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Full Paper

Non-Newtonian Sumudu transform

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Abstract: A novel version of the Sumudu transform is introduced by using the notion of $*$-integral, which is called non-Newtonian Sumudu transform. The fundamental characteristics of the non-Newtonian Sumudu transform are demonstrated. The obtained results are applied to the solving of non-Newtonian differential equations and growth models.

Keywords: non-Newtonian calculus, non-Newtonian Sumudu transform, integral transforms, non-Newtonian differential equations, growth models

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INTRODUCTION

Modelling is essential for solving physics and engineering problems that depend on many physical changes and phenomena observed in nature that affect and guide life. Since differential equations have an important role in mathematical modelling, they are used to model real-life problems in physics, engineering, chemistry, statistics, economics and numerous other disciplines.

Integral transforms are employed to solve a variety of problems including initial-value problems, boundary-value problems, differential equations and integral equations that have taken place in fields such as mathematics, signal theory, physics, chemistry, economics, mechanics and other engineering sciences. Due to the diverse range of applications, many novel integral transforms have been introduced. The most widely used and well-known integral transforms are Laplace, Fourier and Sumudu transforms. These transforms have been studied using different concepts such as fractional, conformable fractional, non-conformable fractional, fuzzy, quantum calculus, multiplicative calculus and non-Newtonian calculus [1-11].

Since Newton and Leibnitz established classical calculus, several calculi have been created, taking into account that a well-known and favoured method for introducing a new mathematical system is to change the axioms of a known system. Moreover, a mathematical issue that is challenging or impossible to answer using one calculus can be easily revealed using another calculus. As an alternative to classical calculus, Grossman and Katz [12] created a new structure called non-Newtonian calculus, which involves some special calculi such as geometric, harmonic, quadratic, bigeometric, biharmonic and biquadratic calculus. They introduced modern forms of derivatives and integrals that convert addition and subtraction into multiplication and division. The pioneering work of Grossman and Katz has led to significant interest in non-Newtonian calculi in recent years owing to their diverse applications in fields such as functional analysis, differential equations, integral equations, probability theory, economy, finance, biology, calculus of variations, computer science including image processing, and signal processing [13-24]. Some definitions and known results in non-Newtonian calculus are reminded in the following.

A generator is an injective function whose domain is $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. $\mathbb{R}_\alpha := \{\alpha(u): u \in \mathbb{R}\}$ is called non-Newtonian real line where α is the generator. For $\upsilon, \nu \in \mathbb{R}_\alpha$, α-arithmetic operations are denoted by

α-addition $\quad \upsilon \dot{+} \nu = \alpha\{\alpha^{-1}(\upsilon) + \alpha^{-1}(\nu)\}$,
α-subtraction $\quad \upsilon \dot{-} \nu = \alpha\{\alpha^{-1}(\upsilon) - \alpha^{-1}(\nu)\}$,
α-multiplication $\quad \upsilon \dot{\times} \nu = \alpha\{\alpha^{-1}(\upsilon) \times \alpha^{-1}(\nu)\}$,
α-division $\quad \upsilon \dot{/} \nu \ (\nu \neq \dot{0}) = \alpha\{\alpha^{-1}(\upsilon) / \alpha^{-1}(\nu)\}$,
α-order $\quad \upsilon \dot{<} \nu \ (\upsilon \dot{\leq} \nu) \Leftrightarrow \alpha^{-1}(\upsilon) < \alpha^{-1}(\nu) \ (\alpha^{-1}(\upsilon) \leq \alpha^{-1}(\nu))$.

$(\mathbb{R}_\alpha, \dot{+}, \dot{\times}, \dot{\leq})$ is totally ordered field. α-Arithmetic is generated by α. The identity function represented by I, generates classical arithmetic. On the other hand, geometric arithmetic is produced by the exponential function. If the α-generator is chosen as exp, i.e. $\alpha(u) = e^u$ for $u \in \mathbb{R}$, then $\alpha^{-1}(\upsilon) = \ln \upsilon$. The concept of α-arithmetic transforms into geometric arithmetic. The definitions of geometric operations are:

geometric addition $\quad \upsilon \oplus \nu = \alpha\{\alpha^{-1}(\upsilon) + \alpha^{-1}(\nu)\} = e^{\{\ln \upsilon + \ln \nu\}} = \upsilon . \nu$,
geometric subtraction $\quad \upsilon \ominus \nu = \alpha\{\alpha^{-1}(\upsilon) - \alpha^{-1}(\nu)\} = e^{\{\ln \upsilon - \ln \nu\}} = \upsilon \div \nu, \nu \neq 0$,
geometric multiplication $\quad \upsilon \odot \nu = \alpha\{\alpha^{-1}(\upsilon) \times \alpha^{-1}(\nu)\} = e^{\{\ln \upsilon \times \ln \nu\}} = \upsilon^{\ln \nu} = \nu^{\ln \upsilon}$,
geometric division $\quad \upsilon \oslash \nu = \alpha\{\alpha^{-1}(\upsilon) / \alpha^{-1}(\nu)\} = e^{\{\ln \upsilon \div \ln \nu\}} = \upsilon^{\frac{1}{\ln \nu}}, \nu \neq 1$.

If $\upsilon \in \mathbb{R}_\alpha$ and $\dot{>} \dot{0}$ $(\upsilon \dot{<} \dot{0})$, then we say that it is α-positive number (α-negative number). Also, $\alpha\{-\alpha^{-1}(\upsilon)\} = \dot{-}\upsilon$ for all $\upsilon \in \mathbb{R}_\alpha$ and $\dot{n} = \alpha(n)$ for all $n \in \mathbb{Z}$. The α-fractional notation $!_\alpha$ is described as

$$\dot{n}!_\alpha = \dot{1} \dot{\times} \dot{2} \dot{\times} \ldots \dot{\times} \dot{n} = \alpha(1) \dot{\times} \alpha(2) \dot{\times} \ldots \dot{\times} \alpha(n) = \alpha(n!).$$

The α-absolute value of $\upsilon \in \mathbb{R}_\alpha$ is determined by

$$|\upsilon|_\alpha = \begin{cases} \upsilon, \upsilon \dot{>} \dot{0} \\ \dot{0}, \upsilon = \dot{0} \\ \dot{0} \dot{-} \upsilon, \upsilon \dot{<} \dot{0} \end{cases}.$$

For any $\upsilon, \nu \in \mathbb{R}_\alpha$, $|\upsilon \dot{+} \nu|_\alpha \dot{\leq} |\upsilon|_\alpha \dot{+} |\nu|_\alpha$. For $\upsilon \in \mathbb{R}_\alpha$, $\upsilon^{n_\alpha} = \alpha\{[\alpha^{-1}(\upsilon)]^n\}$ and $\sqrt[p]{\upsilon}^{\alpha} = \alpha\{\sqrt[p]{\alpha^{-1}(\upsilon)}\}$. The closed α-interval on $\mathbb{R}_\alpha$ is represented by

$$\dot{[}\upsilon, \nu\dot{]} = \{x \in \mathbb{R}_\alpha | \upsilon \dot{\leq} x \dot{\leq} \nu\} = \{x \in \mathbb{R}_\alpha | \alpha^{-1}(\upsilon) \dot{\leq} \alpha^{-1}(x) \dot{\leq} \alpha^{-1}(\nu)\}$$
$$= \alpha([\alpha^{-1}(\upsilon), \alpha^{-1}(\nu)]).$$

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The $*$-calculus is defined using two arbitrarily chosen generators. Let α and β be any generators and $*$ denote the ordered pair of arithmetics (α-arithmetic, β-arithmetic). Table 1 provides the notations used.

