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A note on approximation by some product means in variable exponent Lebesgue spaces

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Abstract: The approximation by product means in the variable exponent Lebesgue spaces is a problem that has not been addressed before. With this motivation, in this study the speed of approximation of the functions of class $Lip(\alpha, p(x))$, $(0 < \alpha \leq 1)$ by some product means is obtained with the parameters in the method used as envisioned by Hardy and Littlewood in 1928.

Keywords: trigonometric approximation, variable exponent Lebesgue space, product means

INTRODUCTION

In 1928 Hardy and Littlewood [1] stated without proof that the speed of approximation by trigonometric polynomials of degree n to the functions of class $Lip(\alpha, p)$ for $p \geq 1$ and $0 < \alpha \leq 1$ is possible with error $O(n^{-\alpha})$, where the big O notation is Landau's symbol. In 1937 Quade [2] discussed this theorem for some trigonometric polynomials and obtained various results. In the following years this problem was developed and handled by many mathematicians [3-8] and some results were obtained in accordance with the theorem laid down [1]. A similar problem in the variable exponent Lebesgue spaces was first addressed by Güven and Israfilov [9] and the speed of approximation to functions from the class $Lip\big(\alpha, p(x)\big)$ was evaluated by using Nörlund and the Riesz averages. There are various studies on approximation with trigonometric polynomials in the variable exponent Lebesgue spaces [10-15].

Another method used in approximation by trigonometric polynomials in the Lipschitz classes is the product means dealt with in the next sections. It is also quite common to use the product means in the study of approximation problems [16-19]. However, approximation problems with product means have not been discussed in the variable exponent Lebesgue spaces before. Therefore, considering the approximation using the product means in these spaces has emerged as a

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problem worth examining. In this study we introduce new ideas about dealing with similar problems in such spaces by other product means.

PRELIMINARIES

Now let us give the concepts we mentioned.

1. Suppose that $\wp := \wp(\mathbb{R})$ is the family of all measurable 2π-periodic functions $p:\mathbb{R}\to[1,\infty]$. The space $L^{p(x)} := L^{p(x)}([0,2\pi])$ is the set of all functions f which are measurable 2π-periodic defined on $[0,2\pi]$ such that $\rho_p(\lambda f)<\infty$ for some $\lambda:=\lambda(f)>0$, where

$$\rho_p(f):=\begin{cases}\int_0^{2\pi}|f(x)|^{p(x)}dx \ , & 1\le p(x)<\infty\\ ess\sup_{x\in[0,2\pi]}|f(x)| \ , & p(x)=\infty\end{cases}$$

and $p\in\wp$. The variable exponent Lebesgue space $L^{p(x)}$ is a Banach space with the norm $\|.\|_{p(x)}$ defined by

$$\|f\|_{p(x)} := \inf\{\lambda>0:\rho_p(f/\lambda)\le 1\}.$$

If $p(x)\equiv p$ is a constant, then the above norm coincides with the usual L^p norm. Some details and further references for the spaces $L^{p(x)}$ can be found [20-22]. Let $\wp_*([0,2\pi])$ be the set of all functions $p\in\wp([0,2\pi])$ that satisfy the condition

$$1<p_*:=ess\inf_{x\in[0,2\pi]} p(x)\le p^*:=ess\sup_{x\in[0,2\pi]} p(x)<\infty.$$

If $q\in\wp_*([0,2\pi])$ satisfies the following condition named *local continuity* (briefly *loc*)

$$-|q(a)-q(b)|\ln|a-b|=O(1),\quad 0<|a-b|\le 1/2,$$

then $q\in\wp_{loc}$, where $\wp_{loc}$ is the set of all the functions satisfying the condition of the local continuity. The *loc* condition ensures that the maximal operator on $L^{p(x)}$ is bounded [23]. The Lipschitz class in the variable exponent Lebesgue spaces is defined [9] as

$$Lip(\alpha,p(x))=\{f\in L^{p(x)}:\Omega_{p(x)}(f,\delta)=O(\delta^\alpha),\ \delta>0,\ 0<\alpha\le 1, p(x)\in\wp_{loc}\},$$

where

$$\Omega_{p(x)}(f,\delta)=\sup_{|h|\le\delta}\|T_h(f)\|_{p(x)}$$

is the integral modulus of continuity of the function $f\in L^{p(x)}$, and here

$$T_h(f)(x)=\frac{1}{h}\int_0^h|f(x+t)-f(x)|dt.$$

2. Suppose that

$$S[f]=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$$

is the Fourier series of a function $f\in L$, where $L:=L(0,2\pi)$ is the space of functions that are 2π-periodic and Lebesgue integrable on $[0,2\pi]$. On the other hand, assume that (p_n) is a non-negative sequence of real numbers. The transformations given by

