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Full Paper

Weighted statistical rough convergence in normed spaces

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Abstract: Statistical convergence is a significant generalisation of the traditional convergence of real or complex valued sequences. Over the years, it has been studied by many authors and found many applications in various problems. In this paper we introduce a new concept about statistical rough convergence for sequences in normed spaces by using weighted density, which is a generalisation of the natural density. We investigate the fundamental properties of g-statistical rough convergence and statistical rough limit points including closeness, convexity and boundedness. We also establish a relationship between statistical rough limit points and g-statistical boundedness. The obtained results provide a new framework for studying statistical rough convergence.

Keywords: statistical rough convergence, g-weight density, limit points, normed spaces

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INTRODUCTION

Statistical convergence is a crucial extension of the traditional convergence for real sequences. First, it was proposed independently by Fast [1] and Steinhaus [2], and then it has been extensively studied by various authors. The theory finds practical applications [3] and has been explored in different contexts such as statistical rough convergence [4, 5], fuzzy number spaces [6], double sequences on L-fuzzy [7] and uncertain sequences [8-11]. In general, sequences exhibiting statistical convergence share many properties with convergent sequences in metric spaces.

In Phu's work [12] a new idea known as rough convergence was presented for normed spaces with finite dimension. The set $\mathrm{LIM}^{r}(\tilde{x})$, denoting the rough limit points of a sequence $\tilde{x} = (x_n)$, was examined for specific properties in normed spaces. Moreover, the idea of a rough Cauchy sequence was presented, and relationships between rough convergence and various other convergence types were investigated. The relationship between $\mathrm{LIM}^{r}(\tilde{x})$ and the degree of roughness denoted by r was also studied. Phu extended the idea of rough continuity to linear operators in subsequent work [13] and further extended the findings on rough convergence in another publication [14] for infinite-dimensional normed spaces.

The primary objective of the present research is to integrate statistical convergence and rough convergence using the g-weight density instead of natural density.

Definition 1 [12]. Let $(X, \|\cdot\|)$ be a normed space. A sequence $\tilde{x} = (x_n)$ in X is defined to be rough convergent (or r-convergent) to $x^* \in X$, denoted by $x_n \xrightarrow{rc} x^*$ for a fixed non-negative real number r if, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\| x_n - x^* \| < r + \varepsilon$ holds for all $n \in \mathbb{N}$ with $n_\varepsilon \leq n \in \mathbb{N}$.

Equivalently, it can be expressed as $\limsup \| x_n - x^* \| \leq r$. Here, the real number r represents the degree of roughness, and rough convergence coincides with norm convergence when $r = 0$. It is important to note that the r-limit point of a sequence $\tilde{x}$ is typically not unique. Therefore, the r-limit set of $\tilde{x}$ is denoted by $\text{LIM}^r(\tilde{x}) := \{x^* \in X : x_n \xrightarrow{rc} x^*\}$. The sequence $\tilde{x}$ is considered rough convergent whenever $\text{LIM}^r(\tilde{x}) \neq \varnothing$. It is evident that a norm convergent sequence is also rough convergent to the norm limit point. However, the converse may not always hold. Rough convergence can be viewed as a less stringent form of convergence compared to norm convergence. On the other hand, the asymptotic density of a set $K \subseteq \mathbb{N}^+$ is denoted by the following limit:

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|.$$

Here, the absolute value bars represent the cardinality of the set $\{k \leq n : k \in K\}$.

Definition 2. Consider a sequence $\tilde{x} = (x_n)$ within a normed space $(X, \|\cdot\|)$. It is termed statistically convergent to $x^* \in X$ if $\delta(\{n : \|x_n - x^*\| \geq \varepsilon\}) = 0$ holds for every $\varepsilon > 0$. We refer to $x^* \in X$ as the statistical limit of the sequence $\tilde{x}$, denoted as $\text{st-}\lim_{n\to\infty} x_n = x^*$.

The concept of rough convergence finds diverse applications in various convergence types. For instance, Aytar [4, 5] specifically adapts it to statistical convergence. Antal et al. [15] study statistical rough Λ-convergence of order α. This broadening and application of rough convergence underscore its relevance and utility in different domains, extending beyond its initial introduction by Phu. Examples include rough ideal convergence [16, 17] and ideal statistically rough convergence introduced by Savaş et al. [18]. Recent notable results on ideal convergence incorporating the concept of roughness are presented by Leonetti [19].

Definition 3 [4]. A sequence $\tilde{x} = (x_n)$ in a normed space $(X, \|\cdot\|)$ is termed statistical rough convergent to $x^* \in X$ for a fixed non-negative scalar r if, for any $\varepsilon > 0$, the density $\delta(\{n : \|x_n - x^*\| \geq r + \varepsilon\}) = 0$.

An element $x^* \in X$ is known as the statistical rough limit (abbreviated as $\text{r-st-}\lim_{n\to\infty} x_n = x^*$ or $x_n \xrightarrow{r-st} x^*$) of the sequence $\tilde{x}$. In the case of $r = 0$, statistical rough convergence coincides with statistical convergence in norm. The x^* is denoted as the r-st-limit point of the sequence $\tilde{x}$, and it is generally not unique. The r-st-limit set of (x_n), represented as $\text{st-LIM}^r(\tilde{x}) := \left\{x^* \in X : x_n \xrightarrow{r-st} x^*\right\}$, points to scenarios where $\text{st - LIM}^r(x_n) \neq \varnothing$, indicating the presence of statistical rough convergence. A modified notion of natural density given by Balcerzak et al. [20] introduces the concept of weight function. Let $g:\mathbb{N}\to[0,\infty)$ be a function satisfying $\lim_{n\to\infty} g(n)=\infty$ and $\lim_{n\to\infty}\frac{n}{g(n)}\neq 0$. The set of all weight functions with these properties is denoted by $\mathcal{G}$.

Definition 4 [20]**.** The density of a set $K\subseteq\mathbb{N}^{+}$ with respect to a weight function $g\in\mathcal{G}$ is defined by the limit

$$\delta_g(K):=\lim_{n\to\infty}\frac{1}{g(n)}|\{k\leq n: k\in K\}|,$$

whenever it exists. This type of density is abbreviated as g-weight density.

The g-weight density coincides with the natural density when $g(n)=n$ and, for example, with $g(n)=n^{\alpha}$ for $\alpha\in(0,1]$ it is reduced to the α-density by Çolak [21]. Thus, the g-weight density serves as a generalised form of natural density while maintaining similar properties. For sets $A,B\subset\mathbb{N}$, properties include $\delta_g(A)=0$ for finite sets A, $\delta_g(\mathbb{N}\setminus A)=\delta_g(\mathbb{N})-\delta_g(A)$, if $\delta_g(A)$ exists for $g\in\mathcal{G}$ and $A\subseteq B$ implies $\delta_g(A)\leq\delta_g(B)$. Following the work of Balcerzak et al. [21], sequences are reconsidered using the g-weight density. Adem and Altınok [22] introduce weight g-statistical convergence for real sequences, and Das and Savaş [23] provide results on the statistical and ideal convergence of metric-valued sequences using the g-weight density.