Table 1. Notations of α-arithmetic and β-arithmetic

	α- Arithmetic	β-Arithmetic
Realm	$C(=\mathbb{R}_\alpha)$	$D(=\mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$(or $-\alpha$)	$\ddot{/}$(or $-\beta$)
Order	$\dot{<}$	$\ddot{<}$

In the $*$-calculus, α-arithmetic is used for arguments and β-arithmetic is used for values. The special calculuses, which are obtained by choosing one of identity function (I) and exponential function (exp) as the generators α and β, are given in Table 2.

Table 2. Special calculuses for generators α and β

Calculus	α	β
Classical	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp.

The isomorphism ι (iota) from α-arithmetic to β-arithmetic uniquely possesses the following characteristics:

(1) ι is one to one;

(2) ι is on C and onto D;

(3) $\iota(\upsilon\dot{+}\nu) = \iota(\upsilon)\ddot{+}\iota(\nu)$

$\iota(\upsilon\dot{-}\nu) = \iota(\upsilon)\ddot{-}\iota(\nu)$

$\iota(\upsilon \dot{\times} \nu) = \iota(\upsilon) \ddot{\times} \iota(\nu)$

$\iota(\upsilon\dot{/}\nu) = \iota(\upsilon)\ddot{/}\iota(\nu), \nu \neq \dot{0}$

$\upsilon \dot{<} \nu \Leftrightarrow \iota(\upsilon) \ddot{<} \iota(\nu)$

for any $\upsilon,\nu \in C$. It appears that $\iota(\upsilon) = \beta\{\alpha^{-1}(\upsilon)\}$ for all $\upsilon \in C$ and $\iota(\dot{n}) = \ddot{n}$ for every integer n [12].

Definition 1 [12, 25]**.** Let $f: X \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be a function and $p \in X'$, $m \in \mathbb{R}_\beta$. If for every $\varepsilon \ddot{>} \ddot{0}$ there is $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $|f(t)\ddot{-}m|_\beta \ddot{<} \varepsilon$ for all $t \in X$ whenever $\dot{0} \dot{<} |t\dot{-}p|_\alpha \dot{<} \delta$, then it is called that the $*$-limit of function f at p is m and it is expressed as $*\lim_{t\to p} f(t) = m$.

Definition 2 [25]**.** Let $f: X \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be a function and $p \in X$. If for every $\varepsilon \ddot{>} \ddot{0}$ there is $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $|f(t)\ddot{-}f(p)|_\beta \ddot{<} \varepsilon$ for all $t \in X$ whenever $|t\dot{-}p|_\alpha \dot{<} \delta$, then it is said that f is $*$-continuous at the point $p \in X$.

Definition 3 [12]. If $* \lim_{t \to p} \frac{f(t) \ddot{-} f(p)}{\iota(t) \ddot{-} \iota(p)} \beta$ occurs, it is described by $(D^* f)(p) = f^*(p)$ and is called the $*$-derivative of f at p and that f is $*$- differentiable at p. $(D^* f)(p) = f^*(p)$ is necessarily in D if it exists.

Definition 4 [12]. The $*$-average of a $*$-continuous function f on $[\dot{v}, \dot{\nu}]$ is described by ${}^*M_v^\nu$ and defined to be the β-limit of the β-convergent sequence whose n^{th} term is β-average of $f(u_1), f(u_2), \dots, f(u_n)$ where $u_1, u_2, \dots, u_n$ is the n-fold α-partition of $[\dot{v}, \dot{\nu}]$. The $*$-integral of a $*$-continuous function f on $[\dot{v}, \dot{\nu}]$ is denoted by $* \int_v^\nu f(t) d^* t$, which is the number $[\iota(\nu) \ddot{-} \iota(v)] \ddot{\times} \; {}^*M_v^\nu$ in D.

Remark 1 [12]. Let $\bar{p} = \alpha^{-1}(p)$ for $p \in C$. Let $\bar{f}(z) = \beta^{-1}(f(\alpha(z)))$ where f is a function whose inputs and outputs are in C and D respectively. Then the following relationships hold:

(1) The limits $* \lim_{t \to p} f(t)$ and $\lim_{z \to \bar{p}} \bar{f}(z)$ coexist and if they do exist, $* \lim_{t \to p} f(t) = \beta\left(\lim_{z \to \bar{p}} \bar{f}(z)\right)$. Furthermore, f is $*$-continuous at p if and only if $\bar{f}$ is continuous at $\bar{p}$.
(2) The derivatives $(Df)(p)$ and $(D^* f)(\bar{p})$ coexist and if they do exist, $(D^* f)(p) = \beta[(D\bar{f})(\bar{p})]$.
(3) If f is $*$-continuous on $[\dot{p}, \dot{q}]$, then ${}^*M_p^q f = \beta\left(M_{\bar{p}}^{\bar{q}} \bar{f}\right)$ and $* \int_p^q f(t) d^* t = \beta\left(\int_{\bar{p}}^{\bar{q}} \bar{f}(z) dz\right)$.

Definition 5 [26]. Let the function $f: [\dot{v}, \dot{+}\infty) \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be $*$-continuous on α-interval $[\dot{v}, \dot{\nu}]$ for each $v \dot{\leq} \nu$. The $*$-limit $* \lim_{\nu \to \dot{+}\infty} * \int_v^\nu f(t) d^* t$ is called improper $*$-integral of the function f on $[\dot{v}, \dot{+}\infty)$ and it is denoted by $* \int_v^{\dot{+}\infty} f(t) d^* t$. If the $* \lim_{\nu \to \dot{+}\infty} * \int_v^\nu f(t) d^* t$ exists and is equal to a number $E \in \mathbb{R}_\beta$, then it is said that the improper $*$-integral $* \int_v^{\dot{+}\infty} f(t) d^* t$ is $*$-convergent.

Definition 6 [8]. If there exist β-constant $\mu \ddot{>} \ddot{0}$ and α-constant γ such that

$$|f(t)|_\beta \ddot{\leq} \mu \ddot{\times} \ddot{e}^{(\alpha^{-1}(\gamma \dot{\times} t))_\beta}$$

for all $t \dot{\geq} t_0$ with $t_0 \dot{\geq} \dot{0}$, then it is said that f is a function of β-exponential order γ.

Definition 7 [8]. A function f is jump $*$-discontinuity at a point t_0 if the right-hand $*$-limit $* \lim_{t \to t_0^+} f(t)$ and the left-hand $*$-limit $* \lim_{t \to t_0^-} f(t)$ exist but are not equal.

Definition 8 [8]. A function f is piecewise (sectionally) $*$-continuous in the closed α-interval $a \dot{\leq} t \dot{\leq} b$ if there is a finite subinterval $[\dot{a}, \dot{t_1}], [\dot{t_1}, \dot{t_2}], \dots, [\dot{t_{n-1}}, \dot{b}]$ such that f is $*$-continuous on each open α-interval $(\dot{t_{i-1}}, \dot{t_i})$ with $t_0 = a, t_n = b$, $i = 1, \dots, n$ and has the one-sided $*$-limits $* \lim_{t \to t_{i-1}^+} f(t)$ and $* \lim_{t \to t_i^-} f(t)$.

