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Full Paper

Lacunary A-statistical convergence of fuzzy variable sequences via Orlicz function

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Abstract: This paper presents and examines lacunary A-statistical convergence for fuzzy variable sequences utilising Orlicz functions. We delve into various facets of credibility theory including mean, measure, almost surely, uniform almost surely and distribution. We aim to illustrate the interrelationships of the constructed sequence spaces through relevant examples.

Keywords: statistical convergence, lacunary convergence, fuzzy variable sequence, credibility theory, Orlicz function

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INTRODUCTION

Zadeh [1] introduced the mathematical concept of fuzzy set theory, which has been widely applied to address various real-world problems. The convergence of fuzzy variables is essential in credibility theory for applications in engineering and finance.

Kaufmann [2] introduced several important concepts such as membership functions, fuzzy variables and possibility distributions. A key concept of possibility theory is the possibility measure, which is often defined as a supremum-preserving set function on the power set of a non-empty set. It is a non-self-dual measure. Recognising the importance of self-dual measures, Liu and Liu [3] developed the credibility measure (Cr), which possesses the essential characteristics of the possibility measure and serves as a substitute in the fuzzy world.

As evidenced by the work of Li and Liu [4, 5] and Liu [6-8], four convergence concepts for fuzzy variable sequences (FVSs) within the credibility theory have been proposed since Liu examined the theory-convergence almost surely, convergence in credibility, convergence in mean, and convergence in distribution. Based on the credibility theory, Jiang [9] and Ma [10] investigated the different convergence properties of credibility distributions for fuzzy variables.

The relationships between mean convergence, distribution convergence, almost surely convergence, credibility convergence, and almost uniform convergence were examined by Wang and Liu [11]. Researchers [12-15] have also emphasised and explored the connections between convergence notions in probability theory, credibility theory and classical measure theory.

In real-life scenarios, individuals often encounter situations in which decisions must be made amid uncertainty. Liu [16] presented two methods for tackling indeterminacy—the probability theory, which is utilised when frequencies closely match the distribution function, and the uncertainty theory. However, the probability theory fails to provide precise solutions in the absence of sample data to approximate probability distribution. For instance, determining the likelihood of a successful medical treatment without sufficient data on dosage poses significant challenges.

To tackle these challenges, Liu [17] delved into the uncertainty theory, which now serves as a mathematical branch for modelling human uncertainty. A refined version of this theory, as presented by Liu [16], extends its application to sequences, demonstrating properties such as mean, distribution, measure, and almost sure convergences. Peng [18] contributed by introducing complex uncertain variables, which were further expanded by Chen et al. [15] and later generalised by Datta and Tripathy [19] with double sequences of complex uncertain variables. Subsequently, Tripathy and Nath [20] employed complex uncertain sequences to delve into statistical convergence, adding depth to the evolving field of uncertainty theory, in which important research on lacunary convergence may be observed [21, 22]. Kolk [23] introduced the concept of A-statistical convergence, and Baliarsingh [24] extended this idea by presenting deferred A-convergence of uncertain sequences.

An Orlicz function, denoted as M, is a mapping from $[0,\infty)$ to $[0,\infty)$ that is continuous, non-decreasing and convex, with $M(0)=0, M(x)>0$ for $x>0$, and $M(x)$ approaching ∞ as x approaches ∞. The idea of Orlicz function to construct sequence spaces was proposed by Lindenstrauss and Tzafriri [25] with the formula below, where l_M is known as an Orlicz sequence space:

$$l_M=\left\{x=(x_k)\in w\colon \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right)<\infty, \text{for some } \rho>0\right\}.$$

Later, Nath and Tripathy [26] introduced statistical convergence for complex uncertain sequences using the Orlicz function. Dowari and Tripathy [27] introduced several concepts of lacunary strongly convergent sequences of uncertain variables with respect to the Orlicz function.