Definition 5 [22]**.** A real sequence $\tilde{x}:=(x_n)$ is weight g-statistical convergent (abbreviated $x_n\xrightarrow{st_g}L$) to the real number L if

$$\delta_g(\{n: |x_n-L|\geq\varepsilon\})=0$$

holds for all $\varepsilon>0$, or equivalently

$$\delta_g(\{n: |x_n-L|<\varepsilon\})=\delta_g(\mathbb{N})$$

holds for all $\varepsilon>0$.

The following definition is a generalisation of Definition 6 which was given by Antal et al. [15], representing the basic concept of this paper.

Definition 6. In a normed space $(X,\|\cdot\|)$, $\tilde{x}=(x_n)$ is g-statistical rough convergent to $x^*\in X$ with roughness degree $r\geq 0$ (or shortly $x_k\xrightarrow{r-st_g}x^*$) if

$$\lim_{n\to\infty}\frac{1}{g(n)}|\{k\leq n: \| x_k-x^* \|\geq r+\varepsilon\}|=0 \qquad (1)$$

holds for every $\varepsilon>0$. The set of all points satisfying (1) is denoted by $LIM_g^r(\tilde{x}):=\{x^*\in X: x_k\xrightarrow{r-st_g}x^*\}$ for the sequence $\tilde{x}$ with respect to $g\in\mathcal{G}$.

It should be noted that in the case of $r=0$, the notation of g-statistical rough convergence is identical to that of g-statistical convergence.

Example 1. Consider the Banach space $X:=L_1(\mathbb{R})$ of Lebesgue integrable functions $f:\mathbb{R}\to\mathbb{R}$, equipped with the L_1-norm defined by

$$\| f \|_{L_1}:=\int_{-\infty}^{\infty}|f(x)|dx<\infty.$$

Let $B(0,1):=\{f\in L_1(\mathbb{R}): \| f \|_{L_1}\leq 1\}$ be the unit ball in X. Take the sequence (f_n) in $B(0,1)$ defined as $f_n(x):=\frac{1}{n}\chi_{[0,n]}(x)$ for all $x\in\mathbb{R}$ and $g(n)=n^2$ for all $n\in\mathbb{N}$. It is observed that $f_n\xrightarrow{r-st_g}0\in L_1(\mathbb{R})$ for all $r\geq 1$.

The concept of ‘rough weighted statistical convergence’ is defined using weighted density, which is defined by the weighted sequence $t = (t_n)_{n\in\mathbb{N}}$ [24, 25]. The use of the multiplier “t_k” in this definition completely distinguishes the sequences considered here from those considered in our definition. Therefore, the definitions are not special cases of each other; in other words, they are not comparable definitions. They only coincide if the weight sequence is specifically $(t_k) = (1,1,1,\dots)$ [24, 25] and specifically when $g(n) = n$ in our definition. Apart from this particular case, there are no overlapping situations.

Example 2. Consider X as a vector space comprising continuous real-valued functions defined on $[0,1]$, characterised by being piecewise polynomial with a finite number of pieces. Define the norm on X as follows:

$$\| f \| := \sup\{|f(t)| + |\tfrac{df}{dt}(t)| : t \in [0,1] \text{ and } \tfrac{df}{dt}(t) \text{ exists}\}$$

for $f \in X$. Let $f_n \in X$ be a linear function on every interval $[\frac{k-1}{2n}, \frac{k}{2n}]$ for $k = 1, \cdots, 2n$ such that $f_n(\frac{k-1}{2n}) = 0$ for odd k, and $f_n(\frac{k-1}{2n}) = \frac{1}{n}$ for even k. Thus, we have $\| f_n \| = \frac{1}{n} + 2$ for all $n \in \mathbb{N}$. Therefore, it is evident that $f_n \xrightarrow{r-st_g} 0$ for $r = 2$ and $g(n) = n$, but it is not norm convergent to zero.

MAIN RESULTS

The majority of the papers and their various generalisations discussing rough convergence and statistical rough convergence share similarities in terms of the results provided [5, 12-14]. However, some differing results were identified by Das et al. [24]. Therefore, the results obtained in this section will be presented in two parts: results on the set of g-statistical rough limit points and results on g-statistical rough convergence.

Results on Set of g-Statistical Rough Limit Points

In this section we primarily explore the fundamental properties of the set $\mathrm{LIM}_g^{r_1}(\tilde{x})$ with respect to $g \in \mathcal{G}$ for a given sequence $\tilde{x}$, such as closeness, convexity and boundedness. Let us begin this section with the following simple properties.

Theorem 1. Let $\tilde{x} = (x_k)$ be a sequence in a normed space X and $g \in \mathcal{G}$ be any weight function.

(i) For non-negative real numbers $r_1 \le r_2$, we have the inclusion $\mathrm{LIM}_g^{r_1}(\tilde{x}) \subseteq \mathrm{LIM}_g^{r_2}(\tilde{x})$.

(ii) If $\tilde{x}$ is norm bounded, then there exists a positive real number r such that $\mathrm{LIM}_g^{r}(\tilde{x}) \neq \varnothing$.

(iii) $x^* \in \mathrm{LIM}_g^{r}(\tilde{x})$ if and only if $0 \in \mathrm{LIM}_g^{r}(\|x_n - x^*\|)$.

(iv) If $x^* \in \mathrm{LIM}^{r}(\tilde{x})$, then $x^* \in \mathrm{LIM}_g^{r}(\tilde{x})$.

Proof: (i) Let $x^* \in \mathrm{LIM}_g^{r_1}(\tilde{x})$ be an arbitrary element and assume that $r_1 \le r_2$ holds. Then from the following inclusion

$$\{k \le n: \|x_k - x^*\| \ge r_2 + \varepsilon\} \subseteq \{k \le n: \|x_k - x^*\| \ge r_1 + \varepsilon\},$$

the proof is obvious.

(ii) The boundedness of the sequence $\tilde{x}$ implies $s := \sup\{\|x_n\| : n \in \mathbb{N}\} < \infty$ exists. Hence for any $\varepsilon > 0$ and all $r \ge s$, we get

$$\delta_g(\{n: \|x_n - 0\| \ge r + \varepsilon\}) = 0.$$

This implies $0 \in \mathrm{LIM}_g^{r}(\tilde{x})$, showing $\mathrm{LIM}_g^{r}(\tilde{x}) \neq \varnothing$ for all $r \ge s$.

Maejo Int. J. Sci. Technol. **2024**, *18*(02), 178-192

(iii) The definition itself implies the conclusion without requiring explicit proof.

