Ω. For $0 < p \le \infty$ and $0 < q \le \infty$, the bicomplex Lorentz spaces, $L^{\mathbb{BC}}_{p,q} = L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$, are the set of all equivalence classes of $\mathbb{BC}$-measurable functions $u = u_1e_1 + u_2e_2 \in \mathfrak{F}(\Omega, \mathfrak{M})$ such that the functional $\|u\|^{\mathbb{BC}}_{p,q}$ is $\mathbb{D}$-finite, where

$$\|u\|^{\mathbb{BC}}_{p,q} = e_1\|u_1\|_{p,q} + e_2\|u_2\|_{p,q}$$

and

$$\| u_i \|_{p,q} = \begin{cases}\left(\dfrac{q}{p}\displaystyle\int_0^\infty \left(t^{1/p}u_i^*(t)\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } 0 < p < \infty, 0 < q < \infty, \\ \sup\limits_{t>0} t^{1/p}u_i^*(t) & \text{if } 0 < p \le \infty, q = \infty\end{cases}$$

for all $i = 1,2$.

Remark 4. For the $\mathbb{BC}$-Lorentz $L^{\mathbb{BC}}_{p,q}$ space, the case $p = \infty$ and $0 < q < \infty$ is not of any interest. The reason for this is that $\|u\|^{\mathbb{BC}}_{\infty,q} \prec \infty_{\mathbb{D}}$ means that $f = 0$ ϑ-*a.e* on Ω. The $\mathbb{BC}$-Lorentz $L^{\mathbb{BC}}_{p,q}$ spaces can be seen as generalisations of the ordinary $\mathbb{BC}$-Lebesgue spaces $L^p_{\mathbb{BC}}$, which are examined by Toksoy and Sağır [11]. The reason for this is that if one writes $q = p$, then we can get $L^{\mathbb{BC}}_{p,p} = L^p_{\mathbb{BC}}$ for $0 < p \le \infty$. In fact, by the definition of $\|\cdot\|^{\mathbb{BC}}_{p,q}$ for $0 < p < \infty$,

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$$\|u\|_{p,p}^{\mathbb{BC}} = e_1\|u_1\|_{p,p} + e_2\|u_2\|_{p,p} = e_1\left(\frac{p}{p}\int_0^\infty \left(t^{\frac{1}{p}}u_1^*(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} + e_2\left(\frac{p}{p}\int_0^\infty \left(t^{\frac{1}{p}}u_2^*(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}}$$

$$= e_1\left(\int_0^\infty \left(u_1^*(t)\right)^p dt\right)^{\frac{1}{p}} + e_2\left(\int_0^\infty \left(u_2^*(t)\right)^p dt\right)^{\frac{1}{p}}$$

$$= e_1\left(\int_\Omega |u_1(x)|^p d\vartheta_1\right)^{\frac{1}{p}} + e_2\left(\int_\Omega |u_2(x)|^p d\vartheta_2\right)^{\frac{1}{p}} = e_1\|u_1\|_p + e_2\|u_2\|_p = \|u\|_{p,\mathbb{BC}}$$

can be obtained. For $p = \infty$, the result can be seen from (4) immediately.

Example 1. For any $\mathfrak{M}$-measurable set E of finite measure according to ϑ_1 and ϑ_2, we have

$$\|\chi_E\|_{p,q}^{\mathbb{BC}} = e_1\|\chi_E\|_{p,q} + e_2\|\chi_E\|_{p,q} = e_1\left(\frac{p}{q}\right)^{\frac{1}{q}}\vartheta_1(E)^{\frac{1}{p}} + e_2\left(\frac{p}{q}\right)^{\frac{1}{q}}\vartheta_2(E)^{\frac{1}{p}} = \left(\frac{p}{q}\right)^{\frac{1}{q}}\vartheta(E)^{\frac{1}{p}}$$

for $0 < p, q < \infty$ by Değirmen and Sağır [12, Definition 2.2]. If $q = \infty$, then

$$\|\chi_E\|_{p,\infty}^{\mathbb{BC}} = e_1\sup_{t>0} t^{\frac{1}{p}}u_1^*(t) + e_2\sup_{t>0} t^{\frac{1}{p}}u_2^*(t) = e_1\vartheta_1(E)^{\frac{1}{p}} + e_2\vartheta_2(E)^{\frac{1}{p}} = \vartheta(E)^{\frac{1}{p}}$$

since $e_1 \cdot e_2 = 0$ in $\mathbb{D}$.

Theorem 4. The $\mathbb{BC}$-Lorentz space $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{p,q}^{\mathbb{BC}}\right)$ is a quasi-normed linear space.

Proof. It is easy by Castillo and Rafeiro [18, Theorem 6.4].

Remark 5. The functional $\|\cdot\|_{p,q}^{\mathbb{BC}}$ is a norm if and only if $1 \leq q \leq p < \infty$ or, in trivial case, $p = \infty = q$.

Definition 8. For any $u \in L_{p,q}^{\mathbb{BC}}$, the functional $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ is defined by

$$\|u\|_{(p,q)}^{\mathbb{BC}} = e_1\|u_1\|_{(p,q)} + e_2\|u_2\|_{(p,q)},$$

where

$$\| u_i \|_{(p,q)} = \begin{cases} \left(\frac{q}{p}\int_0^\infty \left(t^{1/p}u_i^{**}(t)\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } 0 < p < \infty, 0 < q < \infty, \\ \sup_{t>0} t^{1/p}u_i^{**}(t) & \text{if } 0 < p \leq \infty, q = \infty \end{cases}.$$

By using Theorem 3 and Minkowski inequality, it is easy to see that $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ satisfies the triangle inequality for $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Therefore, $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ is a norm on $L_{p,q}^{\mathbb{BC}}$ and hence $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{(p,q)}^{\mathbb{BC}}\right)$ is a normed space if $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = \infty = q$. Moreover, the norm $\|\cdot\|_{(p,q)}^{\mathbb{BC}}$ and the quasi-norm $\|\cdot\|_{p,q}^{\mathbb{BC}}$ are $\mathbb{D}$-equivalent, that is

$$\|\cdot\|_{p,q}^{\mathbb{BC}} \preceq \|\cdot\|_{(p,q)}^{\mathbb{BC}} \preceq \frac{p}{p-1}\|\cdot\|_{p,q}^{\mathbb{BC}},$$

where the first inequality is an immediate consequence of the fact that $u_{\mathbb{BC}}^*(\cdot) \preceq u_{\mathbb{BC}}^{**}(\cdot)$, and the second follows from the Hardy inequality.