In the realm of sequence space analysis, credibility theory and uncertainty theory stand as two distinct concepts. Credibility theory is primarily concerned with the study of sequences involving fuzzy variables, wherein a fuzzy variable is defined as a function that maps from a possibility space to a set of real numbers [28]. On the other hand, uncertainty theory is dedicated to the examination of sequences featuring uncertain variables, wherein an uncertain variable is a function that maps from the uncertainty space to a set of real numbers [6]. Uncertain space is crucial in applications dealing with uncertain data while credibility space is more suitable for situations requiring reliability analysis. Despite the divergence in their focus, both theories share a common interest in investigating the convergence of sequences, highlighting a parallel exploration of similar aspects.

The concept of statistical convergence of fuzzy variables in credibility space was proposed by Savas et al. [29] with further exploration work [30-34].

Our findings notably differ from those of You et al. [14], who emphasised the convergence of FVSs. Our methodology centres on the statistical convergence of FVSs by utilising lacunary sequences, infinite matrices and Orlicz functions. Simultaneously, a key disparity that distinguishes our study lies in our departure from the conventional utilisation of the Cesàro matrix. While the Cesàro matrix is a common choice in similar contexts, we opted for a more inclusive approach by incorporating an arbitrary non-negative regular matrix, denoted as A, in our definition of lacunary A-statistical convergence of FVSs in credibility space. This deliberate departure from the reliance on the Cesàro matrix underscores our commitment to investigating the broader implications and potential advantages associated with the application of a diverse set of matrices within the A-statistical convergence framework.

Our objective is to contribute to the comprehension and applicability of statistical convergence in a manner that transcends the limitations imposed by the use of a specific matrix, thus enriching the scope of related research. We have enhanced the importance of lacunary A-statistical convergence in the realm of FVSs by introducing new illustrative examples. We introduce lacunary A-statistical convergence for FVSs, incorporating considerations for almost surely, uniform almost surely, mean, measure, and distribution aspects as defined by Orlicz functions. This study further explores the interrelationships among these FVSs.

PRELIMINARIES

We present the definitions of several key concepts and relevant results. We introduce the notion of a credibility measure and the concept of the expected value for FVSs based on such measures.

A set function Cr is defined as a credibility measure if it fulfills the following axioms: Let Θ be a non-empty set and $\mathcal{P}(\Theta)$ be the power set of Θ which represents all possible subsets of Θ. Each subset in $\mathcal{P}$ is considered an event. For an arbitrary subset $T \in \mathcal{P}(\Theta)$, a credibility measure $Cr\{T\}$ was put forward by Liu and Liu [3] to express the chance that the fuzzy event T occurs. The following criteria must be met for the set function $Cr\{.\}$ to be categorised as a credibility measure:

Axiom i. (Normality) $Cr(\Theta) = 1$;

Axiom ii. (Monotonicity) $Cr\{T\} \leq Cr\{U\}$ whenever $T \subset U$;

Axiom iii. (Self-Duality) Cr is self-dual, that is, $Cr\{T\} + Cr\{T^c\} = 1$ for every $T \in \mathcal{P}(\Theta)$;

Axiom iv. (Maximality) $Cr\{U_i T_i\} = \sup_i Cr\{T_i\}$ for any collection $\{T_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i Cr\{T_i\} < 0.5$.

A credibility space is defined as the triplet $(\Theta,\ \mathcal{P}(\Theta), C_r)$ [3].

The use of fuzzy variables in credibility theory was examined by Liu and Liu [3], who treated them as functions from the credibility space to the set of real numbers. If at least one of the two integrals is finite, then

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr,$$

the anticipated value of the fuzzy variable ξ [4].

$\{\xi_i\}$ converges almost surely to ξ if the sequence $\{\xi_i\}$ converges in credibility to ξ [11]. The sequence $\{\xi_i\}$ converges in credibility to ξ if it converges in mean to ξ [8].

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NEW CONVERGENCE CONCEPTS IN CREDIBILITY SPACE

This work introduces lacunary A-statistical convergence for FVS defined by Orlicz functions, together with almost surely, uniformly almost surely, mean, measure, and distribution aspects for the FVS. Furthermore, the study investigates the relationships between these FVSs.

Definition 1. The real sequence (w_m) is called lacunary A-statistical convergent with respect to Orlicz function M to w_0 if for each $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : M\left(\frac{|(Aw)_m - w_0|}{\sigma}\right) \geq \epsilon\right\}\right| = 0$$

for some $\sigma > 0$.