In this study, motivated by the extensive applications of both the non-Newtonian calculus and the integral transforms, the non-Newtonian Sumudu transform is established as a new contribution to the literature. The non-Newtonian version of some important properties of the Sumudu transform is obtained. The acquired results are utilised in the determination of solutions of non-Newtonian differential equations, supported by numerical illustration. Hereupon, solutions of some differential equations are found, thank to the relationship between non-Newtonian calculus and classical calculus. The results of the non-Newtonian exponential growth model and Gompertz

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model, which are frequently used in population growth models, are investigated with the aid of the non-Newtonian Sumudu transform.

NON-NEWTONIAN SUMUDU TRANSFORM

Introducing the Sumudu transform from a non-Newtonian perspective, this section provides a fresh viewpoint on the idea of integral transform and a basic explanation of the theory behind the non-Newtonian Sumudu transform.

Definition 9. Let $\mathcal{A}$ be a function set defined by

$$\mathcal{A} = \left\{ f(t): \exists M \ddot{>} \ddot{0}, \tau_1, \tau_2 \dot{>} \dot{0}, |f(t)|_\beta \ddot{<} M \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(|t|_\alpha)/\alpha^{-1}(\tau_j)\right)_\beta}, t \in \left(\dot{0} \dot{-} \dot{1}\right)^{j_\alpha} \times \dot{[}\dot{0}, \dot{+}\infty \dot{)} \right\},$$

where M is a β-constant and τ_1, τ_2 are finite α-constants or infinite. For a given function in the set $\mathcal{A}$, the non-Newtonian Sumudu integral transform is defined as

$$S_N\{f(t)\} = F_N(v) = * \int_{\dot{0}}^{\dot{+}\infty} \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t \tag{1}$$

for $v \in \dot{(}\dot{-}\tau_1, \tau_2\dot{)}$. The equation is also given as

$$S_N\{f(t)\} = * \int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(-\alpha^{-1}(t)\right)_\beta} \ddot{\times} f(v \dot{\times} t) d^*t. \tag{2}$$

Remark 2. For $c, t \in C$, let $\bar{c} = \alpha^{-1}(c), \bar{v} = \alpha^{-1}(v)$. Let $\bar{f}(z) = \beta^{-1}\left(f\left(\alpha(z)\right)\right)$, where f is a function with inputs in C and outputs in D. Then the relationship between the classical Sumudu transform and non-Newtonian Sumudu transform occurs as follows:

$$\begin{aligned}
S_N\{f(t)\} &= * \int_{\dot{0}}^{\dot{+}\infty} \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t \\
&= * \lim_{c \to \dot{+}\infty} * \int_{\dot{0}}^{c} \beta\left(\frac{1}{\alpha^{-1}(v)} \cdot e^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)} \cdot \beta^{-1}\left(f(t)\right)\right) d^*t \\
&= * \lim_{c \to \dot{+}\infty} \beta\left(\int_{0}^{\alpha^{-1}(c)} \frac{1}{\alpha^{-1}(v)} \cdot e^{\left(\frac{-z}{\alpha^{-1}(v)}\right)} \cdot \beta^{-1}\left(f\left(\alpha(z)\right)\right) dz\right) \\
&= \beta\left(\lim_{\bar{c} \to +\infty} \int_0^{\bar{c}} \frac{1}{\bar{v}} \cdot e^{\left(\frac{-z}{\bar{v}}\right)} \cdot \bar{f}(z) dz\right) \\
&= \beta\left(S\left(\bar{f}(z)\right)\right) \\
&= \beta\left(S\left(\beta^{-1}\left(f\left(\alpha(z)\right)\right)\right)\right).
\end{aligned}$$

Hence we get the expression $S_N\{f(t)\} = \beta\left(S\left(\bar{f}(z)\right)\right) = \beta\left(S\left(\beta^{-1}\left(f\left(\alpha(z)\right)\right)\right)\right)$.

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Example 1. The non-Newtonian Sumudu transform of the function $f(t) = \iota(t)$ can be found as follows:

$$S_N\{\iota(t)\} = \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, \iota(t)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, \beta\left(\alpha^{-1}(t)\right)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\int_{\dot{0}}^{\dot{+}\infty} \beta\left(e^{\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}}.\alpha^{-1}(t)\right)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\left(*\int_{\dot{0}}^{c} \beta\left(e^{\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}}.\alpha^{-1}(t)\right)d^*t\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\left(\beta\left[\int_{\alpha^{-1}(\dot{0})}^{\alpha^{-1}(c)} \beta^{-1}\left(\beta\left(e^{\frac{-\alpha^{-1}(\alpha(z))}{\alpha^{-1}(v)}}.\alpha^{-1}\left(\alpha(z)\right)\right)\right)dz\right]\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\beta\left(\int_{0}^{\alpha^{-1}(c)} e^{\frac{-z}{\alpha^{-1}(v)}}.z\,dz\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\beta\left(-\alpha^{-1}(v).e^{\frac{-z}{\alpha^{-1}(v)}}.z\Big|_0^{\alpha^{-1}(c)} + \int_0^{\alpha^{-1}(c)}\alpha^{-1}(v).e^{\frac{-z}{\alpha^{-1}(v)}}dz\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\beta\left(-\alpha^{-1}(v).e^{\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}}.\alpha^{-1}(c) - \left(\left(\alpha^{-1}(v)\right)^2.e^{\frac{-z}{\alpha^{-1}(v)}}\right)\Big|_0^{\alpha^{-1}(c)}\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\beta\left(-\alpha^{-1}(v).e^{\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}}.\alpha^{-1}(c) - \left(\alpha^{-1}(v)\right)^2.e^{\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}} + \left(\alpha^{-1}(v)\right)^2\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}{*}\lim_{c\to\dot{+}\infty}\left(\ddot{-}\ddot{e}^{\left(\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times}\,\iota(c)\,\ddot{\times}\,\iota(v)\ddot{-}\left(\iota(v)\right)^{2_\beta}\ddot{\times}\,\ddot{e}^{\left(\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{+}\iota(v)^{2_\beta}\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\, \ddot{\times}\left(\iota(v)^{2_\beta}\right) = \ddot{1}!_\beta\,\ddot{\times}\,\iota(v).$$

We can generalise this result by using induction as

$$S_N\left\{\iota(t)^{(m)_\beta}\right\} = \ddot{m}!_\beta\,\ddot{\times}\,\iota(v)^{(m)_\beta}\ (m \in \mathbb{N}).$$

Table 3 presents the outcomes of calculating non-Newtonian Sumudu transforms of various basis functions using the provided definition.