Theorem 5 [19]**.** **(**Completeness) The $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}$ with the quasi-norm $\|\cdot\|_{p,q}^{\mathbb{BC}}$ is complete for all $0 < p < \infty, 0 < q \leq \infty$. Nevertheless, if $1 < p < \infty$, $1 \leq q \leq \infty$, $p = q = 1$ or $p = q = \infty$, then the normed space $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{(p,q)}^{\mathbb{BC}}\right)$ is a $\mathbb{BC}$-Banach space.

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Theorem 6 [19]. Let S be the set of all simple integrable functions. Then the set $\mathfrak{S} = \{e_1 s_1 + e_2 s_2 : s_1, s_2 \in S\}$ is dense in $L_{p,q}^{\mathbb{BC}}$ for $0 < p < \infty$ and $0 < q < \infty$.

MAIN RESULTS

Consider the vector space $\mathcal{F}(\Omega)$ comprising all $\mathbb{BC}$-valued functions on a non-empty set Ω. Let $u: \Omega \to \mathbb{BC}$ be a $\mathbb{BC}$-measurable function on Ω such that $u \cdot f \in \mathcal{F}(\Omega)$ whenever $f \in \mathcal{F}(\Omega)$, where $u = u_1 e_1 + u_2 e_2$ and $f = f_1 e_1 + f_2 e_2$. This gives rise to a linear transformation $M_u: \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$, defined as

$$M_u(f) = u \cdot f = u_1 f_1 e_1 + u_2 f_2 e_2,$$

where the product of functions is pointwise. If $\mathcal{F}(\Omega)$ is a topological $\mathbb{BC}$-vector space and M_u is $\mathbb{BC}$-continuous, then it is referred to as a multiplication operator induced by u. Multiplication operators have been scrutinised on various function spaces [20-26]. In line with their arguments, I investigate multiplication operators on the $\mathbb{BC}$-Lorentz spaces $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{(p,q)}^{\mathbb{BC}}\right)$, where $1 < p < \infty$, $1 \le q \le \infty$. I initially establish a characterisation of the boundedness of M_u in terms of u and demonstrate that the set of multiplication operators on $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{(p,q)}^{\mathbb{BC}}\right)$, for $1 < p < \infty$ and $1 \le q \le \infty$, forms a maximal abelian subalgebra of the Banach algebra of all bounded linear operators on $L_{p,q}^{\mathbb{BC}}$. I employ this to characterise the invertibility of M_u on $L_{p,q}^{\mathbb{BC}}$. The compact and Fredholm $\mathbb{BC}$-multiplication operators are also delineated in this paper.

Multiplication Operators on $\mathbb{BC}$-Lorentz Spaces

This section establishes the conditions for the boundedness and invertibility of the $\mathbb{BC}$-multiplication operator M_u. These conditions are expressed in relation to the boundedness and invertibility of the measurable $\mathbb{BC}$-valued function u respectively.

Proposition 1. For any $\mathbb{BC}$-measurable function $u: \Omega \to \mathbb{BC}$, M_u is a $\mathbb{BC}$-linear operator on $\mathcal{F}(\Omega)$.

Theorem 7. The linear transformation $M_u: f \to u \cdot f$ on the $\mathbb{BC}$-Lorentz space $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{(p,q)}^{\mathbb{BC}}\right)$ is bounded for $1 < p \le \infty, 1 \le q \le \infty$ if and only if u is essentially $\mathbb{D}$-bounded. Moreover, $\|M_u\| = \|u\|_\infty^{\mathbb{D}}$.

Proof. Firstly, assume that u is essentially $\mathbb{D}$-bounded and $\|u\|_\infty^{\mathbb{D}} \prec \infty_{\mathbb{D}}$. Since $u \cdot f = u_1 f_1 e_1 + u_2 f_2 e_2$ for any $f \in L_{p,q}^{\mathbb{BC}}$, we have

$$\begin{aligned} D_{u\cdot f}^{\mathbb{BC}}(\lambda) &= D_{u_1 f_1}(\lambda_1) e_1 + D_{u_2 f_2}(\lambda_2) e_2 \\ &= \vartheta_1\{x \in \Omega: |u_1(x) f_1(x)| > \lambda_1\} e_1 + \vartheta_2\{x \in \Omega: |u_2(x) f_2(x)| > \lambda_2\} e_2 \\ &\preceq \|u_1\|_\infty \vartheta_1\{x \in \Omega: |f_1(x)| > \lambda_1\} e_1 + \|u_2\|_\infty \vartheta_2\{x \in \Omega: |f_2(x)| > \lambda_2\} e_2 \\ &= (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)\left(D_{f_1}(\lambda_1) e_1 + D_{f_2}(\lambda_2) e_2\right) = \|u\|_\infty^{\mathbb{D}} D_f^{\mathbb{BC}}(\lambda) \end{aligned}$$

for any $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \succeq 0$. Then $D_{u\cdot f}^{\mathbb{BC}}(\lambda) \preceq \|u\|_\infty^{\mathbb{D}} D_f^{\mathbb{BC}}(\lambda)$ implies

$$\begin{aligned} (u \cdot f)_{\mathbb{BC}}^*(t) &= (u_1 f_1)^*(t_1) e_1 + (u_2 f_2)^*(t_2) e_2 = inf_{\mathbb{D}}\left\{\lambda \succeq 0: D_{u\cdot f}^{\mathbb{BC}}(\lambda) \preceq t\right\} \\ &\preceq \|u\|_\infty^{\mathbb{D}} inf_{\mathbb{D}}\left\{\lambda \succeq 0: D_f^{\mathbb{BC}}(\lambda) \preceq t\right\} \\ &= \|u\|_\infty^{\mathbb{D}} f_{\mathbb{BC}}^*(t) = \|u_1\|_\infty f_1^*(t_1) e_1 + \|u_2\|_\infty f_2^*(t_2) e_2, \end{aligned}$$

and $(u_1 f_1)^*(t_1) \le \|u_1\|_\infty f_1^*(t_1)$, $(u_2 f_2)^*(t_2) \le \|u_2\|_\infty f_2^*(t_2)$ for any $t = t_1 e_1 + t_2 e_2 \succeq 0$. Therefore,

$$(u \cdot f)_{\mathbb{BC}}^{**}(t) = (u_1 f_1)^{**}(t_1) e_1 + (u_2 f_2)^{**}(t_2) e_2$$