Definition 2. Let $\xi, \xi_1, \xi_2, \ldots$ be fuzzy variables defined in the credibility space $(\Theta,\ \mathcal{P}(\Theta), C_r)$. There exists a set $T \in \mathcal{P}(\Theta)$ with a unit credibility measure such that

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : M\left(\frac{\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma}\right) \geq \epsilon\right\}\right| = 0$$

for all $\tau \in T$ and for some $\sigma > 0$ if for any preassigned $\epsilon > 0$ the FVS $\{\xi_m\}$ is said to be lacunary A-statistically convergent, defined by Orlicz function M, almost surely to ξ.

Example 1. Let us examine the credibility space $(\Theta,\ \mathcal{P}(\Theta), C_r)$, where $\mathcal{P}(\Theta) = \{\tau_1, \tau_2, \ldots\}$. Let $M(\xi) = \xi, A = (a_{nk})$ be the constant matrix of 1 and $h_r = 2^r$. Consider

$$Cr\{T\} = \begin{cases} \sup_{\tau_m \in T} \frac{m}{2m+1}, & \text{if } \sup_{\tau_m \in T} \frac{m}{2m+1} < \frac{1}{2} \\ 1 - \sup_{\tau_m \in T^c} \frac{m}{2m+1}, & \text{if } \sup_{\tau_m \in T^c} \frac{m}{2m+1} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

We define the fuzzy variables by

$$\xi_m(\tau) = \begin{cases} m, & \text{if } \tau = \tau_m \\ 0, & \text{otherwise} \end{cases}$$

for $m = 1,2, \ldots$ and $\xi \equiv 0$. Then we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : M\left(\frac{\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma}\right) \geq \epsilon\right\}\right| = \lim_{r\to\infty} \frac{1}{2^r} = 0.$$

Consequently, the sequence $\{\xi_m\}$ is lacunary A-statistical convergent in almost surely to ξ.

Definition 3. An FVS $\{\xi_m\}$ is considered lacunary A-statistical convergent with respect to Orlicz function M in credibility to ξ if for any pre-assigned $\epsilon, \delta > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : Cr\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq \epsilon\right) \geq \delta\right\}\right| = 0$$

for some $\sigma > 0$.

Definition 4. If for all $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq \epsilon\right)\right\}\right| = 0$$

holds for some $\sigma > 0$, then the FVS $\{\xi_m\}$ is considered lacunary A-statistical convergent with respect to Orlicz function M in mean to ξ.

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Definition 5. Let $\{\xi_i\}$ be an FVS defined in a credibility space $(\Theta, \mathcal{P}(\Theta), C_r)$ and ψ_i be the credibility distribution functions for the fuzzy variable ξ_i. Then the sequence $\{\xi_m\}$ is called lacunary A-statistical convergent with respect to Orlicz function M in distribution to ξ, whose credibility distribution function is ψ if for any $\epsilon > 0$ and for some $\sigma > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : M\left(\frac{\|(A\psi)_m(u)-\psi(u)\|}{\sigma}\right) \geq \epsilon\right\}\right| = 0$$

for all $u \in \mathbb{R}$ at which $\psi(u)$ is continuous.

Definition 6. The FVS $\{\xi_m\}$ is considered lacunary A-statistical convergent with respect to Orlicz function M in uniformly almost surely to ξ if for every $\epsilon > 0, \exists\delta > 0$ and a sequence of events $E'_m (m \in \mathbb{N})$ such that

$$\Rightarrow \lim_{r\to\infty} \frac{1}{h_r}|\{m \in I_r\ |Cr(E'_m) - 0| \geq \epsilon\}| = 0$$
$$\Rightarrow \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right) \geq \delta\right\}\right| = 0$$

for all $\tau \in E'_m$ and for some $\sigma > 0$.

RELATIONSHIPS AMONG CONVERGENCE CONCEPTS

We obtain some results that establish interconnections between several types of lacunary A-statistical convergent sequences of FVSs in a credibility space.

Theorem 1. Suppose that an FVS $\{\xi_m\}$ is lacunary A-statistically convergent in mean to ξ. Then it is lacunary A-statistically convergent in credibility to ξ. However, the opposite is not true in general.