(iv) Assuming $x^* \in \mathrm{LIM}^r(\tilde{x})$, for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\| x_n - x^* \| < r + \varepsilon$ holds for all $n \geq n_0$. Consequently, we have the inclusion

$$\{n: \|x_n - x^*\| \geq r + \varepsilon\} \subseteq \{1,2,\cdots,n_0\}$$

for any $\varepsilon > 0$. The monotonicity property of g-density implies the following inequality

$$\delta_g(\{n: \|x_n - x^*\| \geq r + \varepsilon\}) \leq \delta_g(\{1,2,\cdots,n_0\}) = 0.$$

So we get $x^* \in \mathrm{LIM}_g^r(\tilde{x})$.

Remark 1. Undoubtedly, unbounded sequences in normed spaces lack rough limit points. Nevertheless, such sequences may reveal g-statistical rough limit points and statistical rough limit points as illustrated by Aytar [5]. Conversely, for a bounded sequence $\tilde{x} = (x_n)$, it is established that $\mathrm{LIM}^r(\tilde{x}) \neq \varnothing$, consequently implying st-$\mathrm{LIM}^r(\tilde{x}) \neq \varnothing$. This observation remains valid in the context of g-statistical rough convergence as it is shown in Theorem 1(ii). In addition, the converse statement of Theorem 1(iv) may not hold.

Example 3. Consider the normed space $(\mathbb{R}, |.|)$ with the weighted function $g(n) = 2n$ and the sequence $\tilde{x} = (x_n)$ defined by

$$x_n := \begin{cases} m, & n = m^2, m \in \mathbb{N}, \\ 2, & n = m^2 + 1, \\ 3, & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$, the inclusion

$$\{n: |x_n - 2| \geq 1 + \varepsilon\} \subseteq \{m^2: m \in \mathbb{N}\}$$

implies that

$$\delta_{2n}(\{n: |x_n - 2| \geq 1 + \varepsilon\}) \leq \delta_{2n}(\{m^2: m \in \mathbb{N}\})$$

for any $\varepsilon > 0$. Thus, $2 \in \mathrm{LIM}_{2n}^1(\tilde{x})$. However, 2 is not a 1-rough limit point of the sequence $\tilde{x}$ due to the unboundedness of $\tilde{x}$.

Theorem 2. Let $\tilde{x} = (x_k)$ be a sequence in a normed space X. Then we have

$$\operatorname{diam}\left(\mathrm{LIM}_g^r(\tilde{x})\right) \leq 2r$$

for every density $g \in \mathcal{G}$.

Proof: Contrarily, we assume that $\operatorname{diam}\left(\mathrm{LIM}_g^r(\tilde{x})\right) > 2r$. Then in this case there exist some elements $y, z \in \mathrm{LIM}_g^r(x)$ such that $\|y - z\| > 2r$. Moreover, since $y, z \in \mathrm{LIM}_g^r(\tilde{x})$, by taking $\varepsilon = \frac{\|y-z\|}{2} - r > 0$, we obtain $\delta_g(R_1) = 0$ and $\delta_g(R_2) = 0$, where $R_1 = \{k \leq n: \|x_k - y\| \geq r + \varepsilon\}$ and $R_2 = \{k \leq n: \|x_k - z\| \geq r + \varepsilon\}$. Therefore, utilising the equation $\delta_g(R_1^c \cap R_2^c) = 1$, we infer that, for every $k \in R_1^c \cap R_2^c$, we have

$$\|y - z\| \leq \|x_k - y\| + \|x_k - z\| < 2(r + \varepsilon) = \|y - z\|,$$

which is a contradiction.

Now, consider a sequence $\tilde{x} = (x_k)$ such that $x_k \xrightarrow{st_g} x^*$. Then for an arbitrary $\varepsilon > 0$, we have $\delta_g(\{k \leq n: \|x_k - x^*\| \geq \varepsilon\}) = 0$. It follows that

$$\|x_k - y\| \leq \|x_k - x^*\| + \|x^* - y\| \leq \|x_k - x^*\| + r$$

holds for all $y \in \overline{B}(x^*, r) = \{z \in X: \|x^* - z\| \le r\}$. Which shows that $\|x_k - y\| \le r + \varepsilon$ for all $k \in \{k \le n: \|x_k - x^*\| < \varepsilon\}$. Now it follows from $\delta(\{k \le n: \|x_k - x^*\| < \varepsilon\}) = 1$ that we obtain $y \in \mathrm{LIM}_g^r(\tilde{x})$, and so we have

$$\overline{B}(x^*, r) = \mathrm{LIM}_g^r(\tilde{x}).$$

On the other hand, since $\mathrm{diam}\overline{B}(x^*, r) = 2r$, the upper bound $2r$ of the diameter of the set $\mathrm{LIM}_g^r(\tilde{x})$ cannot be decreased anymore.

Remark 2. For $r = 0$, according to Theorem 2, we have $\mathrm{diam}\left(\mathrm{LIM}_g^r(\tilde{x})\right) = 0$ for any sequence $\tilde{x} = (x_n)$ in normed spaces. This implies that $\mathrm{LIM}_g^r(\tilde{x})$ is either empty or a singleton.

Theorem 3. Let $\tilde{x} = (x_k)$ be a sequence in a normed space X. Then $\mathrm{LIM}_g^r(\tilde{x})$ is closed and a convex set for any $g \in \mathcal{G}$.

Proof: If $\mathrm{LIM}_g^r(\tilde{x})$ is empty, the proof is straightforward. Assume $\mathrm{LIM}_g^r(\tilde{x}) \ne \varnothing$. Consider a sequence (y_n) in $\mathrm{LIM}_g^r(\tilde{x})$ converging, within the norm, to $y \in X$. For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\|y_n - y\| < \frac{\varepsilon}{2}$ for all $n > n_\varepsilon$. Let us choose sufficiently large $n_0 \in \mathbb{N}$ such that $n_0 > n_\varepsilon$. Since $y_{n_0} \in \mathrm{LIM}_g^r(\tilde{x})$, the following equality

$$\delta_g\left(\left\{k \le n: \left\|x_k - y_{n_0}\right\| \ge r + \varepsilon/2\right\}\right) = 0 \tag{2}$$

holds. If we take k from $\{k \le n: \left\|x_k - y_{n_0}\right\| < r + \frac{\varepsilon}{2}\}$, then $\|x_k - y\| \le \left\|x_k - y_{n_0}\right\| + \left\|y_{n_0} - y\right\| < r + \varepsilon$. Hence the following inclusion holds:

$$\left\{k \le n: \left\|x_k - y_{n_0}\right\| < r + \varepsilon\right\} \subseteq \{k \le n: \|x_k - y\| < r + \varepsilon/2\},$$

and so we have

$$\{k \le n: \|x_k - y\| \ge r + \varepsilon/2\} \subseteq \left\{k \le n: \left\|x_k - y_{n_0}\right\| \ge r + \varepsilon\right\}.$$

Combining equation (2) with the monotonicity of g-density, we conclude that

$$\delta_g(\{k \le n: \|x_k - y\| \ge r + \varepsilon/2\}) = 0$$

for any $\varepsilon > 0$, implying $y \in \mathrm{LIM}_g^r(\tilde{x})$. Hence $\mathrm{LIM}_g^r(\tilde{x})$ is closed.