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$$= \left(\frac{1}{t_1}\int_0^{t_1}(u_1f_1)^*(s)ds\right)e_1 + \left(\frac{1}{t_2}\int_0^{t_2}(u_2f_2)^*(s)ds\right)e_2$$
$$\leq \|u_1\|_\infty\left(\frac{1}{t_1}\int_0^{t_1}f_1^*(s)ds\right)e_1 + \|u_2\|_\infty\left(\frac{1}{t_2}\int_0^{t_2}f_2^*(s)ds\right)e_2$$
$$= (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)(f_1^{**}(t_1)e_1 + f_2^{**}(t_2)e_2) = \|u\|_\infty^{\mathbb{D}} f_{\mathbb{BC}}^{**}(t)$$

can be written. Consequently, $(u_1f_1)^{**}(t_1) \leq \|u_1\|_\infty f_1^{**}(t_1)$, $(u_2f_2)^{**}(t_2) \leq \|u_2\|_\infty f_2^{**}(t_2)$, and so

$$\|M_u(f)\|_{(p,q)}^{\mathbb{BC}} = e_1\|u_1f_1\|_{(p,q)} + e_2\|u_2f_2\|_{(p,q)}$$
$$= e_1\left(\frac{q}{p}\int_0^\infty\left(s^{\frac{1}{p}}(u_1f_1)^{**}(s)\right)^q\frac{ds}{s}\right)^{\frac{1}{q}} + e_2\left(\frac{q}{p}\int_0^\infty\left(s^{\frac{1}{p}}(u_2f_2)^{**}(s)\right)^q\frac{ds}{s}\right)^{\frac{1}{q}}$$
$$\leq \|u_1\|_\infty e_1\left(\frac{q}{p}\int_0^\infty\left(s^{\frac{1}{p}}f_1^{**}(s)\right)^q\frac{ds}{s}\right)^{\frac{1}{q}} + \|u_2\|_\infty e_2\left(\frac{q}{p}\int_0^\infty\left(s^{\frac{1}{p}}f_2^{**}(s)\right)^q\frac{ds}{s}\right)^{\frac{1}{q}}$$
$$\leq (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)\big(e_1\|f_1\|_{(p,q)} + e_2\|f_2\|_{(p,q)}\big) = \|u\|_\infty^{\mathbb{D}}\|f\|_{(p,q)}^{\mathbb{BC}} \tag{6}$$

is obtained for $1 < p < \infty$ and $1 \leq q < \infty$. If $q = \infty$, then

$$\|M_u(f)\|_{(p,\infty)}^{\mathbb{BC}} = e_1\|u_1f_1\|_{(p,\infty)} + e_2\|u_2f_2\|_{(p,\infty)}$$
$$= e_1\sup_{s>0} s^{\frac{1}{p}}(u_1f_1)^{**}(s) + e_2\sup_{s>0} s^{\frac{1}{p}}(u_2f_2)^{**}(s)$$
$$\leq \|u_1\|_\infty e_1\sup_{s>0} s^{\frac{1}{p}}f_1^{**}(s) + \|u_2\|_\infty e_2\sup_{s>0} s^{\frac{1}{p}}f_2^{**}(s)$$
$$= (\|u_1\|_\infty e_1 + \|u_2\|_\infty e_2)\left(e_1\sup_{s>0} s^{1/p}f_1^{**}(s) + e_2\sup_{s>0} s^{1/p}f_2^{**}(s)\right)$$
$$= \|u\|_\infty^{\mathbb{D}}\big(e_1\|f_1\|_{(p,\infty)} + e_2\|f_2\|_{(p,\infty)}\big) = \|u\|_\infty^{\mathbb{D}}\|f\|_{(p,\infty)}^{\mathbb{BC}} \tag{7}$$

is written. This means M_u is $\mathbb{BC}$-bounded.

Conversely, suppose that M_u is $\mathbb{BC}$-bounded on the $\mathbb{BC}$-Lorentz space $\left(L_{p,q}^{\mathbb{BC}}, \|\cdot\|_{(p,q)}^{\mathbb{BC}}\right)$ for $1 < p \leq \infty, 1 \leq q < \infty$. If u is not essentially $\mathbb{D}$-bounded, then for each $N \succeq 0$, the set

$$E_N = \{x \in \Omega : |u(x)|_k \succ N\}$$

has a $\mathbb{D}$-positive measure. This means there exists N_1, $N_2 \geq 0$ with $N = N_1e_1 + N_2e_2$ such that $|u_1(x)| > N_1$ and $|u_2(x)| > N_2$ for all $x \in E_N$ with $\vartheta(E_N) \succ 0$. Since the decreasing $\mathbb{D}$-rearrangement of $\chi_{E_N} = \chi_{E_N}e_1 + \chi_{E_N}e_2$ is $\left(\chi_{E_N}\right)_{\mathbb{BC}}^*(t) = \chi_{(0,\vartheta_1(E_N))}(t_1)e_1 + \chi_{(0,\vartheta_2(E_N))}(t_2)e_2$, one can get that [15]

$$\left\|\chi_{E_N}\right\|_{(p,q)}^{\mathbb{BC}} = e_1\left\|\chi_{E_N}\right\|_{(p,q)} + e_2\left\|\chi_{E_N}\right\|_{(p,q)}$$
$$= e_1\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\vartheta_1(E_N)^{\frac{1}{p}} + e_2\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\vartheta_2(E_N)^{\frac{1}{p}} = \left(\frac{p}{p-1}\right)^{\frac{1}{q}}\vartheta(E_N)^{\frac{1}{p}}.$$

Now to calculate the norm of $M_u\left(\chi_{E_N}\right)$, if we use the following inequality

$$\left(M_u\left(\chi_{E_N}\right)\right)_{\mathbb{BC}}^*(t) = \left(u\cdot\chi_{E_N}\right)_{\mathbb{BC}}^*(t) = \left(u_1\cdot\chi_{E_N}\right)^*(t_1)e_1 + \left(u_2\cdot\chi_{E_N}\right)^*(t_2)e_2$$
$$= \inf\left\{\alpha_1 \geq 0 : D_{u_1\cdot\chi_{E_N}}(\alpha_1) \leq t_1\right\}e_1 + \inf\left\{\alpha_2 \geq 0 : D_{u_2\cdot\chi_{E_N}}(\alpha_2) \leq t_2\right\}e_2$$