Proof. We obtain for each $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m-\xi\|}{\sigma}\right) \geq \epsilon\right)\right\}\right| = 0$$

if the FVS $\{\xi_m\}$ is lacunary A-statistically convergent in mean to ξ. Applying Markov's inequality, we acquire the following inequality for any given $\delta > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : Cr\left(M\left(\frac{\|(A\xi)_m-\xi\|}{\sigma}\right) \geq \epsilon\right) \geq \delta\right\}\right|$$
$$= \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : Cr\left(\tau: M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right) \geq \epsilon\right) \geq \delta\right\}\right|$$
$$\leq \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \mathbf{E}\left(\tau: M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma\epsilon}\right)\right) \geq \delta\right\}\right| = 0$$
$$= \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m-\xi\|}{\sigma\epsilon}\right)\right) \geq \delta\right\}\right| = 0.$$

This establishes the result.

Currently, we can demonstrate the opposite using the following example:

Example 2. Given the credibility space $(\Theta, \mathcal{P}(\Theta), C_r)$ where $\mathcal{P}(\Theta) = \{\tau_1, \tau_2, ...\}$, with a credibility measure function defined as follows:

$$Cr\{T\} = \begin{cases} \sup_{\tau_m \in T} \frac{1}{2m}, & \text{if } \sup_{\tau_m \in T} \frac{1}{2m} < \frac{1}{2} \\ 1 - \sup_{\tau_m \in T^c} \frac{1}{2m}, & \text{if } \sup_{\tau_m \in T^c} \frac{1}{2m} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

and the fuzzy variables established as

$$\xi_m(\tau) = \begin{cases} 2m, & \text{if } \tau = \tau_m \\ 0, & \text{otherwise} \end{cases}$$

for $m = 1,2, \ldots$ and $\xi \equiv 0,$ then for any given small number $\epsilon > 0$ and $m \geq 2$, we can observe

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ m \in I_r : Cr\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma} \right) \geq \epsilon \right) \geq \delta \right\} \right|$$
$$= \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ m \in I_r : Cr\left(\tau : M\left(\frac{\tau:\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma} \right) \geq \epsilon \right) \geq \delta \right\} \right|$$
$$= \lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ m \in I_r : Cr\left(M\left(\frac{\tau:\|(A\xi)_m(\tau)\|}{\sigma} \right) \geq \epsilon \right) \geq \delta \right\} \right| = 0.$$
$$= \lim_{r\to\infty} Cr\{\tau_m\} = \lim_{r\to\infty} \frac{1}{2m} = 0, (m \in I_r).$$

Therefore, the sequence $\{\xi_m\}$ is lacunary A-statistically convergent in credibility to ξ. However,

$$\psi_m(u) = \begin{cases} 0, & \text{if } u < 0 \\ 1 - \frac{1}{2m}, & \text{if } 0 \leq u \leq 2m \\ 1, & \text{if } u \geq 2m \end{cases}$$

is the credibility distribution of the fuzzy variable $\|\xi_m - \xi\| = \|\xi_m\|$ for $m \geq 2$. Then we obtain

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ m \in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma} \right) - 1 \right) \right\} \right| = \left[\int_0^{2m} 1 - \left(1 - \frac{1}{2m} \right) du - 1 \right] = 0.$$

Hence the sequence $\{\xi_m\}$ does not lacunary-A-statistically converge with respect to Orlicz function M in mean to ξ.

Remark 1. The concepts of lacunary A-statistical convergence in terms of credibility and almost sure convergence with respect to Orlicz function M are independent of each other. This claim is illustrated in the following successive examples.