Table 3. Non-Newtonian Sumudu transform of some elementary functions

$f(t)$	$S_N[f(t)] = F_N(v)$
$\ddot{1}$	$\ddot{1}$
$\iota(t)$	$\iota(v)$
$\iota(t)^{(m)_\beta}, m \in \mathbb{Z}^+$	$\ddot{m}!_\beta \ddot{\times} \iota(v)^{(m)_\beta}$
$\ddot{e}^{\left(\alpha^{-1}(k\dot{\times}t)\right)_\beta}, \frac{\dot{1}}{v}\alpha \dot{>} k, k \in \mathbb{R}_\alpha$	$\frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(k) \ddot{\times} \iota(v)}\beta$
$\iota(t) \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(k\dot{\times}t)\right)_\beta}$	$\frac{\iota(v)}{\left(\ddot{1}\ddot{-}\iota(k) \ddot{\times} \iota(v)\right)^{2_\beta}}\beta$
$*\sin(k \dot{\times} t)$	$\frac{\iota(k) \ddot{\times} \iota(v)}{\ddot{1}\ddot{+}(\iota(k))^{2_\beta} \ddot{\times} (\iota(v))^{2_\beta}}\beta, v \dot{>} \dot{0}$
$*\cos(k \dot{\times} t)$	$\frac{\ddot{1}}{\ddot{1}\ddot{+}(\iota(k))^{2_\beta} \ddot{\times} (\iota(v))^{2_\beta}}\beta, v \dot{>} \dot{0}.$

Note: $*\sin t = \beta\left(\sin\left(\alpha^{-1}(t)\right)\right)$ and $*\cos t = \beta\left(\cos\left(\alpha^{-1}(t)\right)\right)$

Theorem 1 (Existence of non-Newtonian Sumudu transform). The non-Newtonian Sumudu transform $S_N\{f(t)\}$ exists for $\frac{\dot{1}}{v}\alpha \dot{>} \gamma$ and $*$-converges β-absolutely if f is piecewise $*$-continuous on $[\dot{0}, \dot{+}\infty)$ and of β-exponential order γ.

Proof. We can write

$$\frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} * \int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} * \int_{\dot{0}}^{t_0} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t \ddot{+} \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} * \int_{t_0}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t. \quad (3)$$

The function f is $*$-continuous on α-interval $(\dot{0}, t_0\dot{)}$ except possibly at a finite number of points $t_1, t_2, \ldots, t_n$ in $(\dot{0}, t_0\dot{)}$ because f is piecewise $*$-continuous on $[\dot{0}, t_0\dot{]}$. Hence we can write

$$|f(t)|_\beta \ddot{\leq} M_i, \qquad t_i \dot{<} t \dot{<} t_{i+1}\ (i = 1,2, \ldots, n-1)$$

for finite β-constants M_i. To integrate the piecewise $*$-continuous function from $\dot{0}$ to t_0, the β-sum of the $*$-integrals over each of the α-subintervals of f is taken, that is

$$* \int_{\dot{0}}^{t_0} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t = * \int_{\dot{0}}^{t_1} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t$$

$$\ddot{+} * \int_{t_1}^{t_2} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t \ddot{+} \ldots \ddot{+} * \int_{t_n}^{t_0} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t.$$

Given that the function f is $*$-continuous and β-bounded on every α-subinterval, it can be inferred that each $*$-integral is well-defined. Hence the first integral on the right of (3) exists.

Since f has β-exponential order γ, there exist β-constant $M \ddot{>} \ddot{0}$ and α-constant γ such that

$$|f(t)|_\beta \ddot{\leq} M \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(\gamma)\alpha^{-1}(t)\right)_\beta}$$

for all $t \dot{>} t_0$. Thus, we obtain

$$\left| * \int_{t_0}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f(t) d^*t \right|_\beta \ddot{\leq} * \int_{t_0}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} |f(t)|_\beta d^*t$$

$$\ddot{\leq} * \int_{t_0}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} M \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(\gamma)\alpha^{-1}(t)\right)_\beta} d^*t$$

$$= * \int_{t_0}^{\dot{+}\infty} M \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(t).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)\right)_\beta} d^*t$$

$$= M \ddot{\times} * \lim_{c \to \dot{+}\infty} * \int_{t_0}^{c} \ddot{e}^{\left(-\alpha^{-1}(t).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)\right)_\beta} d^*t$$

$$= M \ddot{\times} * \lim_{c \to \dot{+}\infty} \beta \left[\int_{\alpha^{-1}(t_0)}^{\alpha^{-1}(c)} \beta^{-1} \left(\beta \left(e^{-\alpha^{-1}(\alpha(t)).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)} \right) \right) dt \right]$$

$$= M \ddot{\times} * \lim_{c \to \dot{+}\infty} \beta \left[\int_{\alpha^{-1}(t_0)}^{\alpha^{-1}(c)} e^{-t.\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)} dt \right]$$

$$= M \ddot{\times} * \lim_{c \to \dot{+}\infty} \beta \left(\frac{-\alpha^{-1}(v)}{1 - \alpha^{-1}(v)\alpha^{-1}(\gamma)} \cdot \left(e^{-\alpha^{-1}(c).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)} - e^{-\alpha^{-1}(t_0).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)} \right) \right)$$

$$= M \ddot{\times} * \lim_{c \to \dot{+}\infty} \left(\frac{\ddot{-}\iota(v)}{\ddot{1} \ddot{-} \iota(v) \ddot{\times} \iota(\gamma)} \beta \ddot{\times} \left(\ddot{e}^{\left(-\alpha^{-1}(c).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)\right)_\beta} \ddot{-} \ddot{e}^{\left(-\alpha^{-1}(t_0).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)\right)_\beta} \right) \right)$$

$$= M \ddot{\times} \frac{\iota(v)}{\ddot{1} \ddot{-} \iota(v) \ddot{\times} \iota(\gamma)} \beta \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(t_0).\left(\frac{1}{\alpha^{-1}(v)} - \alpha^{-1}(\gamma)\right)\right)_\beta}.$$

Also, the second integral on the right exists for $\frac{\dot{1}}{v}\alpha \dot{>} \gamma$. Therefore, the argument is proven.

Theorem 2 (Non-Newtonian linearity property). If f_1 and f_2 are two $\mathbb{R}_\beta$-valued functions whose non-Newtonian Sumudu transform exists, then

$$S_N\{\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t)\} = \lambda_1 \ddot{\times} S_N\{f_1(t)\} \ddot{+} \lambda_2 \ddot{\times} S_N\{f_2(t)\}$$

where λ_1 and λ_2 are arbitrary β-constants.

Proof. Suppose that

$$|f_1(t)|_\beta \ddot{\leq} M_1 \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(\gamma)\alpha^{-1}(t)\right)_\beta}$$
$$|f_2(t)|_\beta \ddot{\leq} M_2 \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(\gamma)\alpha^{-1}(t)\right)_\beta}.$$

Hence we can write

$$|\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t)|_\beta \ddot{\leq} (\lambda_1 \ddot{\times} M_1 \ddot{+} \lambda_2 \ddot{\times} M_2) \ddot{\times} \ddot{e}^{\left(\alpha^{-1}(\gamma)\alpha^{-1}(t)\right)_\beta},$$

which implies that the non-Newtonian Sumudu transform of the function $\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t)$ exists. By employing β-additive and β-homogeneous properties of improper $*$-integral, we obtain

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$$S_N\{\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t)\} = \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {}_*\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} \left(\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t)\right) d^*t$$

$$= \lambda_1 \ddot{\times} \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {}_*\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f_1(t) d^*t \ddot{+} \lambda_2 \ddot{\times} \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {}_*\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} f_2(t) d^*t$$

$$= \lambda_1 \ddot{\times} S_N\{f_1(t)\} \ddot{+} \lambda_2 \ddot{\times} S_N\{f_2(t)\},$$

which completes the proof.

Theorem 3 (Non-Newtonian first translation theorem). If $S_N\{f(t)\} = F_N(v)$ exists for $\frac{\dot{1}}{v}\alpha \dot{>} \gamma$, then

$$S_N\left\{\ddot{e}^{\left(\alpha^{-1}(k\dot{\times}t)\right)_\beta} \ddot{\times} f(t)\right\} = \frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(k) \ddot{\times} \iota(v)}\beta \ddot{\times} F_N\left(\frac{v}{\dot{1}\dot{-}k \dot{\times} v}\alpha\right)$$

for any α-constant k.