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$$= inf\{\alpha_1 \geq 0: \vartheta_1\{x \in \Omega: |u_1(x)\chi_{E_N}(x)| > \alpha_1\} \leq t_1\}e_1$$
$$+ inf\{\alpha_2 \geq 0: \vartheta_2\{x \in \Omega: |u_2(x)\chi_{E_N}(x)| > \alpha_2\} \leq t_2\}e_2$$
$$\geq inf\left\{\alpha_1 \geq 0: \vartheta_1\left\{x \in \Omega: |\chi_{E_N}(x)| > \frac{\alpha_1}{N_1}\right\} \leq t_1\right\}e_1$$
$$+ inf\left\{\alpha_2 \geq 0: \vartheta_2\left\{x \in \Omega: |\chi_{E_N}(x)| > \frac{\alpha_2}{N_2}\right\} \leq t_2\right\}e_2$$
$$= inf\{N_1\alpha_1 \geq 0: \vartheta_1\{x \in \Omega: |\chi_{E_N}(x)| > \alpha_1\} \leq t_1\}e_1$$
$$+ inf\{N_2\alpha_2 \geq 0: \vartheta_2\{x \in \Omega: |\chi_{E_N}(x)| > \alpha_2\} \leq t_2\}e_2$$
$$= (N_1e_1 + N_2e_2)(\chi_{E_N})^*_{\mathbb{BC}}(t) = N(\chi_{E_N})^*_{\mathbb{BC}}(t),$$

then we get

$$\|M_u(\chi_{E_N})\|^{\mathbb{BC}}_{(p,q)} = e_1\|u_1 \cdot \chi_{E_N}\|_{(p,q)} + e_2\|u_2 \cdot \chi_{E_N}\|_{(p,q)}$$
$$= e_1\left(\frac{q}{p}\int_0^\infty \left(s^{\frac{1}{p}}(u_1\chi_{E_N})^{**}(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} + e_2\left(\frac{q}{p}\int_0^\infty \left(s^{\frac{1}{p}}(u_2\chi_{E_N})^{**}(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}}$$
$$\geq (N_1e_1 + N_2e_2)\|\chi_{E_N}\|^{\mathbb{BC}}_{(p,q)} = N\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\vartheta(E_N)^{\frac{1}{p}}. \tag{8}$$

Besides these, for $q = \infty$, we have

$$\|M_u(\chi_{E_N})\|^{\mathbb{BC}}_{(p,\infty)} = e_1\|u_1\chi_{E_N}\|_{(p,\infty)} + e_2\|u_2\chi_{E_N}\|_{(p,\infty)}$$
$$= e_1\sup_{s>0} s^{\frac{1}{p}}(u_1\chi_{E_N})^{**}(s) + e_2\sup_{s>0} s^{\frac{1}{p}}(u_2\chi_{E_N})^{**}(s)$$
$$\geq N_1e_1\sup_{s>0} s^{\frac{1}{p}}(\chi_{E_N})^{**}(s) + N_2e_2\sup_{s>0} s^{\frac{1}{p}}(\chi_{E_N})^{**}(s) \tag{9}$$
$$= N\left(e_1\sup_{s>0} s^{\frac{1}{p}}(\chi_{E_N})^{**}(s) + e_2\sup_{s>0} s^{\frac{1}{p}}(\chi_{E_N})^{**}(s)\right) = N\|\chi_{E_N}\|^{\mathbb{BC}}_{(p,\infty)} = N\vartheta(E_N)^{\frac{1}{p}}.$$

Both (8) and (9) contradict the boundedness of M_u.

From (6) and (7), it can be seen that $\|M_u\| \preceq \|u\|^{\mathbb{D}}_\infty$. On the other hand, for any $\gamma = e_1\gamma_1 + e_2\gamma_2 \succ 0$, let $G = \{x \in \Omega: |u(x)|_k \succ \|u\|^{\mathbb{D}}_\infty - \gamma\}$. Then

$$\{x \in \Omega: (\|u_i\|_\infty - \gamma_i)\chi_G(x) > \lambda_i\} \subset \{x \in \Omega: |u_i(x)\chi_G(x)| > \lambda_i\}$$

can be written for $i = 1,2$. Therefore, $D^{\mathbb{BC}}_{(\|u\|^{\mathbb{D}}_\infty - \gamma)\chi_G}(\lambda) \preceq D^{\mathbb{BC}}_{u\cdot\chi_G}(\lambda)$ for all $\lambda \in \mathbb{D}^+ \cup \{0\}$ and

$$(M_u(\chi_G))^*_{\mathbb{BC}}(t) \succeq (\|u\|^{\mathbb{D}}_\infty - \gamma)(\chi_G)^*_{\mathbb{BC}}(t)$$

for all $t \in \mathbb{D}^+ \cup \{0\}$. As a result, $\|M_u\| \preceq \|u\|^{\mathbb{D}}_\infty - \gamma$ and $\|M_u\| = \|u\|^{\mathbb{D}}_\infty$.

Remark 6. In general, the multiplication operators on measurable function spaces are not injective. For instance, let $U = \{x \in \Omega: u(x) \neq 0\}$ and $V = \Omega - U$. Then $\mu(V) > 0$ and $(\chi_V \cdot u)(x) = 0$ for all $x \in \Omega$. This implies that $M_u(\chi_V) = 0$ and $Ker(M_u) \neq \{0\}$. Hence M_u is not injective. On the contrary, if M_u is injective, then $\mu(V)$ must be zero. If $\mu(V) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies that $(u \cdot f)(x) = 0$ for all $x \in \Omega$ and so $\{x \in \Omega: f(x) \neq 0\} \subset V$ and $f = 0$ (μ –a.e.) on Ω.

Proposition 2. The multiplication operator M_u is injective on $L^{\mathbb{BC}}_{p,q}(U, \mathfrak{M}, \vartheta) = \{f\chi_U: f \in L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)\}$, where $U = \{x \in \Omega: u(x) \neq 0\}$.

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Proof. Let $f\chi_U$ be an element of $L_{p,q}^{\mathbb{BC}}(U,\mathfrak{M},\vartheta)$ with $M_u(f\chi_U)=0$. Then $0=M_u(f\chi_U)(x)=u(x)f(x)\chi_U(x)$ for all $x\in\Omega$. From this equality, we get $f(x)\cdot u(x)=0$ and so $f(x)=0$ for all $x\in U$. This means $f\chi_U=0$ and $Ker(M_u)=\{0\}$.

Corollary 1. The multiplication operator M_u has a closed range on $L_{p,q}^{\mathbb{BC}}(U,\mathfrak{M},\vartheta)$ if and only if M_u is bounded below on $L_{p,q}^{\mathbb{BC}}(U,\mathfrak{M},\vartheta)$.