Example 3. Lacunary A-statistical convergence in almost surely does not imply lacunary A-statistical convergence in credibility with respect to Orlicz function M. Consider the credibility space $(\Theta,\ \mathcal{P}(\Theta), C_r)$ with $\mathcal{P}(\Theta) = \{\tau_1, \tau_2, \ldots\}$ having a credibility measurable function as follows:

$$Cr\{T\} = \begin{cases} \sup_{\tau_m \in T} \frac{m}{2m+1}, & \text{if } \sup_{\tau_m \in T} \frac{m}{2m+1} < \frac{1}{2} \\ 1 - \sup_{\tau_m \in T^c} \frac{m}{2m+1}, & \text{if } \sup_{\tau_m \in T^c} \frac{m}{2m+1} < \frac{1}{2} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

We establish the fuzzy variables as

$$\xi_m(\tau) = \begin{cases} m, & \text{if } \tau = \tau_m \\ 0, & \text{otherwise} \end{cases}$$

for $m = 1,2, \ldots$ and $\xi \equiv 0$. Then we obtain

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$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right\}\right| = \lim_{r\to\infty}\frac{1}{2^r} = 0.$$

Thus, the sequence $\{\xi_m\}$ is lacunary A-statistical convergent in almost surely to ξ. Additionally, for $\epsilon > 0$, we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(M\left(\frac{\|(A\xi)_m-\xi\|}{\sigma}\right)\geq\epsilon\right)\geq\frac{1}{2}\right\}\right|$$
$$=\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\tau : M\left(\frac{\tau{:}\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq\frac{1}{2}\right\}\right|$$
$$=\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr(\tau_m)\geq\frac{1}{2}\right\}\right| = 0.$$

Thus, $\{\xi_m\}$ is not lacunary A-statistically convergent with respect to Orlicz function M in terms of credibility to ξ.

Example 4. Lacunary A-statistical convergence in credibility does not imply lacunary A-statistical convergence in almost surely with respect to Orlicz function M. We examine the credibility space for FVS $(\Theta,\ \mathcal{P}(\Theta), C_r)$ with $\mathcal{P}(\Theta) = \{\tau_1, \tau_2, ...\}$ using Lebesgue measure and Borel algebra. For $m \in \mathbb{Z}^+$, there exists an integer t such that $m = 2^t + R$. The FVS is defined by

$$\xi_m(\tau) = \begin{cases} 1, & \text{if } \frac{R}{2^t} < \tau \leq \frac{R+1}{2^t} \\ 0, & \text{otherwise} \end{cases}$$

for $m = 1,2, ...$, $R \in \mathbb{Z}$ and $\xi = 0$. For some given $\epsilon, \delta > 0$ and $m \geq 2$, one has

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(M\left(\frac{\|(A\xi)_m-\xi\|}{\sigma}\right)\geq\epsilon\right)\geq\delta\right\}\right|$$
$$=\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\tau : M\left(\frac{\tau{:}\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq\delta\right\}\right|$$
$$=\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(M\left(\frac{\|(A\xi)_m\|}{\sigma}\right)\geq\epsilon\right)\geq\delta\right\}\right| = 0.$$

Consequently, the sequence $\{\xi_m\}$ is lacunary A-statistically convergent in terms of credibility to ξ. Furthermore,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m-\xi\|}{\sigma}\right)\geq\epsilon\right)\right\}\right| = 0$$

holds for $\epsilon > 0$. Therefore, the sequence $\{\xi_m\}$ is lacunary A-statistically convergent in mean to ξ. For any $\tau \in [0,1]$, there exists an infinite number of closed intervals of the form $\left[\frac{R}{2^t}, \frac{R+1}{2^t}\right]$ containing τ. Thus, $\xi_m(\tau)$ is not lacunary A-statistically convergent in almost surely to ξ.

Remark 2. The notions of lacunary A-statistical convergence in almost surely and in mean with respect to Orlicz function M do not imply each other. This assertion is supported by the following two successive cases.

Example 5. Lacunary A-statistical convergence in mean does not imply lacunary A-statistical convergence in almost surely with respect to Orlicz function M. Let us consider the credibility space $(\Theta,\ \mathcal{P}(\Theta), C_r)$, where $\mathcal{P}(\Theta) = \{\tau_1, \tau_2, ...\}$ and the credibility measure for the events is determined by $Cr\{T\} = \sum_{\tau_m\in T}\frac{1}{2^m}$. The fuzzy variables are defined by

$$\xi_m(\tau) = \begin{cases} 2^m, & \text{if } \tau = \tau_m \\ 0, & \text{otherwise} \end{cases}$$

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for $m = 1,2,...$ and $\xi = 0$. Thus, the sequence $\{\xi_m\}$ is lacunary A-statistical convergent in almost surely to ξ. Consider

$$\psi_m(u) = \begin{cases} 0, & \text{if } u < 0 \\ 1 - \frac{1}{2^m}, & \text{if } 0 \le u \le 2^m \\ 1, & \text{if } u \ge 2^m. \end{cases}$$

Here, $\psi_m(u)$ is the credibility distribution of $\|\xi_m\|$. Then we have

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right)\right) \ge 1\right\}\right| = 0.$$

So the sequence $\{\xi_m\}$ does not lacunary-A-statistically converge with respect to Orlicz function M in mean to ξ.

