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Full Paper

An application of generating function for Hermite polynomials

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Abstract: An important application of the Hermite polynomials with the weight function absorbed into them is that they form the solutions of the differential equation for a simple harmonic oscillator. Motivated by this fact, our focus for this work is to investigate a class of singular integral equations whose kernels are formed by the generating function for the Hermite polynomials with the weighting function absorbed into them. We propose an efficient method for solving such a class of singular integral equations. In addition, we establish the results of applications involving the integral of Hermite polynomials and their generating functions.

Keywords: generating functions, Hermite polynomials, integral equations

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INTRODUCTION

Many important mathematical models for phenomena arising from physics and engineering are expressed as integral equations. The history of integral equations goes back to Abel [1], who proposed a generalisation of the tautochrone problem. It was shown that the solution of the tautochrone problem involved the solution of an integral equation [2, 3]. After Abel's work on the subject matter, many great scientists, especially Volterra and Fredholm [4, 5], made an invaluable contribution to the development of the integral equations.

Integral equations have numerous applications across various fields. A vast amount of initial and boundary value problems can be transformed into Volterra or Fredholm integral equations. They can also be used to model population growth, heat transfer, etc [3]. Given their broad range of applications, solving the resulting integral equations, whether analytically or numerically, has consistently been a challenging and evolving problem, and in-depth reviews of the solution methods of the integral equations have been available [6-10].

The history and development of orthogonal polynomials goes back to late 19^{th} century. Chebyshev and Stieltjes [11] are those who first studied and developed the subject. Since then many researchers from different branches of science have contributed to the development of this field. Moreover, it became a vital tool in solving many equations not only in mathematical physics but also in other fields such as computer science, biology, chemistry and engineering. For more detailed study, interested reader is referred to additional references [12, 13].

The Hermite polynomials and their generating function have many applications in different branches of science. One of the most important applications of the Hermite polynomials, $\mathcal{H}_n(x)$, is that they constitute solutions of the simple harmonic oscillator of quantum mechanics. One approach to defining them and developing their properties is to use the generating function:

$$\mathcal{G}(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{\mathcal{H}_n(x)}{n!} t^n$$

For convenient reference, we list the recursion relation and the first several Hermite polynomials:

$$\mathcal{H}_{n+1}(x) = 2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1}(x), \quad n = 1,2,\ldots$$

where $\mathcal{H}_0(x) = 1$ and $\mathcal{H}_1(x) = 2x$.

Hermite polynomials are among the most useful classical orthogonal polynomials along with Legendre and Laguerre polynomials. Classical orthogonal polynomials and their properties have been sufficiently studied. Thus, we will not focus on them further and more information about the properties and alternate approaches on how to obtain them can be found in the literature [14-17].

Following the method of Hermite [18], which he employed to prove the orthogonality of Legendre polynomials, Askey [19] constructed the generating function for the Jacobi polynomials. Thakare and Madhekar [20] showed that this remarkable approach of Hermite can be applied equally well to the remaining classical orthogonal polynomials for obtaining the generating function for them by using a unifying principle.

Many problems in quantum mechanics and optics involve the integrals of Hermite polynomials and Gaussians [21, 22]. To solve such problems, Babusci et al. [23] successfully combined the generating function method and multivariable Hermite polynomials to determine integrals of Gaussian functions and products of Hermite polynomials. Kim et al. [24] showed that the orthogonality property of Hermite polynomials led to derivation of some interesting identities and arithmetic properties of Bernoulli and Euler polynomials. By setting a set formed by the first n Hermite polynomials, they established an effective basis for the set of polynomials with degrees less than or equal to n.

Belafhal et al. [25] derived formulae for many problems involving integrals of Hermite polynomials encountered in physics. These integrals make it possible to evaluate the characteristics of a propagating beam. In addition, they also mentioned some applications to catastrophe optics. Savchenko [26] considered the functions orthogonal with respect to the inner product with the unit weight function in $L_2[a,b]$ and their generating functions. A general method for Fredholm integral equations of the first kind involving generating functions for functions that are orthogonal to the lower degree polynomials is also obtained. Furthermore, based on this, some axially symmetric physical problems are solved.