Now we fix $y_1, y_2 \in \mathrm{LIM}_g^r(\tilde{x})$ and take $\varepsilon > 0$. Then $\delta_g(R_1) = 0$ and $\delta_g(R_2) = 0$ for the sets $R_1 = \{k \le n: \|x_k - y_1\| \ge r + \varepsilon\}$ and $R_2 = \{k \le n: \|x_k - y_2\| \ge r + \varepsilon\}$. For every $k \in R_1^c \cap R_2^c$ and $\lambda \in [0,1]$, the inequality

$$\left\|x_k - \left((1 - \lambda)y_1 + \lambda y_2\right)\right\| = \|(1 - \lambda)(x_k - y_1) + \lambda(x_k - y_2)\| < r + \varepsilon$$

holds. Thus, since $\delta_g(R_1^c \cap R_2^c) = 1$, we can observe that:

$$\delta_g\left(\left\{k \le n: \left\|x_k - \left((1 - \lambda)y_1 + \lambda y_2\right)\right\| \ge r + \varepsilon\right\}\right) = 0,$$

implying $(1 - \lambda)y_1 + \lambda y_2 \in \mathrm{LIM}_g^r(\tilde{x})$. Therefore, $\mathrm{LIM}_g^r(\tilde{x})$ is convex.

Remark 3. Considering Theorem 2 and Theorem 3, and assuming X has finite dimension, we can state: For a sequence $\tilde{x} = (x_k)$, the set $\mathrm{LIM}_g^r(\tilde{x})$ is compact, hence totally bounded and separable.

Definition 7. A sequence $\tilde{x} = (x_n)$ in a normed space X is termed g-statistically bounded if there exists a positive real number $M > 0$ such that

$$\delta_g(\{n \in \mathbb{N}: \|x_n\| \ge M\}) = 0.$$

Evidently a norm bounded sequence is also g-statistical bounded. However, the converse is not true. To see this, let us choose a real number α and use it to define the sequence (x_n) as follows:

$$x_n = \begin{cases} n, & n \in A \\ \alpha, & n \notin A \end{cases}$$

Consequently, for the set $A = \bigcup_{k\geq 1}\{((2k-1)!\,,2(2k-1)!] \cap \mathbb{N}\}$, the following inclusion holds:

$$\{k: |x_k| \geq M\} \subseteq A$$

where $M = |\alpha| + 1$. Furthermore, if we consider the weight function $g \in \mathcal{G}$ (as in Balcerzak [20]) defined by

$$g(n) := \begin{cases} (2k)!, & (2k-1)! < n \leq (2k)! \quad and \quad k = 1,2,\dots \\ n, & (2k)! < n \leq (2k+1)! \quad and \quad k = 1,2,\dots \end{cases},$$

then we have

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: |x_k| \geq M\}| \leq \lim_{n\to\infty} \frac{|A(n)|}{g(n)} = 0,$$

where $A(n) = \{k \in A: k \leq n\}$. This implies that (x_n) is g-statistical bounded but not bounded in the usual sense.

The existence of g-statistical rough limit points does not necessarily imply sequence boundedness, as discussed in Example 3. However, the converse of the previous result holds true for g-statistical boundedness, as stated in the following theorem.

Theorem 4. A sequence $\tilde{x} = (x_n)$ in a normed space X is g-statistically bounded if and only if there exists a non-negative scalar r such that $\mathrm{LIM}_g^r(\tilde{x}) \neq \varnothing$.

Proof: Assume $\tilde{x} := (x_n)$ is a g-statistically bounded sequence. Then there exists a positive number $M > 0$ such that $\delta_g(\{n \in \mathbb{N}: \|x_n\| \geq M\}) = 0$. Let $s := \sup\{\|x_n\|: \|x_n\| < M\}$. For each $\varepsilon > 0$ and every $r \geq s$, we observe

$$\delta_g(\{n: \|x_n - 0\| \geq r + \varepsilon\}) = 0.$$

Hence we have $0 \in \mathrm{LIM}_g^r(\tilde{x})$. Thus, $\mathrm{LIM}_g^r(\tilde{x}) \neq \varnothing$ for all $r \geq s$.

Conversely, if we assume $\mathrm{LIM}_g^r(\tilde{x}) \neq \varnothing$ for some $r \in \mathbb{R}^+$, then for any $x^* \in \mathrm{LIM}_g^r(\tilde{x})$ and fixed $\varepsilon_0 > 0$, we have

$$\delta_g(\{n: \|x_n - x^*\| \geq r + \varepsilon\}) = 0.$$

Clearly, $\delta_g(\{n \in \mathbb{N}: \|x_n\| \geq r + \varepsilon + \| x^* \|\}) = 0$. Thus, (x_n) is g-statistically bounded for $M := r + \varepsilon + \| x^* \|$.

Theorem 5. Consider a sequence $\tilde{x} = (x_n)$ in a normed space X. The element x^* belongs to $\mathrm{LIM}_g^r(\tilde{x})$ if and only if there exists a sequence $\tilde{y} = (y_n)$ (different from $\tilde{x}$) in X such that (y_n) converges generally to x^* and $\| x_n - y_n \| \leq r$ holds for all $n \in \mathbb{N}$.

Proof: Suppose $x_n \xrightarrow{r-st_g} x^*$. Then for every $\varepsilon > 0$, it follows that

$$\delta_g(\{n: \| x_n - x^* \| \geq r + \varepsilon\}) = 0.$$

Let us define the sequence (y_n) as follows:

$$y_n := \begin{cases} x^*, & \| x_n - x^* \| \leq r \\ x_n + r\frac{x^* - x_n}{\|x_n - x^*\|}, & \text{otherwise.} \end{cases}$$

Hence we have

Maejo Int. J. Sci. Technol. **2024**, *18*(02), 178-192

$$y_n - x^* = \begin{cases} 0, & \| x_n - x^* \| \leq r \\ \frac{x_n - x^*}{\| x_n - x^* \|}(\| x_n - x^* \| - r), & \text{otherwise} \end{cases},$$

and so

$$\| y_n - x^* \| = \begin{cases} 0, & \| x_n - x^* \| \leq r \\ \| x_n - x^* \| - r, & \text{otherwise.} \end{cases}$$

Consider the following inclusion:

$$\{k \leq n: \| y_k - x^* \| \geq \varepsilon\} \subseteq \{k \leq n: \| x_k - x^* \| \geq r + \varepsilon\}.$$

Then it follows from the assumption $\delta_g(\{n: \| x_n - x^* \| \geq r + \varepsilon\}) = 0$ that we obtain the following density: $\delta_g(\{k \leq n: \| y_k - x^* \| \geq \varepsilon\}) = 0$. This means that $y_n \xrightarrow{r-st_g} x^*$. Moreover, by considering the definition of (y_n), we observe that

$$\| x_n - y_n \| = \begin{cases} \| x_n - x^* \|, & \| x_n - x^* \| \leq r \\ r, & \text{otherwise} \end{cases}$$

is true; that is $\| x_n - y_n \| \leq r$ for all $n \in \mathbb{N}$. This completes the proof of the sufficiency part.