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$$N_n(f;x) = P_n^{-1} \sum_{v=0}^{n} p_{n-v} s_v(f;x) \tag{1}$$

and

$$R_n(f;x) = P_n^{-1} \sum_{v=0}^{n} p_v s_v(f;x) \tag{2}$$

are respectively called the Nörlund means (or Voronoi means) (N_{p_n}) and Riesz means (R_{p_n}) of the sequence $\{s_n(f;\cdot)\}$, where $s_n(f;\cdot)$ denotes the nth partial sums of $S[f]$. Here,

$$\forall\ n \geq 0,\ P_n = \sum_{k=0}^{n} p_k \neq 0, p_{-1} = 0 = P_{-1}$$

and

$$P_n \to \infty \text{ as } n \to \infty .$$

In (1) and (2) if $p_n = 1$, then these methods give the Cesáro mean denoted by $C_1 := (C,1)$. According to these methods, $C_1.N_{p_n}$ is the product of the means of C_1 and N_{p_n}, and we denote it by [17]

$$t_n^{CN} = \frac{1}{n+1} \sum_{k=0}^{n} P_k^{-1} \sum_{v=0}^{k} p_{k-v} s_v.$$

The $C_1.R_{p_n}$ mean is defined in a similar way. It is given by

$$t_n^{CR} = \frac{1}{n+1} \sum_{k=0}^{n} P_k^{-1} \sum_{v=0}^{k} p_v s_v.$$

In addition to these two means, the following methods take the problem one step further. The concept of deferred Cesáro means has been given by Agnew [24]. The deferred Cesáro mean, $D_{a_n}^{b_n}$, is represented in the form

$$D_{a_n}^{b_n} = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} s_k,$$

where (s_k) is a sequence of real or complex numbers and (a_n) and (b_n) are the sequences of non-negative integers with conditions

$$a_n < b_n, \quad n = 1,2,3, \dots \tag{3}$$

and

$$\lim_{n\to\infty} b_n = \infty. \tag{4}$$

Note that every $D_{a_n}^{b_n}$ is regular with conditions (3) and (4). Detailed information for $D_{a_n}^{b_n}$ has been given [24].

Considering the deferred Cesáro mean, the deferred-Nörlund product mean denoted by $(D_{a_n}^{b_n}.N_{p_k})$ of sequence $\{s_v(f;.)\}$ is given by

$$t_n^{DN} = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} P_k^{-1} \sum_{v=0}^{k} p_{k-v} s_v(f;.).$$

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Similarly, the deferred-Riesz product means $(D_{a_n}^{b_n}.R_{p_k})$ is defined by

$$t_n^{DR} = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} P_k^{-1} \sum_{v=0}^{k} p_v s_v(f;.).$$

3. The monotonicity conditions on the sequence (p_n) characterising the Nörlund and Riesz means are quite important in determining the degree of approximation. For this purpose, let us remember some numerical sequence classes containing the class of sequence satisfying monotonicity conditions [7, 25].

A non-negative sequence (c_n) is called almost monotone decreasing (briefly $(c_n) \in amds$) (increasing (briefly $(c_n) \in amis$)) if there exists a constant $K := K(c_n)$ such that

$$c_n \leq K c_m \quad (c_m \leq K c_n)$$

for all $m \leq n$. If $T \in amds$ $(T \in amis)$ where $T := (T_n) = \left(\frac{1}{n+1}\sum_{m=0}^{n} c_m\right)$, then we say that the sequence (c_n) is almost monotone decreasing (increasing) mean sequence and denote it by $T \in amdms$ $(T \in amims)$. A sequence (c_n) tending to zero is called 'a rest-bounded variation sequence $(rbvs)$', which was first introduced by Leindler [25] if it satisfies

$$\sum_{m=k}^{\infty} |\Delta c_m| \leq K c_k$$

for all natural numbers k where $\Delta c_m = c_m - c_{m+1}$. It is known that $rbvs \subset amds \subset amdms$ and $amis \subset amims$ [7].