In this work we study a class of singular integral equation with a special kernel represented by the equation:

$$\int_{-\infty}^{\infty} K(x,t)\phi(x)\, dx = \mathrm{p}(t), \tag{1}$$

where $\mathcal{K}(x,t) = e^{-(x-t)^2}$, $p(t)$ is a given function and $\phi(x)$ is the desired function. The kernel $\mathcal{K}$ is special in the sense that it is formed by the generating function for the Hermite polynomials

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$\mathcal{G}(x,t) = e^{2xt-t^2}$ with the weight function $w(x) = e^{-x^2}$ absorbed into it. The main aim is to solve equation (1) and establish the results for applications involving the integral of Hermite polynomials.

PRELIMINARIES AND MOTIVATION

The motivation behind this work is to establish new identities and give alternate proofs for some results obtained by solving equation (1). To achieve this, we need some basic definitions and facts which will be stated without going into details. More detailed treatment can be found in the work by Savchenko [26].

Definition 1 [26]**.** A set of functions $\{A_0, A_1, \dots, A_n\}$ are orthogonal to lower degree polynomials if $A_n \perp x^k$ holds for all non-negative integers $k < n$ for any $n = 0,1, \dots$

Definition 2 [26]**.** We call $G(x,t)$ a generating function generated by functions A_n if

$$\mathrm{G}(x,t) = \sum_{n=0}^{\infty} A_n(x)t^n .$$

The relation between generating function and generated functions is

$$A_n(x) = \frac{1}{n!}\frac{\partial^n G(x,t)}{\partial t^n}\Big|_{t=0} .$$

Definition 3 [26]**.** A generating function is called a *deriving* function if its generated functions are orthogonal to lower-degree polynomials.

Before delving into the details of this study, we like to point out a few facts about Fredholm integral equation of the first kind. This is simply because the equation that we consider is a Fredholm integral equation of the first kind.

Remark 1. The first-kind Fredholm integral equations appear and arise naturally in many physical models. We know that many mathematical or physical models involve information from observation. That is one of the reasons that they are often considered as ill-posed problems and there are many difficulties in trying to solve them. A small inaccuracy in the observed data can result in a large change in the solution [3].

This makes it clear that solving or constructing methods for solutions of Fredholm integral equations is quite valuable. Based on the above definitions, some interesting results are obtained by Savchenko [26]. Motivated by this, we show that some similar results can be obtained for one of the classical orthogonal polynomials, namely Hermite polynomials along with their generating function. After obtaining some auxiliary results, we provide a formula for the family of Fredholm integral equations of the first kind. We also provide some examples in order to show the simplicity and accuracy of the obtained results.

MAIN RESULTS

We consider the special set of polynomials H_n which are orthogonal to lower-degree polynomials with respect to the weight function $w(x) = e^{-x^2}$ on $(-\infty, \infty)$. That is,

$$\int_{-\infty}^{\infty} w(x)H_n(x)x^k\,dx = 0, \quad 0 \le k \le n-1. \tag{2}$$

Throughout the article, unless otherwise stated, we assume that

$$w(x) = e^{-x^2} and\ \mathcal{G}(x,t) = e^{2xt-t^2}$$

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Theorem 1. The following equality holds for every non-negative integer k:

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)x^k\,dx = P_k(t), \tag{3}$$

where $P_k(t)$ is a k^{th} degree polynomial.