Corollary 2. If ϑ is a complete $\mathbb{BC}$-measure and $u\neq 0$ $(\vartheta-a.e.)$, then the multiplication operator M_u on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ has a closed range if and only if M_u is bounded below on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$.

Theorem 8. The set of all multiplication operators on the $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$, for $1<p<\infty$, $1\leq q<\infty$, is a maximal abelian subalgebra of $\mathfrak{B}\left(L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)\right)$, the Banach algebra of all $\mathbb{BC}$-bounded $\mathbb{BC}$-linear operators on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$.

Proof. Let $\mathbb{M}=\{M_u: u\in L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)\}$ be the set of all multiplication operators induced by the elements of $L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$. Then it is easy to see that $\mathbb{M}$ is an abelian subalgebra of $\mathfrak{B}\left(L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)\right)$ according to composition. Let T be an operator on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ satisfying $T\circ M_u=M_u\circ T$ for any $u\in L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$, $e:\Omega\to\mathbb{BC}$ be the unit function with $e(x)=e_1+e_2$ for all $x\in\Omega$ and $T(e)=v=v_1e_1+v_2e_2$. Then for any $\mathbb{BC}$-measurable set $E\in\mathfrak{M}$, we get

$$T(\chi_E)=T(e\cdot\chi_E)=T\left(M_{\chi_E}(e)\right)=M_{\chi_E}\big(T(e)\big)=M_{\chi_E}(v)=v\cdot\chi_E=M_v(\chi_E).$$

Now assume that $v\notin L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$. Then the set $F_N=\{x\in\Omega: |v(x)|_k\succ N\}$ has positive $\mathbb{D}$-measure for each $N=N_1e_1+N_2e_2\succeq 0$. Therefore, we get

$$\left\|T(\chi_{F_N})\right\|_{(p,q)}^{\mathbb{BC}}=\left\|M_v(\chi_{F_N})\right\|_{(p,q)}^{\mathbb{BC}}\succeq N\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\vartheta(F_N)^{\frac{1}{p}},$$

which contradicts the boundedness of T. Thus, $v\in L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ and $T=M_v$ by Theorem 6, using the density of simple functions in $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$.

Corollary 3. The multiplication operator M_u on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ for $1<p<\infty$, $1\leq q<\infty$ is invertible if and only if u is $\mathbb{D}$-invertible in $L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$.

Proof. Suppose that u is $\mathbb{D}$-invertible in $L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ with the inverse v. Then $M_v(M_uf)=M_v(u\cdot f)=f$ and $M_u(M_vf)=M_u(v\cdot f)=f$, which means $M_u^{-1}=M_v=M_{u^{-1}}$. If M_u^{-1} exists, then it commutes with all multiplication operators on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$. This means $M_u^{-1}=M_v$ for some $v\in L_\infty^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ by Theorem 8 and v is the $\mathbb{D}$-inverse of u.

Compact Multiplication Operators

A compact multiplication operator on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ is an operator that arises from pointwise multiplication of functions by certain measurable functions, typically $\mathbb{D}$-bounded ones. These operators map $\mathbb{D}$-bounded sets to relatively $\mathbb{D}$-compact sets, exhibiting desirable properties similar to compact operators in functional analysis. In this subsection $\mathbb{D}$-compact multiplication operators are characterised.

Theorem 9. Let M_u be a compact operator, $H_{u,\delta}=\{x\in\Omega: |u(x)|_k\succeq\delta\}$ and $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})=\{f\chi_{H_{u,\delta}}: f\in L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)\}$ for any $\delta=\delta_1e_1+\delta_2e_2\succ 0$. Then $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ is a $\mathbb{D}$-closed, invariant subspace of $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ under M_u and the restriction of M_u to $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ is a compact operator.

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Proof. We first show that $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ is a subspace of $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$. Let $\tilde{f},\tilde{g}\in L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ and $a,b\in\mathbb{BC}$. Since $\tilde{f}=f\chi_{H_{u,\delta}}$ and $\tilde{g}=g\chi_{H_{u,\delta}}$ for any $f,g\in L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$, we get

$$a\tilde{f}+b\tilde{g}=af\chi_{H_{u,\delta}}+bg\chi_{H_{u,\delta}}=(af+bg)\chi_{H_{u,\delta}}.$$

By the definition of $M_u: L_{p,q}^{\mathbb{BC}}(H_{u,\delta})\to L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$, we have $M_u(\tilde{f})=u\cdot\tilde{f}=u\cdot f\chi_{H_{u,\delta}}$ and so $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ is an invariant subspace of $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ under M_u.

Now let us show that the $\mathbb{D}$-closure of $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ satisfies $\overline{L_{p,q}^{\mathbb{BC}}(H_{u,\delta})}^{\mathbb{D}}\subset L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$. Suppose that $\tilde{g}\in\overline{L_{p,q}^{\mathbb{BC}}(H_{u,\delta})}^{\mathbb{D}}$. Then there exists a sequence $\tilde{g}_n$ in $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ such that $\tilde{g}_n\to\tilde{g}$, where $\tilde{g}_n=g_n\chi_{H_{u,\delta}}$ for each $n\in\mathbb{N}$. Since $\tilde{g}_n$ is a $\mathbb{D}$-Cauchy sequence in $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$, it can be written that for all $\varepsilon=\varepsilon_1e_1+\varepsilon_2e_2\succ 0$, there exists an $n_0\in\mathbb{N}$ such that $\|\tilde{g}_n-\tilde{g}_m\|_{(p,q)}^{\mathbb{BC}}\prec\varepsilon$ for all $m,n>n_0$. Hence for all $m,n>n_0$, we can find a $\delta=\delta_1e_1+\delta_2e_2\succ 0$ such that $\delta(g_n-g_m)\preceq(g_n-g_m)\chi_{H_{u,\delta}}$ and $|\delta|_k(g_n-g_m)^*_{\mathbb{BC}}(t)\preceq(g_n-g_m)^*_{\mathbb{BC}}(t)\chi_{[0,\vartheta(H_{u,\delta}))}(t)$. Then

$$\delta\|g_n-g_m\|_{(p,q)}^{\mathbb{BC}}\preceq\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\vartheta(H_{u,\delta})^{\frac{1}{p}}\|\tilde{g}_n-\tilde{g}_m\|_{(p,q)}^{\mathbb{BC}}$$

can be written. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is also a $\mathbb{D}$-Cauchy sequence in $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$. Since $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ is a Banach space, we can write that $g_n\to g$ for an element $g\in L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$. Thus, we have

$$\left\|g_n\chi_{H_{u,\delta}}-g\chi_{H_{u,\delta}}\right\|_{(p,q)}^{\mathbb{BC}}\preceq\|g_n-g\|_{(p,q)}^{\mathbb{BC}}$$

and $\tilde{g}_k\to\tilde{g}$. Consequently $\tilde{g}\in L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ and $M_u|_{L_{p,q}^{\mathbb{BC}}(H_{u,\delta})}$ is a compact operator.

