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Full Paper

k-Mersenne and *k*-Mersenne-Lucas sedenions

Ismail G. Gursoy [1], Ozcan Bektas [2, *] and Suleyman Senyurt [3]

[1] Pazar Science High School, Rize, Turkey
[2] Department of Basic Science, Faculty of Engineering and Nature Science, Samsun University, Samsun, Turkey
[3] Department of Mathematics, Faculty of Arts and Science, University of Ordu, Ordu, Turkey
* Corresponding author, e-mail: ozcan.bektas@samsun.edu.tr

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Abstract: *k*-Mersenne and *k*-Mersenne-Lucas sedenions are specialised extensions of sedenions, a sixteen-dimensional algebraic structure. These variations introduce specific algebraic rules and properties derived from their connection to Mersenne and Lucas numbers. *k*-Mersenne sedenions are defined by their relationship to *k*-Mersenne numbers, while *k*-Mersenne-Lucas sedenions are associated with *k*-Mersenne-Lucas numbers. In this article firstly *k*-Mersenne and *k*-Mersenne-Lucas sedenions are defined. Then the algebraic properties of these sedenions such as norm, conjugate and inner product are examined. The Mersenne and Mersenne-Lucas recurrence relations, Binet's formulas, generating functions and finite sum formulas for these sedenions are derived. These sedenions also reveal interesting connections with established number theory identities such as Catalan's, Cassini's, D'Ocagne's and Vajda's identities, providing further depth to their significance within the mathematical theory and their potential applications across various scientific domains.

Keywords: *k*-Mersenne sedenions, *k*-Mersenne-Lucas sedenions Binet's formula, Catalan's identity

INTRODUCTION

In abstract algebra the term 'sedenion' refers to any 16-dimensional algebra over the real numbers. The Cayley-Dickson construction [1] shows how octonions can be constructed as two-dimensional algebra over quaternions. If this doubling process is applied to octonions, Cayley-Dickson sedenions are obtained. Sedenions are non-commutative, non-associative, non-alternative, but power-associative algebra over the real numbers. Unlike octonions, sedenions are not composition algebras or normed division algebras, because they have zero divisors. Research on sedenions can be summarised as follows. The most fundamental and comprehensive source on sedenions is the study entitled "Sedenionic algebra and analysis" by Imaeda and Imaeda [2]. Carmody [3] dealt with the investigation of circular, hyperbolic quaternions, octonions and sedenions. Cawagas [4] investigated zero divisors within the Cayley-Dickson sedenion algebra. Another research about zero divisors of Cayley-Dickson algebras was studied by Moreno [5]. Chan and Dokovic [6] investigated the conjugacy class of subalgebras in the domain of real sedenions. Kirlak and Kizilates [7] investigated a new generalisation of Fibonacci and Lucas sedenions. Tasyurdu and Akpinar [8] and Akpinar [9] dealt with the study of Padovan, Pell-Padovan, Perrin octonions and sedenions. Bektas [10] introduced C,H,O coefficient sedenions along with their matrix representations. Gursoy and Bektas [11] conducted research on sedenionic matrices. Sumer [12] investigated multifluid plasma equations in terms of sedenions.

The number theory is considered one of the most important branch of pure mathematics. There are various recurrence relations within the theory. The Mersenne recurrence, often expressed in the context of Mersenne numbers and their relation to prime numbers, has been a significant area of study in the number theory. It also has applications in various mathematical fields and computer science. In recent years numerous studies have been carried out on hypercomplex numbers in algebraic geometry and the recurrence relations. The first study in this field dates back to 1963 by Horadam [13] about complex Fibonacci numbers and quaternions. Subsequently, Iyer [14, 15] investigated Fibonacci quaternions and generalised Fibonacci quaternions by showing that Fibonacci and Lucas quaternions can be expressed by Fibonacci and Lucas numbers. Horadam [16] continued his research and obtained recurrence relations for quaternions.