Proof. Since

$$\mathcal{G}(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{\mathcal{H}_n(x)}{n!} t^n,$$

we have

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)x^k\,dx = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x)x^k\,dx. \tag{4}$$

Using the known fact (2), the equation becomes

$$\sum_{n=0}^{k} \frac{t^n}{n!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x)x^k\,dx.$$

It is easy to verify that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)x^k\,dx = \begin{cases} 0 & \text{if } k < n, \\ \sqrt{\pi}k! & \text{if } k = n \end{cases} \tag{5}$$

For the sake of brevity, let

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)x^k\,dx = C_n$$

for $k \geq n$. Thus, equation (4) reads

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)x^k\,dx = \sum_{n=0}^{k} C_n \frac{t^n}{n!},$$

where $C_n \neq 0$ as $n = k$. This completes the proof.

Corollary 1. The following identity holds for every constant A and B.

$$\int_{-\infty}^{\infty} w(x)(A + Bx)\frac{\partial \mathcal{G}}{\partial t} x^k\,dx = P_k(t),$$

where k is any non-negative integer.

Proof. First, consider the case $k = 0$. Since the integrals satisfy the following identities,

$$\int_{-\infty}^{\infty} w(x)\frac{\partial \mathcal{G}}{\partial t}\,dx = 0,$$
$$\int_{-\infty}^{\infty} w(x)x\frac{\partial \mathcal{G}}{\partial t}\,dx = \sqrt{\pi},$$

any linear combination of them must be a constant. On the other hand, taking the derivative of both sides of relation (3) with respect to t yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)x^k\,dx = \int_{-\infty}^{\infty} w(x)\frac{\partial \mathcal{G}}{\partial t} x^k\,dx = P_{k-1}(t)$$

for every positive integer k. The last part of this relation can be rearranged as

$$\int_{-\infty}^{\infty} w\,(x)\left(x\frac{\partial \mathcal{G}}{\partial t}\right)x^{k-1}\,dx = P_{k-1}(t), \text{ for every positive integer k.}$$

Thus, the corollary is proved.

Remark 2. One can produce other deriving functions by following the steps below.

$$(A+Bx)\frac{\partial G}{\partial t} = (A+Bx)\sum_{n=1}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1}$$

$$= (\mathrm{A}+\mathrm{Bx})\sum_{\mathrm{n}=0}^{\infty}\frac{\mathrm{H}_{\mathrm{n}+1}(\mathrm{x})}{\mathrm{n}!}\mathrm{t}^{\mathrm{n}}$$

$$H_n^{(1)}(x) = (A+Bx)H_{n+1}(x)$$

So, the generating function for $(A+Bx)\frac{\partial G}{\partial t}$ can be written as

$$(A+Bx)\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty}\frac{H_n^{(1)}(x)}{n!}t^n\,.$$

Theorem 2. Let $\mathcal{G}(x,t) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$ and $w(x) = e^{-x^2}$ and let $f(t) = \sum_{n=0}^{N} b_n\,t^n$. Then the solution of the integral equation

$$\int_{-\infty}^{\infty} w\,(x)\mathcal{G}(x,t)u(x)\,dx = f(t) \qquad (6)$$

is a polynomial $\mathrm{u(x)} = \sum_{\mathrm{k}=0}^{\mathrm{N}}\mathrm{c_k}\,\mathrm{x^k}$ whose coefficients satisfy $\mathbf{Fc} = \mathbf{b}$, where

$$\mathbf{c} = [c_0, c_1, \dots, c_N]^T, \qquad \boldsymbol{b} = [b_0, b_1, \dots, b_N]^T$$

and **F** is the matrix with components:

$$F_{nk} = \frac{1}{(n-1)!}\int_{-\infty}^{\infty} H_{n-1}\,(x)w(x)x^{k-1}\,dx, \quad n,k = 1,2,\dots N+1. \qquad (7)$$

Proof. Let

$$f(t) = \sum_{k=0}^{N} c_k\,P_k(t),$$

where P_k is a polynomial of degree k. It is obvious that this representation is unique. Using the identity (3),

$$\begin{aligned} f(t) &= \sum_{k=0}^{N} c_k\,P_k(t) \\ &= \sum_{k=0}^{N} c_k \int_{-\infty}^{\infty} w\,(x)\mathcal{G}(x,t)x^k\,dx \\ &= \int_{-\infty}^{\infty} w\,(x)\mathcal{G}(x,t)\left(\sum_{k=0}^{N} c_k\,x^k\,dx\right). \end{aligned}$$