With the perspective given in this section we shall present the results related to trigonometric approximation of the functions belonging to $Lip(\alpha, p(x))$, $(0 < \alpha \leq 1)$ space by the t_n^{CN}, t_n^{CR}, t_n^{DN} and t_n^{DR} means for the partial sums sequence of its Fourier series.

AUXILIARY RESULTS

Here we give some auxiliary results that will be used in the proof of the main conclusions.

Theorem 1 [12]. Let $p \in \wp_{loc}$, $f \in Lip(\alpha, p(x))$, $0 < \alpha < 1$, and let (p_n) be a positive sequence. If one of the following conditions is satisfied:

(i) $(p_n) \in amims$ with

$$(n+1)p_n = O(P_n) \tag{5}$$

(ii) $(p_n) \in amdms$,

then

$$\|f - N_n(f)\|_{p(x)} = O(n^{-\alpha}).$$

Theorem 2 [9]. Let $p \in \wp_{loc}$, $f \in Lip(1, p(x))$ and let (p_n) be a positive sequence. If one of the following conditions is satisfied:

(i) $\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right)$ with condition (5),

(ii) $\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$,

then

$$\|f - N_n(f)\|_{p(x)} = O(n^{-1}).$$

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Theorem 3 [9]**.** Let $p \in \wp_{loc}$, $f \in Lip\big(\alpha, p(x)\big)$, $0 < \alpha \leq 1$, and let (p_n) be a positive sequence. If the following condition is satisfied:

$$\sum_{m=0}^{n-1} \left| \Delta\left(\frac{P_m}{m+1}\right) \right| = O\left(\frac{P_n}{n+1}\right),$$

then

$$\|f - R_n(f)\|_{p(x)} = O(n^{-\alpha}).$$

MAIN RESULTS AND THEIR COROLLARIES

Theorem 4. Let $p \in \wp_{loc}$, $f \in Lip\big(\alpha, p(x)\big)$, $0 < \alpha < 1$, and let (p_n) be a sequence of positive numbers. If $(p_n) \in amims$ with (5) or $(p_n) \in amdms$, then

$$\|f - t_n^{DN}(f)\|_{p(x)} = O((b_n - a_n)^{-\alpha}).$$

Proof. By definition of the deferred-Nörlund $(D_{a_n}^{b_n}.N_{p_k})$ means, we have

$$\|f - t_n^{DN}(f)\|_{p(x)} = \left\| f - \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} s_v(f) \right\|_{p(x)}$$

$$= \left\| \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} f - \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} N_k(f) \right\|_{p(x)}$$

$$= \left\| \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} (f - N_k(f)) \right\|_{p(x)} \leq \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \|f - N_k(f)\|_{p(x)}.$$

Using Theorem 1, we obtain

$$\|f - t_n^{DN}(f)\|_{p(x)} = O(1) \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} k^{-\alpha}$$

$$= O(1) \frac{1}{b_n - a_n} \sum_{k=1}^{b_n - a_n} (k + a_n)^{-\alpha} = O((b_n - a_n)^{-\alpha}).$$

Since $amds \subset amdms$ and $amis \subset amims$, we can derive the following results from Theorem 4.

Corollary 1. Suppose that $p \in \wp_{loc}$, $f \in Lip\big(\alpha, p(x)\big)$, $0 < \alpha < 1$, and let (p_n) be a positive sequence. If $(p_n) \in amis$ with (5) or $(p_n) \in amds$, then

$$\|f - t_n^{DN}(f)\|_{p(x)} = O((b_n - a_n)^{-\alpha}).$$

Since D_0^n is the C_1 transformation as mentioned above, we write the following result from Theorem 4.