The concept of k-Lucas numbers was first introduced by Falcon [17]. Halici [18] investigated the Fibonacci quaternions. Akkus and Kecioglu [19] dealt with Fibonacci octonions. Ipek and Cimen [20, 21] studied Pell quaternions and Pell-Lucas quaternions, as well as Jacobsthal and Jacobsthal-Lucas octonions. Cimen and Ipek [22] continued their research and studied Jacobsthal and Jacobsthal-Lucas sedenions and also Cimen et al. [23] investigated Horadam sedenions. Catarino [24, 25] investigated modified Pell and modified k-Pell quaternions and octonions, as well as k-Pell, k-Pell-Lucas and modified k-Pell sedenions. Fibonacci and Lucas sedenions were studied by Unal et al. [26]. Kizilaslan and Akkus [27] developed new computational methods in quantum mechanics using quaternionic terms. Soykan [28, 29] investigated Tribonacci Tribonacci-Lucas sedenions. Polatli and Kizilates [30] introduced new families of Fibonacci and Lucas octonions with q-integer components.

k-Mersenne and k-Mersenne-Lucas octonions were studied by Kumari et al. [31]. Devi and Devibala [32] studied Mersenne and Mersenne-Lucas sedenions. Boussayoud and Chelgham [33] investigated k-Mersenne-Lucas numbers. Construction of dual-generalised complex Fibonacci and Lucas quaternions and these special quaternions' matrix representations were obtained by Senturk et al. [34]. Yilmaz and Sacli [35] defined one-parameter generalisation of Leonardo dual quaternions and examined their properties. Yilmaz et al. [36] defined six different quaternion-type cyclic-Fibonacci sequence and examined their properties. Uslu and Deniz [37] studied some special identities for k-Mersenne numbers. Uysal et al. [38] introduced the hyperbolic k-Mersenne and k-Mersenne-Lucas octonions.

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In this article general information about k-Mersenne and k-Mersenne-Lucas sequence are given first. Then k-Mersenne and k-Mersenne-Lucas sedenions are introduced. Some properties of these sedenions are examined and recurrence relations of some special identities are obtained.

PRELIMINARIES

Sedenions

A sedenion is constructed over real numbers. Let $E_{16} = \{e_i \in \mathbb{S} \mid i = 0,1,2,\dots,15\}$ be the canonical basis of sedenions, where $e_0 = 1$ is multiplicative scalar element and e_i's $(i = 1,2,\dots,15)$ are imaginary units. Any sedenion can be defined as a linear combination of E_{16} as $\mathcal{X} = \sum_{i=0}^{15} x_i e_i$ [8]. A set of sedenions can be written in the form:

$$\mathbb{S} = \left\{ \mathcal{X} = \sum_{i=0}^{15} x_i e_i \mid x_i \in \mathbb{R}, 0 \leq i \leq 15 \right\}.$$

Sedenionic units satisfy the following properties:

1. $e_0 = 1$ and $e_0 e_i = e_i\, e_0 = e_i$, $(i \neq 0)$
2. $e_i\, e_i = (e_i)^2 = -1,\qquad (i \neq 0)$
3. $e_i\, e_j = -e_j\, e_i,\qquad (i \neq j), (i, j \neq 0).$

The multiplication of sedenionic basis elements is given in Table 1. $Re(\mathcal{X}) = S_{\mathcal{X}} = x_0 e_0$ is called the real part of sedenion and $\vec{V}_{\mathcal{X}} = \sum_{i=1}^{15} x_i e_i = Im(\mathcal{X})$ is called its vectorial part. The sedenion can be expressed as

$$\mathcal{X} = x_0 e_0 + \sum_{i=1}^{15} x_i e_i = S_{\mathcal{X}} + \vec{V}_{\mathcal{X}}.$$

The sum of two sedenions is defined by

$$\mathcal{X} + \mathcal{Y} = \sum_{i=0}^{15} (x_i + y_i) e_i = (S_{\mathcal{X}} + S_{\mathcal{Y}}) + (\vec{V}_{\mathcal{X}} + \vec{V}_{\mathcal{Y}}).$$