The last part of the equation implies that

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$$u(x) = \sum_{k=0}^{N} c_k\, x^k$$

is a solution of the equation (6). On the other hand,

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} w\,(x)\left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n\right)\left(\sum_{k=0}^{N} c_k\, x^k\right) dx \\ &= \sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{N} c_k \int_{-\infty}^{\infty} w\,(x) \frac{H_n(x)}{n!} x^k\, dx\right) \\ &= \sum_{n=0}^{N} b_n\, t^n \end{aligned}$$

Thus, we have

$$b_n = \sum_{k=0}^{N} c_k \int_{-\infty}^{\infty} w\,(x) \frac{H_n(x)}{n!} x^k\, dx, \quad n = 0,1, \ldots, N.$$

We like to note that it is this identity that the entries of ***Fc*=*b*** is formed. Some important corollaries related to ***F*** are in order:

Corollary 2. The diagonal entries of **F** are equivalent. This result directly follows from equations (5) and (7). The entries of ***F*** are defined by

$$F_{nk} = \frac{1}{(n-1)!} \int_{-\infty}^{\infty} H_{n-1}\,(x) w(x) x^{k-1}\, dx, \qquad n, k = 1,2, \ldots N+1.$$

In particular, the diagonal entries are given by

$$F_{kk} = \frac{1}{(k-1)!} \int_{-\infty}^{\infty} H_{k-1}\,(x) w(x) x^{k-1}\, dx,$$

From equation (5),

$$F_{kk} = \sqrt{\pi} \qquad\qquad k = 1,2, \ldots\ N+1.$$

Corollary 3. The matrix **F** is upper triangular. It is clear from the identities (2) and (7). The terms of ***F*** below the main diagonal have $n > k$. The equation (2.1)

$$\int_{-\infty}^{\infty} w\,(x) H_n(x) x^k\, dx = 0, \quad 0 \leq k \leq n-1.$$

implies that k can be at most $n-1$. This, in turn, implies the entries under the main diagonal of the matrix ***F*** are zero.

Corollary 4. The entries in odd upper diagonals of **F** vanish. In other words, $\mathrm{F_{nk}} = 0$ if $\mathrm{n + k}$ is odd.

From the symmetry property of Hermite polynomials obtained from Rodrigues formulae, Hermite polynomials $\mathcal{H}_n$ are either even or odd functions dependent on n, i.e. $\mathcal{H}_n(-x) = (-1)^n \mathcal{H}_n(x)$. This property along with the requisite that $n+k$ be odd makes the integral in equation (7) an odd integral.

If we require the solution to be given in terms of the Hermite polynomials or a combination of them, then the above theorem can be reformulated as Theorem 3. Basically, we make use of the orthogonality property of Hermite polynomials to simplify the result of Theorem 2.

Theorem 3. Let $\mathcal{G}(x,t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$ and $w(x) = e^{-x^2}$ and let $p(t) = \sum_{n=0}^{N} b_n t^n$. Then the solution of the integral equation

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)u(x)\,dx = p(t)$$

is a polynomial $u(x) = \sum_{k=0}^{N} c_k H_k(x)$ whose coefficients are given by

$$c_k = \frac{1}{2^k\sqrt{\pi}} b_k, \quad k = 0,1,2,\dots N.$$

Proof.

$$\begin{aligned} p(t) &= \int_{-\infty}^{\infty} w(x)\left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n\right)\left(\sum_{k=0}^{N} c_k H_k(x)\right) dx \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} w(x)H_n(x)\left(\sum_{k=0}^{N} c_k H_k(x)\,dx\right) \\ &= \sum_{n=0}^{N} c_n \frac{t^n}{n!} \int_{-\infty}^{\infty} w(x)[H_n(x)]^2\,dx \\ &= \sum_{n=0}^{N} c_n 2^n\sqrt{\pi}t^n \end{aligned}$$