Example 6. Regarding Orlicz function M, lacunary A-statistical convergence in mean does not imply lacunary A-statistical convergence in almost surely. Let $\Theta = \{\tau_1, \tau_2, ...\}$ be a set with the credibility measure $Cr\{T\} = 1/t$ for $t = 1,2, ...$ The FVS is defined by

$$\xi_m(\tau_t) = \begin{cases} \frac{(t+1)}{t}, & \text{if } t = m, m+1, m+2, ... \\ 0, & \text{otherwise} \end{cases}$$

for $m = 1,2,...$ and $\xi = 0$. The sequence $\{\xi_m\}$ is not lacunary A-statistical convergent in almost surely to ξ. However,

$$\mathbf{E}[|\xi_m - \xi|] = \frac{k+1}{2k^2} \to 0.$$

Therefore, for every $\varepsilon > 0$ and for some $\sigma > 0$ we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \mathbf{E}\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \ge \epsilon\right)\right\}\right| = 0,$$

which shows that $\{\xi_m\}$ lacunary-A-statistically converges in mean to ξ.

Concerning Orlicz function M, lacunary A-statistical convergence in distribution does not imply lacunary A-statistical convergence in credibility. Let us now consider an example to illustrate this.

Example 7. Assume that $\{\tau_1, \tau_2, ...\}$ represents the credibility space $(\Theta, \mathcal{P}(\Theta), C_r)$, with $Cr\{\tau_1\} = Cr\{\tau_2\} = \frac{1}{2}$. We establish the fuzzy variable as follows:

$$\xi_m(\tau) = \begin{cases} 1, & \text{if } \tau = \tau_1 \\ -1, & \text{if } \tau = \tau_2. \end{cases}$$

We also take $\{\xi_m\} = -\xi$ for $m \in \mathbb{N}$. Then $\{\xi_m\}$ and ξ have the same distribution and $\{\xi_m\}$ lacunary-A-statistically converges in distribution to ξ with respect to Orlicz function M. However, for any given $\epsilon, \delta > 0$ we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : Cr\left(M\left(\frac{\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma}\right) \ge \epsilon\right) \ge \delta\right\}\right|$$
$$= \lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : Cr\left(M\left(\frac{\|2(A\xi)_m(\tau)\|}{\sigma}\right) \ge \epsilon\right) \ge \delta\right\}\right| \ne 0.$$

Hence the sequence $\{\xi_m\}$ does not lacunary-A-statistically converge in terms of credibility to ξ with respect to Orlicz function M.

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Theorem 2. Suppose that $\psi, \psi_1, \psi_2, \ldots$ are the credibility distributions of the fuzzy variables $\xi, \xi_1, \xi_2, \ldots$, respectively. If the sequence $\{\xi_m\}$ converges lacunary-A-statistically in credibility to ξ, then $\{\xi_m\}$ converges lacunary-A-statistically in distribution to ξ.

Proof. For credibility distribution ψ, let x represent any given continuity point. One way to look at it is

$$\{\psi : (A\xi)_m(\psi) \leq x\} = \{\psi : (A\xi)_m(\psi) \leq x, \xi(\psi) \leq y\} \cup \{\psi : (A\xi)_m(\psi) \leq x, \xi(\psi) > y\}$$
$$\subset \{\psi : \xi(\psi) \leq y\} \cup \left\{\psi : M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq y - x\right\},$$

which implies that

$$(A\psi)_m(x) \leq \psi(y) + Cr\left\{\psi : M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq y - x\right\}$$

for every $y > x$. For any $\delta > 0$ we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r : Cr\left(M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq y - x\right) \geq \delta\right\}\right| = 0$$