The multiplication of two sedenions is defined by

$$\mathcal{X}\mathcal{Y} = \left(\sum_{i=0}^{15} x_i e_i\right)\left(\sum_{i=0}^{15} y_i e_i\right) = \sum_{i,j,k=1}^{15} f_{ij} \gamma_{ij}^{k} e_k,$$

where $e_i, e_j, e_k \in E_{16}$, $f_{ij} = x_i y_j$ and $\gamma_{ij}^{k} \in \{-1, 0, +1\}$. The coefficient γ_{ij}^{k} is called the field parameter. The list of all triplet indices of the ordered triplets (i, j, k), which provide the loops here, can be found in Table 2.

$\bar{\mathcal{X}}$ is called the conjugate of sedenion and is defined as

$$\bar{\mathcal{X}} = x_0 e_0 - \sum_{i=1}^{15} x_i e_i = S_{\mathcal{X}} - \vec{V}_{\mathcal{X}}.$$

$\mathcal{X}$ non-zero sedenion, the inverse of sedenion, is defined as

$$\mathcal{X}^{-1} = \frac{\bar{\mathcal{X}}}{\|\mathcal{X}\|^2}, \ (\|\mathcal{X}\| \neq 0).$$

Corollary 1. $(\mathbb{S}, +)$ is an Abelian group.

Corollary 2. The set of sedenions is 16-dimensional vector space over real numbers.

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Table 1. Multiplication table of unit sedenion basic elements

e_ie_j		e_j															
		e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_i	e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
	e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
	e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
	e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
	e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
	e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	$-e_1$
	e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

Table 2. Sedenionic Triplets

(i, j, k)				
(1,2,3)	(1,4,5)	(1,7,6)	(1,8,9)	(1,11,10)
(1,13,12)	(1,14,15)	(2,4,6)	(2,5,7)	(2,8,10)
(2,9,11)	(2,14,12)	(2,15,13)	(3,4,7)	(3,6,5)
(3,8,11)	(3,10,9)	(3,13,14)	(3,15,12)	(4,8,12)
(4,9,13)	(4,10,14)	(4,11,15)	(5,8,13)	(5,10,15)
(5,12,9)	(5,14,11)	(6,8,14)	(6,11,13)	(6,12,10)
(6,15,9)	(7,8,15)	(7,9,14)	(7,12,11)	(7,13,10)

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Let $\mathcal{X} = \left(\sum_{i=0}^{15} y_i e_i\right)$ and $\mathcal{Y} = \left(\sum_{i=0}^{15} x_i e_i\right)$ be two sedenions. Then the inner product of two sedenions is defined by [8]:

$$< \mathcal{X}, \mathcal{Y} > = \frac{1}{2}(\mathcal{X}\bar{\mathcal{Y}} + \mathcal{Y}\bar{\mathcal{X}}) = x_0 y_0 + x_1 y_1 + \cdots + x_{15} y_{15}.$$

Hence the norm of sedenion $\mathcal{X}$ denoted by $\|\mathcal{X}\| = \sqrt{\mathcal{X}\bar{\mathcal{X}}} = \sqrt{\sum_{i=0}^{15}(x_i)^2}$. If $\|\mathcal{X}\| = 1$, then $\mathcal{X}$ is called unit sedenion. The inner product and the norm operations mentioned above provide the following properties:

1. $< \mathcal{X}, \mathcal{Y} > = < \mathcal{Y}, \mathcal{X} >$,
2. $< \mathcal{X}, \mathcal{X} > = \|\mathcal{X}\|^2 \geq 0$,
3. $< \mathcal{X}\mathcal{Y}, \mathcal{Z} > = < \mathcal{Y}, \bar{\mathcal{X}}\mathcal{Z} > = < \mathcal{X}, \mathcal{Z}\bar{\mathcal{Y}} >$,
4. $\|\mathcal{X} + \mathcal{Y}\| \leq \|\mathcal{X}\| + \|\mathcal{Y}\|$,
5. $\|\mathcal{X}\| = \|-\mathcal{X}\| = \|\bar{\mathcal{X}}\| = \|-\bar{\mathcal{X}}\|$,
6. $\|\mathcal{X}\|^2 + \|\mathcal{Y}\|^2 = \frac{1}{2}(\|\mathcal{X} + \mathcal{Y}\|^2 + \|\mathcal{X} - \mathcal{Y}\|^2)$,
7. $\|\mathcal{X}\mathcal{Y}\| = \|\mathcal{Y}\mathcal{X}\| = \|\bar{\mathcal{X}}\mathcal{Y}\| = \|\mathcal{X}\bar{\mathcal{Y}}\|$.

Mersenne Sequences

In this section some information about Mersenne sequences is given.

Definition 1. The Mersenne sequence $\{M_n\}_{n\geq 2}$ is defined by the following equation [32, 33]:

$$M_n = 3M_{n-1} - 2M_{n-2},\ M_0 = 0,\ M_1 = 1. \tag{1}$$

By altering the initial conditions of definition, Mersenne-Lucas sequence $\{ML_{k,n}\}_{n\geq 2}$ is defined as

$$ML_n = 3ML_{n-1} - 2ML_{n-2},\ ML_0 = 2,\ ML_1 = 3. \tag{2}$$

Definition 2. The *k*-Mersenne sequence $\{M_{k,n}\}_{n\geq 2}$ is defined by the following equation [32, 33]:

$$M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2},\ M_{k,0} = 0,\ M_{k,1} = 1. \tag{3}$$

Similarly, by altering the initial conditions of the *k*-Mersenne sequence, *k*-Mersenne-Lucas sequence $\{ML_{k,n}\}_{n\geq 2}$ is defined as

$$ML_{k,n} = 3kML_{k,n-1} - 2ML_{k,n-2},\ ML_{k,0} = 2,\ ML_{k,1} = 3k. \tag{4}$$

Definition 3. The n^{th} terms of *k*-Mersenne and *k*-Mersenne-Lucas numbers with a negative indices are as follows [32, 33]:

$$M_{k,-n} = -\frac{1}{2^n}M_{k,n},\ ML_{k,-n} = \frac{1}{2^n}ML_{k,n}. \tag{5}$$

The characteristic equation corresponding to Eq.(3) and Eq.(4) is $\alpha^2 - 3k\alpha + 2 = 0$. The roots are $\alpha = \frac{3k+\sqrt{9k^2-8}}{2}$, $\beta = \frac{3k-\sqrt{9k^2-8}}{2}$ and hold the following properties: $\alpha + \beta = 3k, \alpha\beta = 2$ and $\alpha - \beta = \sqrt{9k^2 - 8}$. For all $n \in \mathbb{N}$, the Binet's formulas of k-Mersenne and k-Mersenne-Lucas sequence are given by

$$M_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, ML_{k,n} = \alpha^n + \beta^n. \tag{6}$$

The generating functions of *k*-Mersenne and *k*-Mersenne-Lucas sequence are given by

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$$\sum_{n=0}^{\infty} M_{k,n}x^n = \frac{x}{1-3kx+2x^2}, \ \sum_{n=0}^{\infty} ML_{k,n}x^n = \frac{2-3kx}{1-3kx+2x^2}. \tag{7}$$

k-MERSENNE SEDENIONS AND *k*-MERSENNE-LUCAS SEDENIONS

In this section *k*-Mersenne sedenions and *k*-Mersenne-Lucas sedenions are introduced and their norm, conjugate, recurrence relation, sum formulas and Binet's formulas are obtained.