The last two steps follow from the fact that

$$\int_{-\infty}^{\infty} w(x)H_n(x)H_m(x)\,dx = \begin{cases} 0 & \text{if } m \neq n, \\ 2^n\sqrt{\pi}n! & \text{if } m = n. \end{cases}$$

It is easy to verify that

$$c_n = \frac{b_n}{2^n\sqrt{\pi}}, \quad n = 0,1,\dots,N.$$

EXAMPLES

Example 1. Consider the following singular integral equation:

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)u(x)\,dx = 1 + t,$$

where $\mathcal{G}(x,t) = e^{2xt-t^2}$ and $w(x) = e^{-x^2}$. Setting ***Fc*=*b*** to find **b**, where

$$\mathbf{F} = \begin{bmatrix} \int_{-\infty}^{\infty} w(x)H_0(x)dx & \int_{-\infty}^{\infty} w(x)H_0(x)xdx \\ \int_{-\infty}^{\infty} w(x)H_1(x)dx & \int_{-\infty}^{\infty} w(x)H_1(x)xdx \end{bmatrix} = \begin{bmatrix} \sqrt{\pi} & 0 \\ 0 & \sqrt{\pi} \end{bmatrix} \text{ and}$$

$$\boldsymbol{b} = [1, 1]^T,$$

we obtain

$$c = \frac{\sqrt{\pi}}{\pi}[\,1, 1] \quad \Rightarrow \quad u(x) = \frac{\sqrt{\pi}}{\pi}(1 + x).$$

Alternatively, if we use Theorem 3, we obtain

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$$c_0 = \frac{1}{\sqrt{\pi}}, \quad c_1 = \frac{1}{2\sqrt{\pi}}$$

and

$$u(x) = \sum_{k=0}^{1} c_k H_k(x) = \frac{1}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} 2x = \frac{\sqrt{\pi}}{\pi}(1 + x),$$

which is the same as above.

Example 2. Consider the following integral equation:

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)u(x)\,dx = 1 + t + t^2.$$

Setting ***Fc***=***b*** to find **b**, where

$$\mathbf{F} = \begin{bmatrix} \sqrt{\pi} & 0 & \frac{\sqrt{\pi}}{2} \\ 0 & \sqrt{\pi} & 0 \\ 0 & 0 & \sqrt{\pi} \end{bmatrix} \text{ and } \boldsymbol{b} = [1,1,1]^T,$$

we obtain

$$\boldsymbol{c} = \frac{\sqrt{\pi}}{\pi}[1/2,1,1] \Rightarrow u(x) = \frac{\sqrt{\pi}}{\pi}(1/2 + x + x^2).$$

Alternatively, from Theorem 3,

$$c_0 = \frac{1}{\sqrt{\pi}}, \quad c_1 = \frac{1}{2\sqrt{\pi}}, \quad c_2 = \frac{1}{4\sqrt{\pi}}$$

and

$$\begin{aligned} u(x) = \sum_{k=0}^{2} c_k H_k(x) \quad &= \frac{1}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}} 2x + \frac{1}{4\sqrt{\pi}}(4x^2 - 2) \\ &= \frac{\sqrt{\pi}}{\pi}\left(\frac{1}{2} + x + x^2\right). \end{aligned}$$

Example 3. Consider the following integral equation:

$$\int_{-\infty}^{\infty} w(x)\mathcal{G}(x,t)u(x)\,dx = 1 + t + t^2 + t^3.$$

Setting ***Fc***=***b*** to find **b**, where

$$F = \begin{bmatrix} \sqrt{\pi} & 0 & \frac{\sqrt{\pi}}{2} & 0 \\ 0 & \sqrt{\pi} & 0 & \frac{3\sqrt{\pi}}{2} \\ 0 & 0 & \sqrt{\pi} & 0 \\ 0 & 0 & 0 & \sqrt{\pi} \end{bmatrix} \text{ and } b = [1,1,1,1]^T,$$

we obtain

$$\boldsymbol{c} = \frac{\sqrt{\pi}}{2\pi}[1,-1,2,2] \Rightarrow u(x) = \frac{\sqrt{\pi}}{\pi}(1 - x + 2x\text{^}2 + 2x^3)$$