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Corollary 2. Let $p \in \wp_{loc}$, $f \in Lip(\alpha, p(x))$, $0<\alpha<1$, and let (p_n) be a sequence of positive numbers. If $(p_n) \in amims$ with (5) or $(p_n) \in amdms$, then

$$\|f - t_n^{CN}(f)\|_{p(x)} = O(n^{-\alpha}).$$

The subsequent result is obtained for the more general sequence class than the class of monotone sequences. If (p_n) is a non-decreasing sequence satisfying condition (5), it is obvious that

$$\sum_{m=0}^{n-1} |\Delta p_m| = O\left(\frac{P_n}{n}\right).$$

If (p_n) is a non-increasing sequence, then

$$\sum_{m=1}^{n-1} m|\Delta p_m| = O(P_n).$$

Theorem 5. Let $p \in \wp_{loc}$, $f \in Lip(1, p(x))$, and let (p_n) be a sequence of positive numbers. If

$$\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$$

or

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right) \text{ with (5)},$$

then the evaluation

$$\|f - t_n^{DN}(f)\|_{p(x)} = O\left(\frac{\ln\left(\frac{b_n}{a_n+1}\right)}{b_n - a_n}\right)$$

holds for $n = 1,2, \ldots$.

Proof. According to the definition of the product method, we obtain

$$\begin{aligned}\|f - t_n^{DN}(f)\|_{p(x)} &= \left\| f - \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \frac{1}{P_k} \sum_{v=0}^{k} p_{k-v} s_v(f) \right\|_{p(x)} \\ &= \left\| \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} (f - N_k(f)) \right\|_{p(x)} \le \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \|f - N_k(f)\|_{p(x)}.\end{aligned}$$

Therefore, the expected result is obtained by Theorem 2:

$$\|f - t_n^{DN}(f)\|_{p(x)} = O(1) \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} \frac{1}{k} = O\left(\frac{\ln\left(\frac{b_n}{a_n+1}\right)}{b_n - a_n}\right).$$

Corollary 3. With the conditions of Theorem 5, we have the following result:

$$\|f - t_n^{CN}(f)\|_{p(x)} = O\left(\frac{\ln n}{n}\right).$$

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It is clear that if $(p_n) \in rbvs$ satisfies condition (5), then

$$\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n).$$

Therefore, according to the first condition of Theorem 5 and this last statement, the next result can be written as follows.

Corollary 4. Let $p \in \wp_{loc}$, $f \in Lip(1, p(x))$, $0 < \alpha < 1$. If $(p_n) \in rbvs$ with condition (5), then

$$\|f - t_n^{DN}(f)\|_{p(x)} = O\left(\frac{\ln\left(\frac{b_n}{a_n + 1}\right)}{b_n - a_n}\right)$$

and

$$\|f - t_n^{CN}(f)\|_{p(x)} = O\left(\frac{\ln n}{n}\right).$$

Considering the deferred-Riesz product means for $0 < \alpha \le 1$, we get the following result on approximation of functions from the class $f \in Lip(\alpha, p(x))$ with an additional condition.

Theorem 6. Let $p \in \wp_{loc}$, $f \in Lip(\alpha, p(x))$ for $0 < \alpha \le 1$, and (p_n) be a sequence of positive numbers. If

$$\sum_{m=0}^{n-1} \left|\Delta\left(\frac{P_m}{m + 1}\right)\right| = O\left(\frac{P_n}{n + 1}\right),$$

then the following estimate is obtained:

$$\|f - t_n^{DR}(f)\|_{p(x)} = O((b_n - a_n)^{-\alpha}).$$

We omit the proof here, since it is similar to the proof of Theorem 4. According to Theorem 6, we obtain the following result.

Corollary 5. Under the conditions of Theorem 6, the following estimate holds:

$$\|f - t_n^{CR}(f)\|_{p(x)} = O(n^{-\alpha}).$$

CONCLUSIONS

When examining approximation problems with trigonometric polynomials, the parameters in the polynomials used reveal the characteristics of the method and to what extent they affect the speed of approximation. In this sense trigonometric polynomials created with the product means are worth examining. Such problems, especially related to the product means, become important as they have not been addressed before in variable exponent Lebesgue spaces. The results obtained in this study fill the gap in this field.

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