Proof. Considering the definition of the non-Newtonian Sumudu transform in equation (2), we find

$$S_N\left\{\ddot{e}^{\left(\alpha^{-1}(k\dot{\times}t)\right)_\beta} \ddot{\times} f(t)\right\} = {}_*\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(-\alpha^{-1}(t)\right)_\beta} \ddot{\times} \ddot{e}^{\left(-\alpha^{-1}(k\dot{\times}v\dot{\times}t)\right)_\beta} \ddot{\times} f(v \dot{\times} t) d^*t$$

$$= {}_*\int_{\dot{0}}^{\dot{+}\infty} \beta\left(e^{\left(-\alpha^{-1}(t)\right).[1-\alpha^{-1}(k).\alpha^{-1}(v)]}\right) \ddot{\times} f(v \dot{\times} t) d^*t$$

$$= {}_*\lim_{c\to\dot{+}\infty} {}_*\int_{\dot{0}}^{c} \beta\left(e^{\left(-\alpha^{-1}(t)\right).[1-\alpha^{-1}(k).\alpha^{-1}(v)]}\right) \ddot{\times} f(v \dot{\times} t) d^*t$$

$$= {}_*\lim_{c\to\dot{+}\infty} \beta\left[\int_0^{\alpha^{-1}(c)} \beta^{-1}\left(\beta\left(e^{(-t)[1-\alpha^{-1}(k)\alpha^{-1}(v)]}\right)\right).\beta^{-1}\left(f\left(\alpha(\alpha^{-1}(v).t)\right)\right) dt\right]$$

for $\frac{\dot{1}}{v}\alpha \dot{>} \gamma$. If it is taken as $t[1 - \alpha^{-1}(k)\alpha^{-1}(v)] = w$, then we find

$$S_N\left\{\ddot{e}^{\left(\alpha^{-1}(k\dot{\times}t)\right)_\beta} \ddot{\times} f(t)\right\}$$

$$= {}_*\lim_{c\to\dot{+}\infty} \beta\left[\int_0^{\alpha^{-1}(c).[1-\alpha^{-1}(k)\alpha^{-1}(v)]} e^{-w} \cdot \beta^{-1}\left(f\left(\alpha\left(\frac{\alpha^{-1}(v)}{1-\alpha^{-1}(k)\alpha^{-1}(v)}.w\right)\right)\right) \cdot \frac{dw}{1-\alpha^{-1}(k)\alpha^{-1}(v)}\right]$$

$$= {}_*\lim_{c\to\dot{+}\infty} \beta\left[\frac{1}{1-\alpha^{-1}(k)\alpha^{-1}(v)} \int_0^{\alpha^{-1}(c).[1-\alpha^{-1}(k)\alpha^{-1}(v)]} e^{-w} \cdot \beta^{-1}\left(f\left(\alpha\left(\frac{\alpha^{-1}(v)}{1-\alpha^{-1}(k)\alpha^{-1}(v)}\, w\right)\right)\right) dw\right]$$

$$= \frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(k) \ddot{\times} \iota(v)}\beta \ddot{\times} {}_*\lim_{c\to\dot{+}\infty} {}_*\int_{\dot{0}}^{c\dot{\times}[\dot{1}\dot{-}k\dot{\times}v]} \ddot{e}^{\left(-\alpha^{-1}(w)\right)_\beta} \ddot{\times} f\left(\frac{v}{\dot{1}\dot{-}k \dot{\times} v}\alpha \dot{\times} w\right) d^*w$$

$$= \frac{\ddot{1}}{\ddot{1}\ddot{-}\iota(k) \ddot{\times} \iota(v)}\beta \ddot{\times} F_N\left(\frac{v}{\dot{1}\dot{-}k \dot{\times} v}\alpha\right).$$

Theorem 4 (Non-Newtonian second translation theorem). If $S_N\{f(t)\} = F_N(v)$ and $h(t) = \begin{cases} \ddot{0}, \dot{0} \dot{<} t \dot{<} k \\ f(t\dot{-}k) \quad , t \dot{>} k \end{cases}$, then $S_N\{h(t)\} = \ddot{e}^{\left(\frac{-\alpha^{-1}(k)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} F_N(v)$.

Proof. Using the definition of non-Newtonian Sumudu transform in equation (1), it is straightforward to observe that

$$S_N\{h(t)\} = \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, h(t)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\int_{k}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, f(t\dot{-}k)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty}\left({*}\int_{k}^{c} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, f(t\dot{-}k)d^*t\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty}\left(\beta\left[\int_{\alpha^{-1}(k)}^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.\beta^{-1}\left(f\left(\alpha\left(t-\alpha^{-1}(k)\right)\right)\right)dt\right]\right).$$

If we substitute $t-\alpha^{-1}(k)=w$, then we find

$$S_N\{h(t)\} = \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty}\left(\beta\left[\int_{0}^{\alpha^{-1}(c)-\alpha^{-1}(k)} e^{\frac{-w-\alpha^{-1}(k)}{\alpha^{-1}(v)}}.\beta^{-1}\left(f\left(\alpha(w)\right)\right)dw\right]\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty}\left(\beta\left[e^{\frac{-\alpha^{-1}(k)}{\alpha^{-1}(v)}}.\int_{0}^{\alpha^{-1}(c)-\alpha^{-1}(k)} e^{\frac{-w}{\alpha^{-1}(v)}}.\beta^{-1}\left(f\left(\alpha(w)\right)\right)dw\right]\right)$$

$$= \ddot{e}^{\left(\frac{-\alpha^{-1}(k)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times} \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty}\left({*}\int_{\dot{0}}^{c\dot{-}k} \ddot{e}^{\left(\frac{-\alpha^{-1}(w)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, f(w)d^*w\right)$$

$$= \ddot{e}^{\left(\frac{-\alpha^{-1}(k)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, F_N(v)\,,$$

which ends the proof.

Theorem 5 (Non-Newtonian derivative theorem). If $f(t)$ is $*$-continuous on $\dot{[}\dot{0},\dot{+}\infty\dot{)}$ and of β-exponential order γ, and also $f^*(t)$ is piecewise $*$-continuous on $\dot{[}\dot{0},\dot{+}\infty\dot{)}$, then

$$S_N\{f^*(t)\} = \frac{S_N\{f(t)\}\ddot{-}f(\dot{0})}{\iota(v)}\beta$$

for $\frac{\dot{1}}{v}\alpha \dot{>} \gamma$.