Theorem 10. A multiplication operator M_u on $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ is $\mathbb{D}$-compact for $1<p\le\infty, 1\le q\le\infty$ if and only if $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})$ is finite dimensional for each $\delta=\delta_1e_1+\delta_2e_2\succ 0$, where $H_{u,\delta}=\{x\in\Omega: |u(x)|_k\succeq\delta\}$ and $L_{p,q}^{\mathbb{BC}}(H_{u,\delta})=\{f\chi_{H_{u,\delta}}: f\in L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)\}$.

Proof. Assume that M_u is a compact operator. Then $L_{p,q}^{\mathbb{BC}}(u,\delta)$ is a $\mathbb{D}$-closed and M_u-invariant subspace of $L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ and so the restriction of M_u to $L_{p,q}^{\mathbb{BC}}(u,\delta)$ is compact by Theorem 9. Now take any $x\in\Omega$ and let $x\notin H_{u,\delta}$. Then $|u(x)|_k\prec\delta$, $\left(M_u|_{L_{p,q}^{\mathbb{BC}}(H_{u,\delta})}(f)\right)^*_{\mathbb{BC}}(t)=\left(u\cdot f\cdot\chi_{H_{u,\delta}}\right)^*_{\mathbb{BC}}(t)=0$ and so $M_u|_{L_{p,q}^{\mathbb{BC}}(H_{u,\delta})}=0$ for any $f\in L_{p,q}^{\mathbb{BC}}(\Omega,\mathfrak{M},\vartheta)$ and $t=t_1e_1+t_2e_2\succ 0$. If $x\in H_{u,\delta}$, then $|u_1(x)|\ge\delta_1$, $|u_2(x)|\ge\delta_2$ and we have

$$\begin{aligned}\left|(u\cdot f\cdot\chi_{H_{u,\delta}})(x)\right|_k&\succeq\delta_1\left|(f\cdot\chi_{H_{u,\delta}})(x)\right|e_1+\delta_2\left|(f\cdot\chi_{H_{u,\delta}})(x)\right|e_2\\&=\delta\left|(f\cdot\chi_{H_{u,\delta}})(x)\right|_k,\end{aligned}$$

and $\delta D_{u\cdot f\cdot\chi_{H_{u,\delta}}}^{\mathbb{BC}}(\lambda)\succeq D_{f\cdot\chi_{H_{u,\delta}}}^{\mathbb{BC}}(\lambda)$ for any $\lambda=\lambda_1e_1+\lambda_2e_2\succeq 0$. Therefore, $\delta\left(f\cdot\chi_{H_{u,\delta}}\right)^*_{\mathbb{BC}}(t)\preceq\left(u\cdot f\cdot\chi_{H_{u,\delta}}\right)^*_{\mathbb{BC}}(t)$, $\delta\left(f\cdot\chi_{H_{u,\delta}}\right)^{**}_{\mathbb{BC}}(t)\preceq\left(u\cdot f\cdot\chi_{H_{u,\delta}}\right)^{**}_{\mathbb{BC}}(t)$ and

$$\begin{aligned}\left\|M_u(f\chi_{H_{u,\delta}})\right\|_{(p,q)}^{\mathbb{BC}}&=e_1\left\|u_1\cdot f_1\cdot\chi_{H_{u,\delta}}\right\|_{(p,q)}+e_2\left\|u_2\cdot f_2\cdot\chi_{H_{u,\delta}}\right\|_{(p,q)}\\&\succeq e_1\delta_1\left\|f_1\cdot\chi_{H_{u,\delta}}\right\|_{(p,q)}+e_2\delta_2\left\|f_2\cdot\chi_{H_{u,\delta}}\right\|_{(p,q)}=\delta\left\|f\chi_{H_{u,\delta}}\right\|_{(p,q)}^{\mathbb{BC}}.\end{aligned}$$

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Thus, in either case $M_u|_{L^{\mathbb{BC}}_{p,q}(H_{u,\delta})}$ possesses a $\mathbb{D}$-closed range in $L^{\mathbb{BC}}_{p,q}(H_{u,\delta})$ and hence $\mathbb{D}$-invertible. Since M_u is a compact operator, it can be written that $L^{\mathbb{BC}}_{p,q}(H_{u,\delta})$ is finite dimensional.

Conversely, suppose that $L^{\mathbb{BC}}_{p,q}(H_{u,\delta})$ is finite dimensional for each $\delta \succ 0$. Then $L^{\mathbb{BC}}_{p,q}\left(H_{u,\frac{1}{n}}\right)$ is finite dimensional for each $n \in \mathbb{N}$. Let $u_n\colon \Omega \to \mathbb{BC}$ be a sequence defined as $u_n(x) = u(x)$ if $|u_n(x)|_k \succeq \frac{1}{n}$, $u_n(x) = 0$ otherwise. Since u is essentially $\mathbb{D}$-bounded, all members of u_n is essentially $\mathbb{D}$-bounded. Moreover, for any $f \in L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$, we have

$$D^{\mathbb{BC}}_{(u_n-u)f}(\lambda) = D_{\left(u_n^{(1)}-u_1\right)f_1}(\lambda_1)e_1 + D_{\left(u_n^{(2)}-u_2\right)f_2}(\lambda_2)e_2$$

and

$$\left((u_n - u)f\right)^*_{\mathbb{BC}}(t) = \left(\left(u_n^{(1)} - u_1\right)f_1\right)^*(t_1)e_1 + \left(\left(u_n^{(2)} - u_2\right)f_2\right)^*(t_2)e_2.$$