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By using Theorem 3,

$$c_0 = \frac{1}{\sqrt{\pi}}, \quad c_1 = \frac{1}{2\sqrt{\pi}}, \quad c_2 = \frac{1}{4\sqrt{\pi}}, \quad c_3 = \frac{1}{8\sqrt{\pi}}$$

and

$$u(x) = \sum_{k=0}^{3} c_k H_k(x) = \frac{1}{\sqrt{\pi}} + \frac{1}{2\sqrt{\pi}}2x + \frac{1}{4\sqrt{\pi}}(4x^2 - 2) + \frac{1}{8\sqrt{\pi}}(8x^3 - 12x)$$

$$= \frac{\sqrt{\pi}}{2\pi}(1 - x + 2x^2 + 2x^3).$$

Example 4. Consider the following Fredholm integral equation:

$$\int_{-\infty}^{\infty} w(x) G(x,t) u(x)\, dx = 1 + t/2,$$

where $G(x,t) = e^{2xt - t^2}$ and $w(x) = e^{-x^2}$.

Setting ***Fc*=*b*** to find **b**, where

$$\mathbf{F} = \begin{bmatrix} \sqrt{\pi} & 0 \\ 0 & \sqrt{\pi} \end{bmatrix} \text{ and } \mathbf{b} = [1, 1/2]^T,$$

we obtain

$$c = \frac{\sqrt{\pi}}{\pi}[\,1,\, 1/2] \quad \Longrightarrow \quad u(x) = \frac{\sqrt{\pi}}{2\pi}(2 + x).$$

Remark 3. It is easily observed from the examples that the matrix $\boldsymbol{F}$ is independent of the right-hand side $\boldsymbol{b}$ if the dimension of $\boldsymbol{b}$ is not changed. To be more precise, for every **n** dimensional $\boldsymbol{b}$, there is one and only one $\boldsymbol{F} = \boldsymbol{F_n}$ of dimension nxn. If $\boldsymbol{b}$ is a vector of dimension $\mathrm{n} + 1$, the corresponding $\boldsymbol{F_{n+1}}$ is formed by adding one row and column to $\boldsymbol{F}_n$.

INTEGRAL EVALUATIONS

We claim that Theorem 1 can be used to evaluate some difficult integrals. Belefhal et al. [25] evaluated integrals in terms of Hermite polynomials. According to them, some integrals that are formed by a product of Hermite polynomials and Gaussian weight in caustic optics need some techniques to compute or evaluate. Furthermore, three classes of family of integrals were investigated. We will focus only on one of them here.

Theorem 4 [25]**.** Suppose that p>0. Then it is true that

$$\int_{-\infty}^{\infty} x^l e^{-px^2 + 2qx}\, dx = \mathrm{e}^{\mathrm{q}^2/\mathrm{p}} \sqrt{\frac{\pi}{p}} \left(\frac{1}{2\mathrm{i}\sqrt{\mathrm{p}}}\right)^{\mathrm{l}} \mathrm{H}_{\mathrm{l}}\left(\frac{\mathrm{iq}}{\sqrt{\mathrm{p}}}\right), \tag{8}$$

where H_l is the l^{th} Hermite polynomial. If we consider this integral for $\mathrm{p} = 1$ and $q = t$, the integral above boils down to

$$I_l(1,t) = \int_{-\infty}^{\infty} x^l e^{-x^2 + 2xt}\, dx.$$

Reconsidering Theorem 3, we reveal the connection between equation (6) and $I(1,t)$. The matrix $\boldsymbol{F}$ being non-singular and assuming there is a polynomial solution, one can find the desired polynomial solution $u(x)$ for every selection of constants c_k (see Theorem 3). This in turn implies