because $\{\xi_m\}$ is lacunary statistically convergent in terms of credibility to ξ. Thus, we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r \sup_m\left(M\left(\frac{|(A\psi)_m(x) - \psi(x)|}{\sigma}\right)\right) \geq \delta\right\}\right| = 0 \qquad (1)$$

for all δ as $y \to x$. However, for any $z < x$ we have

$$\{\psi : \xi(\psi) \leq z\} = \{\psi : \xi(\psi) \leq z, \xi_m(\psi) \leq x\} \cup \{\psi : \xi(\psi) \leq z, \xi_m(\psi) > x\}$$
$$\subset \{\psi : \xi_m(\psi) \leq x\} \cup \left\{\phi : M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq \mathrm{x} - \mathrm{z}\right\},$$

which implies that

$$\psi(z) \leq (A\psi)_m(x) + Cr\left\{\psi : M\left(\frac{\|(A\xi)_m - \xi\|}{\sigma}\right) \geq \mathrm{x} - \mathrm{z}\right\}.$$

Since

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r : Cr\left(M\left(\frac{\psi\!: \|(A\xi)_m - \xi\|}{\sigma}\right) \geq x - z\right) \geq \delta\right\}\right| = 0,$$

thus, we obtain

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r \inf_m\left(M\left(\frac{|(A\psi)_m(x) - \psi(x)|}{\sigma}\right)\right) \geq \delta\right\}\right| = 0 \qquad (2)$$

for all $\delta > 0$ as $z \to x$. It follows from (1) and (2) that $\{\xi_m\}$ is lacunary A-statistically convergent in distribution to ξ.

Proposition 1. A sequence $\{\xi_m\}$ is lacunary A-statistically convergent in almost surely if and only if

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : Cr\left(\cap_{m=1}^{\infty} \cup_{r=m}^{\infty} \tau \in \mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma} \geq \epsilon\right)\right) \geq \delta\right\}\right| = 0.$$

Proof. According to the definition of lacunary A-statistical convergence in almost surely, there exists a set $T \in \mathcal{P}(\Theta)$ with a unit credibility measure such that

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{m \in I_r : \left(\tau \in \mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma}\right) \geq \epsilon\right)\right\}\right| = 0$$

for each $\tau \in T$ and for some $\sigma > 0$. Then for any $\epsilon > 0$ there exists an integer m such that $M\left(\frac{\|(A\xi)_m(\tau) - \xi(\tau)\|}{\sigma}\right) < \epsilon$, where $r > m$ and $\tau \in T$ that is equivalent to

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$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcap_{m=1}^{\infty}\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)<\epsilon\right)\geq 1\right\}\right|=0.$$

It follows from the self-duality axiom of credibility measure that

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcap_{m=1}^{\infty}\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq \delta\right\}\right|=0.$$

Conversely, suppose that for $\epsilon>0$, $\sigma>0$ and for $v\in T$ we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcap_{m=1}^{\infty}\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq \delta\right\}\right|=0.$$

From the self-duality axiom of credibility measure and for any $\epsilon>0$, there exists an integer m where $r>m$ and for all $\tau\in T$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcap_{m=1}^{\infty}\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)<\epsilon\right)\geq 1\right\}\right|=0.$$

That is, for any $\epsilon>0$ there exists an integer m such that $M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)<\epsilon$, where $r>m$ and $T\in\mathcal{P}(\Theta)$ with $Cr\{T\}=1$. Hence we can write

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : \left(\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\right\}\right|=0.$$

Thus, $\{\xi_m\}$ is lacunary A-statistically convergent in almost surely to ξ.