If $x \in H_{u,\frac{1}{n}}$, then $|u(x)|_k \succeq \frac{1}{n}$ and so $(u_n - u)f = 0$. If $x \notin H_{u,\frac{1}{n}}$, then $|u(x)|_k \prec e_1\frac{1}{n} + e_2\frac{1}{n}$, $u_n(x) = 0$ and so $\left((u_n - u)f\right)^*_{\mathbb{BC}}(t) \preceq \frac{1}{n}f_1^*(t_1)e_1 + \frac{1}{n}f_2^*(t_2)e_2 = \frac{1}{n}f^*_{\mathbb{BC}}(t)$ with $\left((u_n - u)f\right)^{**}_{\mathbb{BC}}(t) \preceq \frac{1}{n}f^{**}_{\mathbb{BC}}(t)$ for all $t = t_1e_1 + t_2e_2 \succ 0$. Consequently,

$$\begin{aligned}\left\|(M_{u_n} - M_u)f\right\|^{\mathbb{BC}}_{(p,q)} &= \|(u_n - u)f\|^{\mathbb{BC}}_{(p,q)}\\ &= e_1\left\|\left(u_n^{(1)} - u_1\right)f_1\right\|_{(p,q)} + e_2\left\|\left(u_n^{(2)} - u_2\right)f_2\right\|_{(p,q)} \preceq \frac{1}{n}\|f\|^{\mathbb{BC}}_{(p,q)}.\end{aligned}$$

This last inequality means that M_{u_n} $\mathbb{D}$-converges to M_u uniformly. Since $L^{\mathbb{BC}}_{p,q}\left(H_{u,\frac{1}{n}}\right)$ is finite-dimensional, M_{u_n} constitutes a finite-rank operator. Therefore M_{u_n} is a sequence of compact operators. Uniform convergence reveals the result: M_u is a compact operator, which completes the proof.

Corollary 4. If ϑ_1 and ϑ_2 are non-atomic measures, then the only compact multiplication operator on the $\mathbb{BC}$-Lorentz space $L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$ is the zero operator.

Corollary 5. If the set $H_{u,\delta}$ contains only finitely many atoms for each $\delta \succ 0$, then M_u is a compact multiplication operator on the $\mathbb{BC}$-Lorentz space $L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$.

Fredholm Multiplication Operators

In this section we initially set forth a criterion for a multiplication operator to possess a closed range. Subsequently, we leverage this criterion to delineate Fredholm multiplication operators on $L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$, where $1 < p, q < \infty$ and ϑ_1 and ϑ_2 are non-atomic measures. In this context the operator M_u is considered Fredholm if its range $\mathfrak{R}(M_u)$ is closed and $dimN(M_u)$ and $codim\mathfrak{R}(M_u)$ are finite.

Theorem 11. Let M_u be a multiplication operator on $L^{\mathbb{BC}}_{p,q}(\Omega, \mathfrak{M}, \vartheta)$, where $1 < p \leq \infty, 1 \leq q \leq \infty$ and $W = \{x \in \Omega\colon u(x) \neq 0\}$. Then M_u has a closed range if and only if there exists a $\gamma = \gamma_1e_1 + \gamma_2e_2 \succ 0$ such that $|u(x)|_k = |u_1(x)|e_1 + |u_2(x)|e_2 \succeq \gamma$ $(\vartheta - a.e.)$ on W.

Proof. If there exists a $\gamma = \gamma_1e_1 + \gamma_2e_2 \succ 0$ such that $|u(x)|_k \succeq \gamma$ $(\vartheta - a.e.)$ on W, then

$$\begin{aligned}\|M_uf\chi_W\|^{\mathbb{BC}}_{(p,q)} &= \|u \cdot f \cdot \chi_W\|^{\mathbb{BC}}_{(p,q)} = e_1\|u_1 \cdot f_1 \cdot \chi_W\|_{(p,q)} + e_2\|u_2 \cdot f_2 \cdot \chi_W\|_{(p,q)}\\ &\succeq e_1\gamma_1\|f_1 \cdot \chi_W\|_{(p,q)} + e_2\gamma_2\|f_2 \cdot \chi_W\|_{(p,q)} = \gamma\|f\chi_W\|^{\mathbb{BC}}_{(p,q)}\end{aligned}$$

for all $f \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$. Therefore, M_u has a closed range.

Conversely, let $L_{p,q}^{\mathbb{BC}}(W) = \{f\chi_W : f \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)\}$. If M_u has a closed range, then there exists an $\varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2 \succ 0$ such that $\|M_u f\|_{(p,q)}^{\mathbb{BC}} \succeq \varepsilon \parallel f \parallel_{(p,q)}^{\mathbb{BC}}$ for all $f \in L_{p,q}^{\mathbb{BC}}(W)$. Now assume that $V = \{x \in W : |u(x)|_k \prec \frac{\varepsilon_1}{2} e_1 + \frac{\varepsilon_2}{2} e_2\}$ and $\vartheta(V) = e_1\vartheta_1(V) + e_2\vartheta_2(V) \succ 0$. In that case a $\mathbb{BC}$-measurable set $E \subset V$ can be found such that $\chi_E \in L_{p,q}^{\mathbb{BC}}(W)$. Then we have

$$\begin{aligned} D_{u\chi_E}^{\mathbb{BC}}(\lambda) &= D_{u_1\chi_E}(\lambda_1)e_1 + D_{u_2\chi_E}(\lambda_2)e_2 \\ &= \vartheta_1\{x \in \Omega : |u_1\chi_E(x)| > \lambda_1\}e_1 + \vartheta_2\{x \in \Omega : |u_2\chi_E(x)| > \lambda_2\}e_2 \\ &\succeq \frac{\varepsilon_1}{2}\vartheta_1\{x \in \Omega : |\chi_E(x)| > \lambda_1\}e_1 + \frac{\varepsilon_2}{2}\vartheta_2\{x \in \Omega : |\chi_E(x)| > \lambda_2\}e_2 = \frac{\varepsilon}{2} D_{\chi_E}^{\mathbb{BC}}(\lambda) \end{aligned}$$

and $(u \cdot \chi_E)_{\mathbb{BC}}^*(t) \preceq \left(\frac{\varepsilon_1}{2} e_1 + \frac{\varepsilon_2}{2} e_2\right)(\chi_E)_{\mathbb{BC}}^*(t)$, which implies $(u \cdot \chi_E)_{\mathbb{BC}}^{**}(t) \preceq \frac{\varepsilon}{2}(\chi_E)_{\mathbb{BC}}^{**}(t)$ for all $t = t_1 e_1 + t_2 e_2 \succ 0$. Thus, $\|M_u\chi_E\|_{(p,q)}^{\mathbb{BC}} \preceq \frac{\varepsilon}{2} \parallel \chi_E \parallel_{(p,q)}^{\mathbb{BC}}$, which is a contradiction. This implies $\vartheta(V) = 0$ and completes the proof.