Proof. By definition of non-Newtonian Sumudu transform, one gets

$$S_N\{f^*(t)\} = \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\int_{\dot{0}}^{\dot{+}\infty} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, f^*(t)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty} {*}\int_{\dot{0}}^{c} \ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta} \ddot{\times}\, f^*(t)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta \ddot{\times} {*}\lim_{c\to\dot{+}\infty}\left[\beta\left(\int_{0}^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.\beta^{-1}\left(f^*\left(\alpha(t)\right)\right)dt\right)\right].$$

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From the relationship between classical calculus and non-Newtonian calculus, it can be seen that $\beta^{-1}\left(f^*(\alpha(t))\right) = (\beta^{-1}f\alpha)'(t)$. Consequently, by using partial integration method, the following expression is derived:

$$\int_0^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.\beta^{-1}\left(f^*(\alpha(t))\right)dt = \int_0^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.(\beta^{-1}f\alpha)'(t)dt$$

$$= e^{\frac{-t}{\alpha^{-1}(v)}}.(\beta^{-1}f\alpha)(t)\Big|_0^{\alpha^{-1}(c)} + \frac{1}{\alpha^{-1}(v)}.\int_0^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.(\beta^{-1}f\alpha)(t)dt$$

$$= e^{\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}}.\left(\beta^{-1}f(c)\right) - \beta^{-1}f\left(\alpha(0)\right) + \frac{1}{\alpha^{-1}(v)}.\int_0^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.(\beta^{-1}f\alpha)(t)dt.$$

As a consequence, we obtain

$$S_N\{f^*(t)\} = \frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} * \lim_{c\to\dot{+}\infty}\beta\left(e^{-\frac{\alpha^{-1}(c)}{\alpha^{-1}(v)}}.\left(\beta^{-1}f(c)\right) - \beta^{-1}f(\dot{0}) + \frac{1}{\alpha^{-1}(v)}.\int_0^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.(\beta^{-1}f\alpha)(t)dt\right)$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} * \lim_{c\to\dot{+}\infty}\left[\ddot{e}^{\left(\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(c)\ddot{-}f(\dot{0})\ddot{+}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times}\ \beta\left(\int_0^{\alpha^{-1}(c)} e^{\frac{-t}{\alpha^{-1}(v)}}.(\beta^{-1}f\alpha)(t)dt\right)\right]$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times}\left[* \lim_{c\to\dot{+}\infty}\ddot{e}^{\left(\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(c)\right]\ddot{-}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} f(\dot{0})$$

$$\ddot{+}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} * \lim_{c\to\dot{+}\infty}\int_{\dot{0}}^{c}\ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(t)d^*t\,.$$

Since f is of β-exponential order γ, then there is β-constant $\mu \ddot{>} \ddot{0}$ and α-constant γ such that

$$|f(t)|_\beta \ddot{\leq} M\ \ddot{\times}\ \ddot{e}^{\left(\alpha^{-1}(\gamma\dot{\times}t)\right)_\beta}.$$

Hence one obtains

$$\left|\ddot{e}^{\left(\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(c)\right|_\beta \ddot{\leq} M\ \ddot{\times}\ \ddot{e}^{\left(\alpha^{-1}(\gamma).\alpha^{-1}(c)-\frac{\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}$$

$$= M\ \ddot{\times}\ \ddot{e}^{\left(-\alpha^{-1}(c)\left(-\alpha^{-1}(\gamma)+\frac{1}{\alpha^{-1}(v)}\right)\right)_\beta}.$$

Since $* \lim_{c\to\dot{+}\infty}\ddot{e}^{\left(-\frac{\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(c) = \ddot{0}$ for $\frac{\dot{1}}{v}\alpha \dot{>} \gamma$, we find

$$S_N\{f^*(t)\} = \frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times}\left[* \lim_{c\to\dot{+}\infty}\ddot{e}^{\left(\frac{-\alpha^{-1}(c)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(c)\right]\ddot{-}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} f(\dot{0})$$

$$\ddot{+}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} * \lim_{c\to\dot{+}\infty}\int_{\dot{0}}^{c}\ddot{e}^{\left(\frac{-\alpha^{-1}(t)}{\alpha^{-1}(v)}\right)_\beta}\ddot{\times} f(t)d^*t$$

$$= \frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times}\ \ddot{0}\ddot{-}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} f(\dot{0})\ddot{+}\frac{\ddot{1}}{\iota(v)}\beta\ \ddot{\times} S_N\{f(t)\}$$

$$= \frac{\ddot{1}}{\iota(\upsilon)}\beta \ddot{\times} \left[S_N\{f(t)\}\ddot{-}f(\dot{0})\right]$$

as desired.

In the general situation, the following outcome is obtained.

Corollary 1. Supposing that $f(t), f^*(t), \cdots, f^{*(n-1)}(t)$ are $*$-continuous functions on the α-interval $[\dot{0}, \dot{+}\infty)$ and possess β-exponential order γ and further assuming that $f^{*(n)}(t)$ is piecewise $*$-continuous on $[\dot{0}, \dot{+}\infty)$, then it is concluded that

$$S_N\{f^{*(n)}(t)\} = \frac{S_N\{f(t)\}}{\iota^{n_\beta}(\upsilon)}\beta \ddot{-} \frac{f(\dot{0})}{\iota^{n_\beta}(\upsilon)}\beta \ddot{-} \frac{f^*(\dot{0})}{\iota^{(n-1)_\beta}(\upsilon)}\beta \ddot{-} \dots \ddot{-} \frac{f^{*(n-1)}(\dot{0})}{\iota(\upsilon)}\beta$$

for $\frac{\dot{1}}{\upsilon}\alpha \dot{>} \gamma$.

Definition 10. If $S_N\{f(t)\} = F_N(\upsilon)$, then the inverse non-Newtonian Sumudu transform is defined by $S_N^{-1}\{F_N(\upsilon)\} = f(t)$.

Theorem 6. The inverse non-Newtonian Sumudu transform is linear, i.e.

$$S_N^{-1}\{\lambda_1 \ddot{\times} F_{1_N}(\upsilon) \ddot{+} \lambda_2 \ddot{\times} F_{2_N}(\upsilon)\} = \lambda_1 \ddot{\times} S_N^{-1}\{F_{1_N}(\upsilon)\} \ddot{+} \lambda_2 \ddot{\times} S_N^{-1}\{F_{2_N}(\upsilon)\}$$

where λ_1 and λ_2 are arbitrary β-constants.

Proof. Let $f_1(t)$ and $f_2(t)$ be some functions such that $S_N\{f_1(t)\} = F_{1_N}(\upsilon)$ and $S_N\{f_2(t)\} = F_{2_N}(\upsilon)$. Since non-Newtonian Sumudu transform is linear, we have

$$S_N\{\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t)\} = \lambda_1 \ddot{\times} S_N\{f_1(t)\} \ddot{+} \lambda_2 \ddot{\times} S_N\{f_2(t)\} = \lambda_1 \ddot{\times} F_{1_N}(\upsilon) \ddot{+} \lambda_2 \ddot{\times} F_{2_N}(\upsilon).$$

Applying the inverse non-Newtonian Sumudu transform to this expression gives

$$\lambda_1 \ddot{\times} f_1(t) \ddot{+} \lambda_2 \ddot{\times} f_2(t) = S_N^{-1}\{\lambda_1 \ddot{\times} F_{1_N}(\upsilon) \ddot{+} \lambda_2 \ddot{\times} F_{2_N}(\upsilon)\}$$

which is the equivalent to

$$S_N^{-1}\{\lambda_1 \ddot{\times} F_{1_N}(\upsilon) \ddot{+} \lambda_2 \ddot{\times} F_{2_N}(\upsilon)\} = \lambda_1 \ddot{\times} S_N^{-1}\{F_{1_N}(\upsilon)\} \ddot{+} \lambda_2 \ddot{\times} S_N^{-1}\{F_{2_N}(\upsilon)\}.$$

This demonstrates the linearity of the inverse non-Newtonian Sumudu transform.