Conversely, assume that $y_n \xrightarrow{r-st_g} x^*$ and $\| x_n - y_n \| \leq r$ for all $n \in \mathbb{N}$. The following simple triangle inequality

$$\| x_n - x^* \| \leq \| x_n - y_n \| + \| y_n - x^* \|$$

states that the following inclusion holds:

$$\{k \leq n: \| y_k - x_k \| \leq r\} \cap \{k \leq n: \| y_k - x^* \| < \varepsilon + r\} \subseteq \{k \leq n: \| x_k - x^* \| \leq r + \varepsilon\}.$$

Thus, it follows that $\delta_g(\{n: \| x_n - x^* \| \geq r + \varepsilon\} = 0$, i.e. $x^* \in \mathrm{LIM}_g^r(\tilde{x})$.

Remark 4. Let A be a bounded subset of a normed space X. Then there exists a positive scalar M such that $\| x \| \leq M$ for all $x \in A$. Let us take a scalar $r \geq 2M$. Thus, for an arbitrary sequence $\tilde{x} = (x_n)$ and element x in A, we have the following inequality:

$$\| x_n - x \| \leq \| x_n \| + \| x \| \leq M + M \leq r < r + \varepsilon$$

for every $\varepsilon > 0$ and for all $n \in \mathbb{N}$. Thus, we can get $x_n \xrightarrow{r-st_g} x$ for some weight functions $g \in \mathcal{G}$. Therefore, every sequence is g-statistical rough convergent to any element on a bounded subset.

Now, let us take a scalar $0 \leq r < 2M$, a weight function $g \in \mathcal{G}$, and a sequence $\tilde{x}$ in A. Thus, for any $\varepsilon > 0$ and $x \in X$, let us choose n_ε as the smallest positive integer such that

$$\frac{1}{g(n_\varepsilon)} |\{k \leq n_\varepsilon: \| x_k - x \| \geq r + \varepsilon\}| < \varepsilon.$$

We define a new sequence (y_k) as follows:

$$y_k := \begin{cases} x_k, & k \leq n_\varepsilon \\ x, & k > n_\varepsilon \end{cases}.$$

Then we have

$$|\{k \leq n: \| y_k - x \| \geq r + \varepsilon\}| = |\{k \leq n_\varepsilon: \| x_k - x \| \geq r + \varepsilon\}|.$$

Therefore, we get

$$\frac{1}{g(n)} |\{k \leq n: \| y_k - x \| \geq r + \varepsilon\}| = \frac{1}{g(n_\varepsilon)} |\{k \leq n_\varepsilon: \| x_k - x \| \geq r + \varepsilon\}| < \varepsilon.$$

This shows that $y_k \xrightarrow{r-st_g} x$.

Maejo Int. J. Sci. Technol. **2024**, *18*(02), 178-192

We complete this section with the following basic result.

Proposition 1. Let $\tilde{x} = (x_n)$ be a sequence in the normed space X. Then $x_n \xrightarrow{st_g} x^*$ implies that $\overline{B_r(x^*)} = LIM_g^r(\tilde{x})$.

Proof: Assume that $x_n \xrightarrow{st_g} x^*$ in X and take an arbitrary $z \in \overline{B_r(x^*)}$. Then $\| z - x^* \| \leq r$ holds. It follows that

$$\| x_n - z \| \leq \| x_n - x^* \| + \| x^* - z \| \leq \| x_n - x^* \| + r.$$

Therefore, for any $\varepsilon > 0$, $\| x_n - x^* \| < \varepsilon$ implies $\| x_n - z \| < r + \varepsilon$. Hence we get the following inequality:

$$\frac{1}{g(n)} |\{k \leq n : \| x_k - z \| \geq \varepsilon + r\}| \leq \frac{1}{g(n)} |\{k \leq n : \| x_k - x^* \| \geq \varepsilon\}|.$$

By applying $x_n \xrightarrow{st_g} x^*$, the limit of the right-hand side of the above inequality is zero. Hence we get

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : \| x_k - z \| \geq \varepsilon + r\}| = 0.$$

It follows that $z \in LIM_g^r(\tilde{x})$, that is $\overline{B_r(x^*)} \subseteq LIM_g^r(\tilde{x})$. For the converse inclusion, we can consider the second part of the proof of Theorem 2.

Findings on g-Statistical Rough Convergence

It is a known fact that the convergence of a subsequence in norm does not necessarily imply norm convergence in the context of statistical convergence. In this section we explore this phenomenon in the framework of g-statistical rough convergence, as stated in the following result.

Theorem 6. For an arbitrary element x^* in $LIM_g^r(x_i)$, there exists a subset $K = \{k_1 < k_2 < \cdots\}$ of $\mathbb{N}$ with $\delta_g(K) = \delta_g(\mathbb{N})$ such that $x^* \in LIM_g^r(x_{k_n})$.

Proof: Consider an arbitrary element $x^* \in LIM_g^r(x_i)$. For any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : \|x_k - x^*\| < r + \varepsilon\}| = \delta_g(\mathbb{N}). \tag{3}$$

By setting $\varepsilon = \frac{1}{j}$ for every $j \in \mathbb{N}$, we get $\delta_g(K_j^c) = 0$ for the set $K_j^c := \{n \in \mathbb{N} : \|x_n - x^*\| \geq r + \frac{1}{j}\}$. This fact implies also that $\delta_g(K_j) = \delta_g(\mathbb{N})$. Moreover, the inclusions$K_1 \supset K_2 \supset \cdots \supset K_j \supset K_{j+1} \supset \cdots$ are also satisfied. Let (t_j) be a strictly increasing sequence of positive real numbers such that $\lim_{n\to\infty} t_j = \delta_g(\mathbb{N})$. By the definition of K_1, choose any $b_1 \in K_1$ such that $\frac{K_1(n)}{g(n)} > t_1$ holds for all $n > b_1$. Also choose $b_2 \in K_2$ such that $b_2 > b_1$, and for all $n \geq b_2$ we have $\frac{K_2(n)}{g(n)} > t_2$. Continuing this iterative procedure leads to a sequence (b_j) of natural numbers such that $b_j \in K_j$ for all $j = 1,2,\cdots$, and also

$$\frac{K_j(n)}{g(n)} > t_j \tag{4}$$

for all $n \geq b_j$, where $K_j(n)$ denotes the cardinality of $\{b_j \leq k : k \in K_j\}$. Thus, we have identified the sequence (b_j), which we will utilise to form the set K purportedly existing in the theorem.