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that by making proper selections for c_k, the integrals given in equation (8) can be evaluated by noticing the equivalency between equation (6) and the following equation

$$e^{-t^2} I_l(1,t) = f(t) \tag{9}$$

Then the corresponding identity $\boldsymbol{Fc} = \boldsymbol{b}$ becomes

$$\boldsymbol{b} = (l+1)^{th} \text{ column of the matrix } \boldsymbol{F},$$

where $c = [0, \dots, 0, 1, 0, \dots, 0]$ (1 in l^{th} position, 0 elsewhere). To conclude,

$$\boldsymbol{b} = \begin{pmatrix} F_{1,l+1} \\ F_{2,l+1} \\ \vdots \\ F_{N+1,l+1} \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^{\infty} \mathrm{w(x)H_0(x)} x^l \, \mathrm{dx} \\ \int_{-\infty}^{\infty} \mathrm{w(x)H_1(x)} x^l \, \mathrm{dx} \\ \vdots \\ \frac{1}{\mathrm{N!}} \int_{-\infty}^{\infty} \mathrm{w(x)H}_N\mathrm{(x)} x^l \, \mathrm{dx} \end{pmatrix}.$$

Let us evaluate the following integral and verify the result. Assuming that $l = 2$, then

$$I_2(1,t) = \int_{-\infty}^{\infty} x^2 e^{-x^2+2xt} \, d\mathrm{x}.$$

Consider Example 2 from the previous section. Following the steps explained above,

$$\boldsymbol{F} = \begin{bmatrix} \sqrt{\pi} & 0 & \frac{\sqrt{\pi}}{2} \\ 0 & \sqrt{\pi} & 0 \\ 0 & 0 & \sqrt{\pi} \end{bmatrix}$$

and $\boldsymbol{b}$ is the 3rd column of the matrix $\boldsymbol{F}$. Thus,

$$\boldsymbol{b} = \begin{pmatrix} \frac{\sqrt{\pi}}{2} \\ 0 \\ \sqrt{\pi} \end{pmatrix}.$$

The entries are used to form the right-hand side function $f(t)$. In other words,

$$\mathrm{f(t)} = \frac{\sqrt{\pi}}{2} + \sqrt{\pi}\mathrm{t}^2.$$

From equation (9), it follows that

$$I_2(1,t) = e^{t^2} f(t) = \frac{\sqrt{\pi}}{2} e^{t^2} (1 + 2\mathrm{t}^2).$$

This coincides with the result obtained by Belefhal et al. [25]. The verification is easy by substituting the values for p, q and l into equation (8).

CONCLUSIONS

It is known that for the classical orthogonal polynomials $\mathcal{P}_n(x)$ there exists a generating function for each, defined as

$$G(x,t) = \sum_{n=0}^{\infty} a_n \mathcal{P}_n(x) t^n.$$

where a_n's are some real scalars. For Hermite polynomials, this amounts to

$$\mathcal{G}(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{H}_n(x) t^n.$$

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The importance of Hermite polynomials follows simply from their occurrence in solutions of the simple harmonic oscillator of quantum mechanics. An important application with the weight function absorbed into them is that they form the differential equation for a quantum mechanical simple harmonic oscillator. In this work we study a singular integral equation where the kernel is special in the sense that it is formed by the generating function for the Hermite polynomials $\mathcal{G}(x,t) = e^{2xt-t^2}$ with the weight function $w(x) = e^{-x^2}$ absorbed into it. We solve singular integral equations when the data function is a polynomial. An algorithm is also given where the desired function is expressed directly in terms of Hermite polynomials. Numerical examples are provided to test the given algorithm. In addition, we also show that the results can be used to compute some difficult integrals in catastrophe optics.

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