Application to Ordinary Non-Newtonian Differential Equations

The non-Newtonian Sumudu transform approach may be used to solve linear non-Newtonian differential equations with β-coefficients. It effectively converts the problem of solving non-Newtonian differential equations into an algebraic problem. We shall now provide an example.

Example 2. Consider the second-order non-Newtonian differential equation

$$y^{**}(t) \ddot{+} y(t) = \iota(t) \quad (4)$$

with $y(\dot{0}) = \ddot{1}$, $y^*(\dot{0}) = \ddot{0}$. By applying the non-Newtonian Sumudu transform to either side of equation (4), we obtain

$$S_N\{y^{**}(t) \ddot{+} y(t)\} = S_N\{\iota(t)\}$$
$$S_N\{y^{**}(t)\} \ddot{+} S_N\{y(t)\} = S_N\{\iota(t)\}$$
$$\frac{S_N\{y(t)\}}{\iota^{2_\beta}(\upsilon)}\beta \ddot{-} \frac{y(\dot{0})}{\iota^{2_\beta}(\upsilon)}\beta \ddot{-} \frac{y^*(\dot{0})}{\iota(\upsilon)}\beta \ddot{+} S_N\{y(t)\} = \iota(\upsilon)$$

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$$S_N\{y(t)\} \ddot{\times} \left(\ddot{1} \ddot{+} \frac{\ddot{1}}{\iota^{2\beta}(v)}\beta\right) = \iota(v) \ddot{+} \frac{\ddot{1}}{\iota^{2\beta}(v)}\beta.$$

If we adjust the equation based on the variable $S_N\{y(t)\}$, we have

$$\begin{aligned} S_N\{y(t)\} &= \frac{\iota^{3\beta}(v) \ddot{+} \ddot{1}}{\iota^{2\beta}(v) \ddot{+} \ddot{1}}\beta \\ &= \iota(v) \ddot{+} \frac{\ddot{1} \ddot{-} \iota(v)}{\iota^{2\beta}(v) \ddot{+} \ddot{1}}\beta \\ &= \iota(v) \ddot{+} \frac{\ddot{1}}{\iota^{2\beta}(v) \ddot{+} \ddot{1}}\beta \ddot{-} \frac{\iota(v)}{\iota^{2\beta}(v) \ddot{+} \ddot{1}}\beta. \end{aligned}$$

The solution is obtained as

$$y(t) = \iota(t) \ddot{+} * \cos(t) \ddot{-} * \sin(t)$$

by applying the inverse transform.

Especially, when the non-Newtonian differential equation (4) is considered in the sense of geometric calculus, the equivalent geometric differential equation problem for $\alpha = I, \beta = \exp$ is equal to

$$y''(t).y(t) - \left(y'(t)\right)^2 + y(t)\, y^2(t) - ty^2(t) = 0, y(0) = e, y'(0) = e,$$

and the solution is $y(t) = e^{t + \cos t - \sin t}$.

Particularly, by choosing the generators $\alpha = \exp$, $\beta = \exp$, we obtain the equivalent bigeometric differential equation problem of (4) as follows:

$$t.y(t)y'(t) + t^2.y(t)y''(t) - t^2.\left(y'(t)\right)^2 + \left(y(t)\right)^2.(\ln y(t) - \ln t) = 0, y(1) = e, y'(1) = 0$$

and the solution is $y(t) = t.e^{(\cos \ln t - \sin \ln t)}$.

Application to Growth Models

Thomas Malthus introduced one of the earliest mathematical models illustrating the dynamic change of populations. The Malthusian model posits that the rate of population growth in a country is directly proportional to its total population, denoted as $W(t)$, at any given time t. This concept is commonly used to explain how the more people there are at any given time, the more there will be in the future. In mathematical terms, this assumption can be expressed such that κ is a constant of proportionality. In classical calculus, the mathematical representation of this model is described as $\frac{dW(t)}{dt} = \kappa W(t)$ [27]. Güngör [18] generalised this model in the non-Newtonian sense, and called it the non-Newtonian exponential growth model, which is expressed by

$$\frac{d^*W}{dt^*} = \kappa \ddot{\times} W(t) \tag{5}$$

with inital condition $W(\dot{0}) = W_0$, where W represents the population size at time t and κ is a β-positive number. Taking into account the non-Newtonian differential equation, then

$$\begin{cases} W^*(t) = \kappa \ddot{\times} W(t) \\ W(\dot{0}) = W_0 \end{cases}.$$

From here, we use the non-Newtonian Sumudu transform to calculate the population size at time t in the context of non-Newtonian calculus. We obtain

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$$S_N\left\{\frac{d^*W}{dt^*}\right\} = \kappa \ddot{\times} S_N\{W(t)\}$$

by taking the non-Newtonian Sumudu transform to both sides of (5). Benefiting from the property of the non-Newtonian Sumudu of $*$-derivative, we get

$$\frac{S_N\{W(t)\}\ddot{-}W(\dot{0})}{\iota(v)}\beta = \kappa \ddot{\times} S_N\{W(t)\}$$

$$S_N\{W(t)\}\ddot{-}W(\dot{0}) = \kappa \ddot{\times} \iota(v) \ddot{\times} S_N\{W(t)\}$$

$$(\ddot{1}\ddot{-}\kappa \ddot{\times} \iota(v)) \ddot{\times} S_N\{W(t)\} = W_0$$

$$S_N\{W(t)\} = \frac{W_0}{\ddot{1}\ddot{-}\kappa \ddot{\times} \iota(v)}\beta. \tag{6}$$

Applying inverse non-Newtonian Sumudu transform on either side of (6), we get

$$\begin{aligned} W(t) &= S_N^{-1}\left\{\frac{W_0}{\ddot{1}\ddot{-}\kappa \ddot{\times} \iota(v)}\beta\right\} \\ &= W_0 \ddot{\times} S_N^{-1}\left\{\frac{\ddot{1}}{\ddot{1}\ddot{-}\kappa \ddot{\times} \iota(v)}\beta\right\} \\ &= W_0 \ddot{\times} \ddot{e}^{(\beta^{-1}(\kappa)\alpha^{-1}(v))_\beta}, \end{aligned}$$

which is the required amount of the population at time t.

The Gompertz model is among the most prevalent models of population growth. Adapted applications of this model include plant growth, the growth of some animals, tumour growth and bacterial growth. The mathematical expression for this model is denoted by the differential equation:

$$\frac{dW(t)}{dt} = \mu W(t) \ln \frac{\kappa}{W(t)}, \tag{7}$$

where $W(t)$ represents the population size at time t, μ is the growth rate and κ is the carrying capacity [28]. To solve equation (7) using the non-Newtonian Sumudu transform, we shall use the connection between classical calculus and non-Newtonian calculus. By setting $\alpha = I$ and $\beta = \exp$ in the definition of the $*$-derivative, we get $W^*(t) = e^{\frac{W'(t)}{W(t)}}$ in geometric calculus. Thus, we may express equation (7) as

$$W^*(t) = \exp\left\{\ln\left(\frac{\kappa}{W(t)}\right)^\mu\right\}.$$

Therefore, we can represent this equation as

$$W^*(t) = \kappa^\mu \oplus e^{-\mu} \odot W(t) \tag{8}$$

in the sense of geometric calculus. In particular, if we apply the geometric calculus version of the non-Newtonian Sumudu transform to either side of (8), we get

$$S_N\{W^*(t)\} = S_N\{\kappa^\mu \oplus e^{-\mu} \odot W(t)\}$$

$$(S_N\{W(t)\} \ominus W(0)) \oslash e^v = \kappa^\mu \oplus e^{-\mu} \odot S_N\{W(t)\}$$

$$(e \ominus (e^{-\mu} \odot e^v)) \odot S_N\{W(t)\} = \kappa^\mu \odot e^v \oplus W(0).$$