Now, let us construct the set $K \subseteq \mathbb{N}$ as follows: include every natural number within $[1, b_1]$ in K, and include every natural number within $[b_j, b_{j+1}] \cap K_j$ for each $j = 1,2,\cdots$ in K. Consequently, it follows from (3) and (4) that

Maejo Int. J. Sci. Technol. **2024**, *18*(02), 178-192

$$\frac{K(n)}{g(n)} \geq \frac{K_j(n)}{g(n)} > t_j$$

for all $b_j \leq n < b_{j+1}$. As n approaches infinity, $\delta_g(K) = \delta_g(\mathbb{N})$. For any chosen $\varepsilon > 0$, there exists a natural number $j_0 \in \mathbb{N}$ such that $\frac{1}{j_0} < \varepsilon$. Consider $n \geq b_j$, where n belongs to K. Then there exists a unique $t \geq j$ such that $b_t \leq n < b_{t+1}$, and also

$$\|x_k - x^*\| < r + \frac{1}{t} < r + \frac{1}{j_0} < r + \varepsilon$$

holds. This implies $x_n \xrightarrow{r-st_g} x^*$ on the set K.

Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ represent two sequences in the normed space X. If $\delta_g(n: x_n \neq y_n) = 0$, they are considered equivalent with respect to $g \in \mathcal{G}$, denoted by $\tilde{x} \simeq \tilde{y}$.

Theorem 7. For any sequence $\tilde{x} = (x_n)$ in normed space X and $x^* \in LIM_g^r(\tilde{x})$, there exists a sequence $\tilde{y} = (y_n)$ (different from $\tilde{x}$) in X such that $\tilde{x} \simeq \tilde{y}$ and $x^* \in LIM_g^r(\tilde{y})$.

Proof: Consider an arbitrary element $x^* \in LIM_g^r(\tilde{x})$. By Theorem 1(iii), we have $0 \in LIM_g^r(\|x_n - x^*\|)$. Consequently, Theorem 6 implies the existence of $K = \{n_k : k \in \mathbb{N}\}$ with $\delta_g(K) = \delta_g(\mathbb{N})$ such that $0 \in LIM_g^r(\|x_{n_k} - x^*\|)$. Let us define the sequence $\tilde{y} = (y_n)$ as

$$y_n := \begin{cases} x_n, & n \in K, \\ x^*, & n \in \mathbb{N} - K. \end{cases}$$

Clearly,

$$\delta_g(\{n \in \mathbb{N}: x_n \neq y_n\}) = \delta_g(\mathbb{N} - K) = 0$$

holds. This implies that $\tilde{x} \simeq \tilde{y}$ and

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \|y_k - x^*\| \geq r + \varepsilon\}| = 0$$

holds. Thus, the proof is concluded.

Theorem 8. For a sequence $\tilde{x} = (x_n)$ in a normed space X and $x^* \in X$, we have $\lim_{n\to\infty} x_n = x^*$ if and only if $x^* \in LIM^r(\tilde{x})$ for all $r > 0$.

Proof: The proof of this basic theorem is straightforward and so it is omitted.

The following theorem is g-version of Theorem 8.

Theorem 9. For a sequence $\tilde{x} = (x_n)$ in a normed space X and $x^* \in X$, $x_n \xrightarrow{st_g} x^*$ if and only if $x^* \in LIM_g^r(\tilde{x})$ for all $r > 0$.

Proof: Assuming $x_n \xrightarrow{st_g} x^*$, for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \|x_k - x^*\| \geq \varepsilon\}| = 0.$$

Let us take an arbitrary positive real number $r > 0$. The inclusion

$$\{k \leq n: \| x_k - x^* \| \geq r + \varepsilon\} \subseteq \{k \leq n: \| x_k - x^* \| \geq \varepsilon\}$$

implies

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \| x_k - x^* \| \geq r + \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \| x_k - x^* \| \geq \varepsilon\}|.$$

Since $\lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n: \|x_k - x^*\| \ge \varepsilon\}| = 0$, we see that the equality $\lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n: \| x_k - x^* \| \ge r+\varepsilon\}| = 0$ holds. Thus, this equality shows that x^* belongs to the set $LIM_g^r(\tilde{x})$. This completes the proof of the sufficiency part.

Conversely, assuming $x_n \xrightarrow{r-st_g} x^*$ for all $r > 0$, we obtain

$$\lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n: \| x_k - x^* \| \ge r+\varepsilon\}| = 0.$$

Now let us choose an arbitrary $\varepsilon_0 > 0$, set $r := \frac{\varepsilon_0}{2}$ and $\varepsilon = \frac{\varepsilon_0}{2}$. This leads to

$$\lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n: \| x_k - x^* \| \ge r+\varepsilon\}| = \lim_{n\to\infty}\frac{1}{g(n)}|\{k\le n: \| x_k - x^* \| \ge \varepsilon_0\}| = 0.$$

Thus, the desired result $x_n \xrightarrow{st_g} x$ is obtained.

Let us consider a sequence $\tilde{x} = (x_k) \subset X$ and a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$, and define $\tilde{x}' = (x_{n_k})$ and $K_{\tilde{x}'} = \{n_k : k \in \mathbb{N}\}$. As a natural consequence of Theorem 6, we can pose the following query.

Remark 5. If a specific number serves as a weighted statistical rough limit for a sequence, how can we verify its role as a weighted statistical rough limit for a subsequence as well?

Theorem 10. Let $\tilde{x} := (x_k)$ be a sequence and $\tilde{x}' := (x_{n_k})$ be a subsequence of $\tilde{x}$ in a normed space X such that

$$\liminf_{n\to\infty}\frac{|K_{\tilde{x}'}(n)|}{g(n)} > 0.$$

If $x^* \in LIM_g^r(\tilde{x})$, then $x^* \in LIM_g^r(\tilde{x}')$.

Proof: Assume $x^* \in LIM_g^r(\tilde{x})$. Then the inclusion

$$\{m_k \le n: \|x_{m_k} - x_*\| \ge r+\varepsilon\} \subseteq \{m \le n: \|x_m - x^*\| \ge r+\varepsilon\}$$

holds for all $\varepsilon > 0$. This implies

$$\frac{1}{|K_{\tilde{x}'}(n)|}|\{m_k \le n: \|x_{m_k} - x_*\| \ge r+\varepsilon\}| \le \frac{1}{|K_{\tilde{x}'}(n)|}|\{m \le n: \|x_m - x^*\| \ge r+\varepsilon\}|. \quad (5)$$

The condition x^* belonging to $LIM_g^r(\tilde{x})$ is expressed by

$$\limsup_{n\to\infty}\frac{1}{|K_{\tilde{x}'}(n)|}|\{m_k \le n: \|x_{m_k} - x^*\| \ge r+\varepsilon\}| = 0 \quad (6)$$

for every $\varepsilon > 0$. To prove (6), it suffices to show that the right part of the inequality in (5) tends to zero. Utilising the inequalities given by Antal et al. [15] and defining sequences y_n and z_n as specified, we achieve the desired result.