Proposition 2. An FVS $\{\xi_m\}$ is lacunary A-statistically convergent uniformly almost surely to ξ if and only if

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq \delta\right\}\right|=0.$$

Proof. Let us assume that the sequence $\{\xi_m\}$ is lacunary A-statistically convergent uniformly almost surely to ξ. Thus, for every $\varsigma>0$, there exists a set T such that $Cr(T)<\varsigma$ and $\{\xi_m\}$ exhibits lacunary A-statistical convergence uniformly to ξ on $\mathcal{P}(\Theta)-T$. Therefore, for any $\epsilon,\sigma>0$, there exists a positive integer m such that $M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)<\epsilon$ for $r\leq m$ and for $\tau\in\mathcal{P}(\Theta)-T$, we can write

$$\left(\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\subset T.$$

From the sub-additivity axiom, we obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\right\}\right|\leq\delta(Cr\{T\})<\varsigma.$$

Then

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq \delta\right\}\right|=0.$$

Conversely, consider

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r : Cr\left(\bigcup_{r=m}^{\infty}\tau\in\mathcal{P}(\Theta) : M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq \delta\right\}\right|=0.$$

Additionally, for some given $\delta>0$ and $\mathfrak{I}\geq 1$, there exists a set $\mathfrak{I}_m$ such that

$$\delta\left(Cr\left(\bigcup_{r=\mathfrak{I}_m}^{\infty}\left\{\tau\in\mathcal{P}(\Theta): M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\frac{1}{\mathfrak{I}}\right\}\right)\right)\leq\frac{\delta}{2^{\mathfrak{I}}}.$$

Suppose

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$$T=\bigcup_{\Im=1}^{\infty}\bigcup_{r=\Im_m}^{\infty}\left\{\tau\in\mathcal{P}(\Theta):M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\frac{1}{\Im}\right\}.$$

Then

$$\delta(Cr\{T\})\leq\sum_{\Im=1}^{\infty}\delta\left(Cr\left(\bigcup_{r=\Im_m}^{\infty}\left\{\tau\in\mathcal{P}(\Theta):M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\frac{1}{\Im}\right\}\right)\right)\leq\sum_{\Im=1}^{\infty}\frac{\delta}{2^{\Im}}.$$

Furthermore, we have

$$\sup_{\tau\in\mathcal{P}(\Theta)-T}\left(\tau\in\mathcal{P}(\Theta):M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\right)<\frac{1}{\Im}$$

for any $\Im=1,2,3,\ldots$ and $r>\Im_m$. Thus, $\{\xi_m\}$ is lacunary A-statistically convergent uniformly almost surely to ξ.

Theorem 3. If FVS $\{\xi_m\}$ is lacunary A-statistically convergent uniformly almost surely to ξ, then the sequence $\{\xi_m\}$ is lacunary A-statistically convergent to ξ in almost surely.

Proof. We obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r:Cr\left(\bigcup_{r=m}^{\infty}\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq\delta\right\}\right|=0$$

by applying Proposition 2. Additionally,

$$\delta\left(Cr\left(\bigcap_{m=1}^{\infty}\bigcup_{r=m}^{\infty}\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\right)$$
$$\leq\delta\left(Cr\left(\bigcup_{r=m}^{\infty}\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\right).$$

Since $r\to\infty$ on both sides, we have

$$\delta\left(Cr\left(\bigcap_{m=1}^{\infty}\bigcup_{r=m}^{\infty}\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(v\tau)\|}{\rho}\right)\geq\epsilon\right)\right)=0.$$

Using Proposition 1, we obtain the desired result.

Theorem 4. FVS $\{\xi_m\}$, which lacunary-A-statistically converges uniformly almost surely to ξ, also lacunary-A-statistically converges in credibility to ξ.

Proof. We obtain

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{m\in I_r:Cr\left(\bigcup_{r=m}^{\infty}\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\geq\delta\right\}\right|=0$$

and

$$\delta\left(Cr\left(\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right)\right)\leq\delta\left(Cr\left(\bigcup_{r=m}^{\infty}\left\{\tau:M\left(\frac{\|(A\xi)_m(\tau)-\xi(\tau)\|}{\sigma}\right)\geq\epsilon\right\}\right)\right)$$

by applying Proposition 2. This approach gives the desired result. Therefore, the sequence $\{\xi_m\}$ is lacunary A-statistical convergent to ξ in terms of credibility.

CONCLUSIONS

This article explores lacunary A-statistical convergence in relation to Orlicz function M, focussing on almost surely convergence, uniform almost surely convergence, mean convergence, measure convergence, and distribution convergence for FVS. We suggest further investigation into the specific conditions under which lacunary A-statistical convergence holds as well as its implications for different sequence spaces.

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