Thus, we obtain

$$S_N\{W(t)\} = [W(0) \oslash (e \ominus (e^{-\mu} \odot e^v))] \oplus \left[\frac{(\kappa^\mu \odot e^v)}{(e \ominus (e^{-\mu} \odot e^v))}\right]. \tag{9}$$

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Using the inverse non-Newtonian Sumudu transform on either side of (9) yields

$$\begin{aligned} W(t) &= S_N^{-1}\{W(0) \oslash (e \ominus (e^{-\mu} \odot e^{v}))\} \oplus S_N^{-1}\{\kappa\} \ominus S_N^{-1}\{\kappa \oslash (e \ominus (e^{-\mu} \odot e^{v}))\} \\ &= W(0) \odot e^{e^{-\mu t}} \oplus \kappa \odot e^{e^{-\mu t}} \ominus \kappa \\ &= \kappa \cdot \left(e^{e^{-\mu t}}\right)^{\ln\left(\frac{W(0)}{\kappa}\right)}. \end{aligned}$$

As a result, we get the solution to equation (7).

CONCLUSIONS

The definition of the Sumudu transform has been generalised to non-Newtonian calculus through the use of *-integral. The relationship between the classical Sumudu transform and non-Newtonian Sumudu transform is examined. Many significant characteristics of the non-Newtonian Sumudu transform have been proven. The solution of non-Newtonian differential equation is investigated with the use of the acquired findings. In particular, when the forms of this equation in geometric and bigeometric calculus, which are popular classes of non-Newtonian calculus, are considered, it is seen that the relationship between non-Newtonian calculus and classical calculus can be exploited to obtain results for some difficult differential equations. The results of the non-Newtonian exponential growth models have been obtained by applying the non-Newtonian Sumudu transform. The result of the Gompertz model, one of the most popular population growth models, has been obtained with the help of the geometric form of the non-Newtonian Sumudu transform.

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REFERENCES

1. T. Abdeljawad, "On conformable fractional calculus", *J. Comput. Appl. Math.*, **2015**, *279*, 57-66.
2. T. Allahviranloo and M. B. Ahmadi, "Fuzzy Laplace transforms", *Soft Comput.*, **2010**, *14*, 235-243.
3. Z. Al-Zhour, F. Alrawajeh, N. Al-Mutairi and R. Alkhasawneh, "New results on the conformable fractional Sumudu transform: Theories and applications", *Int. J. Anal. Appl.*, **2019**, *17,* 1019-1033.
4. N. Alp and M. Z. Sarıkaya, "q-Laplace transform on quantum integral", *Kragujevac J. Math.*, **2023**, *47*, 153-164.
5. A. H. Bhat, J. Majid and I. A. Wani, "Multiplicative Sumudu transform and its applications", *J. Emerg. Technol. Innov. Res.*, **2019**, *6*, 579-589.
6. A. H. Bhat, J. Majid, T. R. Shah, I. A. Wani and R. Jain, "Multiplicative Fourier transform and its applications to multiplicative differential equations", *J. Comput. Math. Sci.*, **2019**, *10*, 375-383.
7. V. G. Gupta, B. Shrama and A. Kiliçman, "A note on fractional Sumudu transform", *J. Appl. Math.*, **2010**, Art.no.154189.
8. N. Güngör, "Non-Newtonian Laplace transform", Proceedings of 7th International Conference on Recent Advances in Pure and Applied Mathematics, **2020**, Mugla, Turkey.

Maejo Int. J. Sci. Technol. **2024**, *18*(02), 130-145

9. N. A. Khan, O. A. Razzaq and M. Ayyaz, "On the solution of fuzzy differential equations by fuzzy Sumudu transform", *Nonlinear Eng.*, **2015**, *4*, 49-60.
10. F. Martinez González, P. O. Mohammed and J. E. N. Valdes, "Non-conformable fractional Laplace transform", *Kragujevac J. Math.*, **2022**, *46*, 341-354.
11. N. A. A. Rahman and M. Z. Ahmad, "Fuzzy Sumudu transform for solving fuzzy partial differential equations", *J. Nonlinear Sci. Appl.*, **2016**, *9*, 3226-3239.
12. M. Grosmann and R. Katz, "Non-Newtonian Calculus", Lee Press, Pigeon Cove (MA), **1972**.
13. F. Córdova-Lepe, "The multiplicative derivative as a measure of elasticity in economics", *TMAT Revista Latinoamer. Ciencias Ingeneriria*, **2006**, *2*, 1-8.
14. A. F. Çakmak and F. Başar, "On line and double integrals in the non-Newtonian sense", *AIP Conf. Proc.*, **2014**, *1611*, 415-423.
15. C. Duyar, B. Sağır and O. Oğur, "Some basic topological properties on non-Newtonian real line", *British J. Math. Comput. Sci.*, **2015**, *9*, 300-307.
16. N. Değirmen and C. Duyar, "A new perspective on Fibonacci and Lucas numbers", *Filomat*, **2023**, *37*, 9561-9574.
17. D. A. Filip and C. Piatecki, "A non-Newtonian examination of the theory of exogenous economic growth", *Math. Aeterna*, **2014**, *4*, 101-117.
18. N. Güngör, "A note on linear non-Newtonian Volterra integral equations", *Math. Sci.*, **2022**, *16*, 373-387.
19. N. Güngör, "Some geometric properties of the non-Newtonian sequence spaces $l_p(N)$", *Math. Slovaca*, **2020**, *70*, 689-696.
20. N. Güngör, "The succesive approximations method for solving non-Newtonian Fredholm integral equations of the second kind", *Conf. Proc. Sci. Technol.*, **2020**, *3*, 166-175.
21. U. Kadak and M. Özlük, "Generalized Runge-Kutta methods with respect to the non-Newtonian calculus", *Abstr. Appl. Anal.*, **2015**, Art.no.594685.
22. M. Rybaczuk and P. Stoppel, "The fractal growth of fatigue defects in materials", *Int. J. Fracture*, **2000**, *103*, 71-94.
23. N. Sager and B. Sağır, "Some inequalities in quasi-Banach algebra of non-Newtonian bicomplex numbers", *Filomat*, **2021**, *35*, 2231-2243.
24. D. F. M. Torres, "On a non-Newtonian calculus of variations", *Axioms*, **2021**, *10*, Art.no.171.
25. B. Sağır and F. Erdoğan, "On the function sequences and series in the non-Newtonian calculus", *J. Sci. Arts*, **2019**, *4*, 915-936.
26. M. Erdoğan and C. Duyar, "Non-Newtonian improper integrals", *J. Sci. Arts*, **2018**, *1*, 49-74.
27. W. S. Weigelhofer and K. A. Lindsay, "Ordinary Differential Equations and Applications: Mathematical Methods for Applied Mathematicians, Physicists, Engineers and Bioscientists", Woodhead Publishing, Cambridge, **1999**.
28. R. K. Nagle, E. B. Saff and A. D. Snider, "Fundamentals Differential Equations", Pearson, Upper Saddle River (NJ), **2008**.