Corollary 1. Let $(X, \|.\|)$ be a normed space and $\tilde{x} = (x_n)$ be a sequence in X. The following statements are equivalent:

(i) $x^* \in LIM_g^r(\tilde{x})$;

(ii) $x^* \in LIM_g^r(\tilde{x}')$ when $\liminf_{n\to\infty}\frac{|K_{\tilde{x}'}(n)|}{g(n)} > 0$;

(iii) $x^* \in LIM_g^r(\tilde{x}')$ when $\liminf_{n\to\infty}\frac{|K_{\tilde{x}'}(n)|}{g(n)} = \delta_g(\mathbb{N})$.

Theorem 11. Let $\tilde{x} = (x_n)$ be a sequence in a normed space X, and $g, h \in \mathcal{G}$. Then the following assertions hold:

(i) If there exist $M > 0$ and $k_0 \in \mathbb{N}$ such that $\frac{g(n)}{h(n)} \leq M$ holds for all $n \geq k_0$, then we have $LIM_g^r(\tilde{x}) \subseteq LIM_h^r(\tilde{x})$.

(ii) If there exist $m > 0$ and $k_0 \in \mathbb{N}$ such that $m \leq \frac{g(n)}{h(n)}$ holds for all $n \geq k_0$, then we have $LIM_h^r(\tilde{x}) \subseteq LIM_g^r(\tilde{x})$.

Proof: (i) Let $x^* \in LIM_g^r(\tilde{x})$ be any element, satisfying

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \|x_k - x^*\| \geq r + \varepsilon\}| = 0 \tag{7}$$

for all $\varepsilon > 0$. By considering $\frac{g(n)}{h(n)} \leq M$ and $n \geq k_0$, we obtain the inequality

$$\frac{1}{h(n)} |\{k \leq n: \|x_k - x^*\| \geq r + \varepsilon\}| \leq M \frac{1}{g(n)} |\{k \leq n: \|x_k - x^*\| \geq r + \varepsilon\}|.$$

Thus, by taking the limit, equation (7) implies $x^* \in LIM_h^r(\tilde{x})$. Therefore, $LIM_g^r(\tilde{x}) \subseteq LIM_h^r(\tilde{x})$ is established. The proof of (ii) follows a similar approach and is omitted here.

Corollary 2. For a sequence $\tilde{x} = (x_n)$ in a normed space X and functions $g, h \in \mathcal{G}$, if constants $m, M > 0$ and $k_0 \in \mathbb{N}$ satisfy $m \leq \frac{g(n)}{h(n)} \leq M$ for all $n \geq k_0$, then

$$LIM_h^r(\tilde{x}) = LIM_g^r(\tilde{x}).$$

Theorem 12. For sequences $\tilde{x} := (x_n)$ and $\tilde{y} := (y_n)$ in a normed space X, if $(\tilde{x} - \tilde{y}) \to 0$, then $LIM_g^r(\tilde{x}) = LIM_g^r(\tilde{y})$.

Proof: Let $x^* \in LIM_g^r(\tilde{x})$ be arbitrary. For any $\varepsilon > 0$, the inclusion

$$\{n \in \mathbb{N}: \|y_n - x^*\| \geq r + \varepsilon\} \subseteq \{n \in \mathbb{N}: \|x_n - y_n\| \geq r + \frac{\varepsilon}{2}\} \cup \{n \in \mathbb{N}: \|x_n - x^*\| \geq r + \frac{\varepsilon}{2}\}$$

implies that the following inequality

$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \|y_k - x^*\| \geq r + \varepsilon\}| \leq \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \|x_k - y_k\| \geq r + \frac{\varepsilon}{2}\}| + \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n: \|x_k - x^*\| \geq r + \frac{\varepsilon}{2}\}|$$

holds. Thus, $x^* \in LIM_g^r(\tilde{y})$. The reverse inclusion is proven similarly and is omitted.

Theorem 13. Let X and Y be normed spaces, $T: X \to Y$ be a linear bounded operator, and $\tilde{x} = (x_n)$ be a sequence in X. If $x^* \in LIM_g^r(\tilde{x})$, then $Tx^* \in LIM_g^{\|T\|r}(T\tilde{x})$, where $T\tilde{x} := (Tx_n)$.

Proof: Let $x^* \in LIM_g^r(\tilde{x})$. For $T = 0$, the result is obvious. Assume T is nonzero. For any $\varepsilon > 0$, due to the boundedness and linearity of T, we have

$$\|Tx_n - Tx^*\| \leq \|T\|\|x_n - x^*\| < \|T\| \left(r + \frac{\varepsilon}{\|T\|}\right).$$

Thus,

$$\{k \leq n: \|Tx_n - Tx^*\| \geq \|T\|r + \varepsilon\} \subseteq \{k \leq n: \|x_n - x^*\| \geq r + \frac{\varepsilon}{\|T\|}\}.$$

This implies $\delta_g(\{k \leq n: \|Tx_n - Tx^*\| \geq \|T\|r + \varepsilon\}) = 0$, that is $Tx^* \in LIM_g^{\|T\|r}(T\tilde{x})$.

Corollary 3. If $\|T\| \leq 1$, then we have $T\left(LIM_g^r(\tilde{x})\right) \subseteq LIM_g^r(T\tilde{x})$.

Let $\ell_\infty(X)$, $gS(B)$ and gS_0 be the sets of all bounded sequences, g-statistical bounded sequences and g-statistical null sequences on X respectively. The following theorem generalises the result [24] and provides a decomposition for gS(B).

Theorem 14. A decomposition exists in the form of

$$gS(B) = \ell_\infty(X) + gS_0,$$

where X is a normed space. However, this decomposition is not unique.

Proof: Consider $\tilde{x} := (x_n)$ in $gS(B)$. Let $M > 0$ such that $\delta_g(P) = 0$, where $P := \{n \in \mathbb{N} : \|x_n\| \geq M\}$. Let the sequences $\tilde{y} := (y_n)$ and $\tilde{z} = (z_n)$ be defined as follows:

$$y_n = \begin{cases} x_n, & n \in P^c \\ 0, & n \in P \end{cases} \quad \text{and} \quad z_n = \begin{cases} 0, & n \in P^c \\ x_n, & n \in P \end{cases}.$$

The equality $x_n = y_n + z_n$ holds for each n. We have $\tilde{y} \in l_\infty(X)$. For $r \in \mathbb{R}^+$ and $\varepsilon > 0$, the inclusion

$$\{n \in \mathbb{N} : \|z_n\| \geq r + \varepsilon\} \subseteq P$$

implies $\delta_g(\{n \in \mathbb{N} : \|z_n\| \geq r + \varepsilon\}) = 0$, that is $z_n \xrightarrow{r-st_g} 0$. Therefore, $gS(B) \subseteq \ell_\infty(X) + gS_0$.