Theorem 12. Suppose that ϑ_1 and ϑ_2 are non-atomic component measures of ϑ. Let u be an essentially $\mathbb{D}$-bounded $\mathbb{BC}$-measurable function and M_u be a multiplication operator on the $\mathbb{BC}$-Lorentz space $L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$ for $1 < p, q < \infty$. Then the following conditions are equivalent:

(1) $\mathfrak{R}(M_u)$ is closed and codim $\mathfrak{R}(M_u) < \infty$.
(2) $|u(x)|_k \succeq \delta$ $(\vartheta - a.e.)$ on Ω for some $\delta = e_1\delta_1 + e_2\delta_2 \succ 0$.
(3) M_u is an invertible operator.
(4) M_u is a Fredholm operator.

Proof. $(1 \Longrightarrow 2)$ Assume that $\mathfrak{R}(M_u)$ is closed and codim$\mathfrak{R}(M_u) < \infty$. Then by the preceding theorem, there exists a $\delta = e_1\delta_1 + e_2\delta_2 \succ 0$ such that $|u(x)|_k \succeq \delta$ $(\vartheta - a.e.)$ on $W = \{x \in \Omega : u(x) \neq 0\}$. Therefore, it suffices to demonstrate that $\vartheta(\Omega \backslash W) = 0$.

Now suppose that M_u is not surjective and let $h = e_1 h_1 + e_2 h_2 \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta) \backslash \mathfrak{R}(M_u)$. Since $\mathfrak{R}(M_u)$ is closed, we can find a function $g = e_1 g_1 + e_2 g_2 \in L_{p',q'}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$ such that

$$\int_\Omega h_1 g_1 d\vartheta_1 = e_1, \int_\Omega h_2 g_2 d\vartheta_2 = e_2 \tag{10}$$

and $e_1 \int_\Omega u_1 f_1 g_1 d\vartheta_1 + e_2 \int_\Omega u_2 f_2 g_2 d\vartheta_2 = 0$ for all $f = e_1 f_1 + e_2 f_2 \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$, where $1/p + 1/p' = 1/q + 1/q' = 1$. From (10), for $i = 1,2$, the sets $E_{\varepsilon_i} = \{x \in \Omega : (h_i g_i)(x) \geq \varepsilon_i\}$ must have positive measure for some $\varepsilon = e_1\varepsilon_1 + e_2\varepsilon_2 \succ 0$. Since ϑ_1 and ϑ_2 are non-atomic component measures of ϑ, we can select disjoint sequences $\left\{E_n^{(i)}\right\}$ of subsets of E_{ε_i} with $0 < \vartheta_i\left(E_n^{(i)}\right) < \infty$ for $i = 1,2$. If we choose $g_n = e_1 g_n^{(1)} + e_2 g_n^{(2)} = \chi_{E_n^{(1)}} g_1 e_1 + \chi_{E_n^{(2)}} g_2 e_2$, then $g_n \in L_{p',q'}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$ and non-zero because

$$\begin{aligned} e_1 \int_\Omega h_1 g_n^{(1)} d\vartheta_1 + e_2 \int_\Omega h_2 g_n^{(2)} d\vartheta_2 &= e_1 \int_{E_n^{(1)}} h_1 g_1 d\vartheta_1 + e_2 \int_{E_n^{(2)}} h_2 g_2 d\vartheta_2 \\ &\succeq e_1\varepsilon_1\vartheta_1\left(E_n^{(1)}\right) + e_2\varepsilon_2\vartheta_2\left(E_n^{(2)}\right) = \varepsilon\left(\vartheta_1\left(E_n^{(1)}\right) + \vartheta_2\left(E_n^{(2)}\right)\right) \succ 0. \end{aligned}$$

Moreover, $\chi_{E_n^{(1)}} f_1 e_1 + \chi_{E_n^{(2)}} f_2 e_2$ is in $L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$ for each $f = e_1 f_1 + e_2 f_2 \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta)$. Therefore,

$$\langle M_u^* g_n, f \rangle = \langle g_n, M_u f \rangle = e_1 \int_\Omega g_n^{(1)}(u_1 f_1) d\vartheta_1 + e_2 \int_\Omega g_n^{(2)}(u_2 f_2) d\vartheta_2$$

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$$= e_1 \int_\Omega \chi_{E_n^{(1)}} g_1 (u_1 f_1) d\vartheta_1 + e_2 \int_\Omega \chi_{E_n^{(2)}} g_2 (u_2 f_2) d\vartheta_2$$
$$= \int_\Omega M_u f \left(e_1 \chi_{E_n^{(1)}} + e_2 \chi_{E_n^{(2)}} \right) (g_1 e_1 + g_2 e_2) d\vartheta = 0,$$

where M_u^* is the conjugate operator of M_u. This implies $\{g_n\}$ is a sequence in $N(M_u^*)$ and so the sequence $\{g_n\}$ forms a linearly independent subset of $N(M_u^*)$. This contradicts the fact that $\dim N(M_u^*) = \operatorname{codim} \Re(M_u) < \infty$. Hence M_u is surjective and $\vartheta(\Omega \backslash W) = 0$. If $\vartheta(\Omega \backslash W) \succ 0$, then there exists a subset V of $\Omega \backslash W$ with $0 \prec \vartheta(V) \prec \infty_{\mathbb{D}}$. Then $\chi_V = e_1 \chi_V + e_2 \chi_V \in L_{p,q}^{\mathbb{BC}}(\Omega, \mathfrak{M}, \vartheta) \backslash \Re(M_u)$, which contradicts the fact that M_u is surjective. Therefore, $\vartheta(\Omega \backslash W) = 0$ and $|u(x)|_k \succeq \delta$ a.e. on Ω.

The other implications are easy.

CONCLUSIONS

In this study I analysed multiplication operators acting on bicomplex Lorentz spaces, a mathematical structure that extends classical Lorentz spaces to the bicomplex setting. By utilising tools from functional analysis and operator theory, I derived necessary and sufficient conditions for these operators to be compact and Fredholm. My findings reveal that the compactness of the multiplication operators depends critically on the behaviour of the symbol functions within the bicomplex framework while the Fredholm property is governed by both the invertibility conditions and the interplay of bicomplex components.

These results contribute to a broader understanding of operator theory in bicomplex spaces, offering insights that may prove useful for further theoretical developments or applications in mathematical physics and engineering. Future research could explore the spectral properties of these operators or extend the analysis to other classes of bicomplex function spaces.

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