Conversely, consider $\tilde{y} := (y_n)$ in $\ell_\infty(X)$ and $\tilde{z} := (z_n)$ in gS_0. There exists M_y such that $\| y_n \| \leq M_y$ for every n. For $r \in \mathbb{R}^+$ and $\varepsilon = 1$, we have $\delta_g(\{n \in \mathbb{N} : \| z_n \| \geq r + 1\}) = 0$. The containment

$$\{n \in \mathbb{N} : \|y_n + z_n\| \geq M_y + r + 1\} \subseteq \{n \in \mathbb{N} : \|z_n\| \geq r + 1\}$$

implies $\tilde{v} \in gS_0$, completing the proof.

CONCLUSIONS AND FURTHER REMARKS

Statistical convergence stands as a significant extension of the traditional convergence observed in sequences of real or complex numbers, drawing considerable attention from researchers over the years due to its wide-ranging applications across various fields. This study introduces the concept of g-statistical rough convergence for sequences within normed spaces, leveraging a g-weight density as a broader form of the natural density. Within this framework, it delves into convergence results and outlines fundamental properties of the g-rough set of statistical limit points associated with a sequence. Additionally, it explores the interconnection between statistical rough boundary points and the notion of g-statistical boundedness.

Let X be a normed vector space and $\tilde{x} := (x_n)$ be a sequence in X. From the definition of rough convergence, if $x_n \xrightarrow{rc} x^* \in X$, then for any $\varepsilon > 0$, the following set

$$\{n \in \mathbb{N} : \|x_n - x^*\| \geq r + \varepsilon\}$$

is finite. This implies that $x^* \in LIM_g^r(\tilde{x})$ for all $g \in \mathcal{G}$ because of

$$\delta_g(\{n \in \mathbb{N} : \|x_n - x^*\| \geq r + \varepsilon\}) = 0.$$

The relation between the rough limit point and g-statistical rough limit point of a sequence naturally causes one to ask the following question: Is it true that if $x^* \in X$ is g-statistical rough limit point of the sequence $\tilde{x}$ for every $g \in \mathcal{G}$, then it is rough limit point of $\tilde{x}$?

REFERENCES

1. H. Fast, “Sur la convergence statistique”, *Colloq. Math.*, **1951**, *2*, 241-244 (in French).

2. H. Steinhaus, "Sur la convergence ordinaire et la convergence asymptotique", *Colloq. Math.*, **1951**, *2*, 73-74 (in French).
3. I. J. Maddox, "Statistical convergence in a locally convex spaces", *Math. Proc. Cambridge Philos. Soc.*, **1988**, *104*, 141-145.
4. S. Aytar, "The rough limit set and the core of a real sequence", *Numer. Func. Anal. Optim.*, **2008**, *29*, 283-290.
5. S. Aytar, "Rough statistical convergence", *Numer. Func. Anal. Optim.*, **2008**, *29*, 291-303.
6. F. Nuray and E. Savaş, "Statistical convergence of sequences of fuzzy numbers", *Math. Slovaca,* **1995**, *45*, 269-273.
7. R. Yapalı and U. Gürdal, "Pringsheim and statistical convergence for double sequences on *L*-fuzzy normed space", *AIMS Math.*, **2021**, *6*, 13726-13733.
8. B. Das, B. C. Tripathy and P. Debnath, "A new type of statistically convergent complex uncertain triple sequence", *Thai J. Math.*, **2022**, *20*, 1441-1450.
9. B. Das, B. C. Tripathy and P. Debnath, "Some results on statistically convergent triple sequences in an uncertainty space", *Annals Univ. Craiova Math. Comput. Sci. Ser.*, **2022**, *49*, 120-134.
10. B. Das and B. C. Tripathy, "On λ^2-statistical convergence of complex uncertain sequences", *Asian-European J. Math.*, **2023**, *16*, Art.no.2350083.
11. J. Nath, B. Das, B. Bhattacharya and B. C. Tripathy, "On statistically pre-Cauchy sequences of complex uncertain variables defined by Orlicz functions", *Int. J. Uncertain. Fuzz. Knowl.-Based Syst.*, **2023**, *31*, 191-207.
12. H. X. Phu, "Rough convergence in normed linear spaces", *Numer. Func. Anal. Optim.*, **2001**, *22*, 199-222.
13. H. X. Phu, "Rough continuity of linear operators", *Numer. Func. Anal. Optim.*, **2002**, *23*, 139-146.
14. H. X. Phu, "Rough convergence in infinite dimensional normed spaces", *Numer. Func. Anal. Optim.*, **2003**, *24*, 285-301.
15. R. Antal, M. Chawla and V. Kumar, "Some remarks on rough statistical Λ-convergence of order α", *Ural Math. J.*, **2021**, *7*, 16-25.
16. S. K. Pal, D. Chandra and S. Dutta, "Rough ideal convergence", *Hacet. J. Math. Stat.*, **2013**, *42*, 633-640.
17. E. Dündar and C. Çakan, "Rough I-convergence", *Demonstratio Math.*, **2014**, *47*, 638-651.
18. E. Savaş, S. Debnath and D. Rakshit, "On I-statistically rough convergence", *Pub. l'Institut Math.*, **2019**, *105*, 145-150.
19. P. Leonetti, "Rough families, cluster points, and cores", *arXiv*, **2023**, https://doi.org/10.48550/arXiv.2305.15928 (Accessed: May 2023).
20. M. Balcerzak, P. Das, M. Filipczak and J. Swaczyna, "Generalized kinds of density and the associated ideals", *Acta Math. Hungar.*, **2015**, *147*, 97-115.
21. R. Çolak and C. Bektas, "Statistical convergence of order α", in "Modern Methods in Analysis and Its Applications" (Ed. M. Mursaleen), Anamaya Publication, New Delhi, **2010**, pp.121-129.
22. A. A. Adem and M. Altınok, "Weighted statistical convergence of real valued sequences", *Facta Univ. Ser. Math. Inform.*, **2020**, *35*, 887-898.
23. P. Das and E. Savaş, "On generalized statistical and ideal convergence of metric-valued sequences", *Ukrain. Math. J.*, **2017**, *68*, 1849-1859.

Maejo Int. J. Sci. Technol. **2024**, *18*(02), 178-192

24. P. Das, S. Ghosal, A. Ghosh and S. Som, "Characterization of rough weighted statistical limit set", *Math. Slovaca*, **2018**, *68*, 881-896.
25. S. Ghosal and M. Banerjee, "Rough weighted statistical convergence on locally solid Riesz spaces", *Positivity*, **2021**, *25*, 1789-1804.