Definition 4. For $n \geq 0$, the n^{th} $k-$Mersenne sedenion, $\widehat{M_{k,n}}$, and n^{th} $k-$Mersenne$-$Lucas sedenion, $\widehat{ML_{k,n}}$, are defined as

$$\widehat{M_{k,n}} = \sum_{s=0}^{15} M_{k,n+s}e_s = M_{k,n} + M_{k,n+1}e_1 + M_{k,n+2}e_2 + \cdots + M_{k,n+15}e_{15}, \tag{8}$$

$$\widehat{ML_{k,n}} = \sum_{s=0}^{15} ML_{k,n+s}e_s = ML_{k,n} + ML_{k,n+1}e_1 + ML_{k,n+2}e_2 + \cdots + ML_{k,n+15}e_{15}. \tag{9}$$

Definition 5. The n^{th} $k-$Mersenne sedenion and n^{th} $k-$Mersenne$-$Lucas sedenion written as negative indices of n, $\widehat{M_{k,-n}}$ and $\widehat{ML_{k,-n}}$, are

$$\widehat{M_{k,-n}} = -\frac{1}{2^n}M_{k,n} - \frac{1}{2^{n-1}}M_{k,n-1}e_1 - \frac{1}{2^{n-2}}M_{k,n-2}e_2 - \cdots - \frac{1}{2^{n-15}}M_{k,n-15}e_{15}, \tag{10}$$

$$\widehat{ML_{k,-n}} = \frac{1}{2^n}ML_{k,n} + \frac{1}{2^{n-1}}ML_{k,n-1}e_1 + \frac{1}{2^{n-2}}ML_{k,n-2}e_2 + \cdots + \frac{1}{2^{n-15}}ML_{k,n-15}. \tag{11}$$

Definition 6. The scalar and vectorial parts of the $k-$ Mersenne sedenions $\widehat{M_{k,n}}$ and $k-$Mersenne$-$Lucas sedenions $\widehat{ML_{k,n}}$ are denoted by

$S(\widehat{M_{k,n}}) = M_{k,n}$ and $V(\widehat{M_{k,n}}) = (M_{k,n+1}, M_{k,n+2}, \dots, M_{k,n+15})$,

$S(\widehat{ML_{k,n}}) = ML_{k,n}$ and $V(\widehat{ML_{k,n}}) = (ML_{k,n+1}, ML_{k,n+2}, \dots, ML_{k,n+15})$,

Thus, $\widehat{M_{k,n}} = S(\widehat{M_{k,n}}) + V(\widehat{M_{k,n}})$ and $\widehat{ML_{k,n}} = S(\widehat{ML_{k,n}}) + V(\widehat{ML_{k,n}})$.

Definition 7. For $n \geq 0$, the conjugates of $k-$Mersenne sedenions $\widehat{M_{k,n}}$ and $k-$Mersenne$-$Lucas sedenions $\widehat{ML_{k,n}}$ are defined by

$$\overline{\widehat{M_{k,n}}} = M_{k,n} - \sum_{s=1}^{15} M_{k,n+s}e_s = S(\widehat{M_{k,n}}) - V(\widehat{M_{k,n}}), \tag{12}$$

$$\overline{\widehat{ML_{k,n}}} = ML_{k,n} - \sum_{s=1}^{15} ML_{k,n+s}e_s = S(\widehat{ML_{k,n}}) - V(\widehat{ML_{k,n}}). \tag{13}$$

Theorem 1. For $n \geq 0$, we can write following features:

$\widehat{M_{k,n}} + \overline{\widehat{M_{k,n}}} = 2S(\widehat{M_{k,n}}) = 2M_{k,n}$,

$\widehat{ML_{k,n}} + \overline{\widehat{ML_{k,n}}} = 2S(\widehat{ML_{k,n}}) = 2ML_{k,n}$.

Proof: They follow immediately from Definition 7.

Definition 8. According to the norm of definition of sedenions, the norms of $k-$Mersenne sedenions $\mathrm{N}(\widehat{M_{k,n}})$ and $k-$Mersenne$-$Lucas sedenions $\mathrm{N}(\widehat{ML_{k,n}})$ are defined as

$$\mathrm{N}(\widehat{M_{k,n}}) = \sqrt{M_{k,n}^2 + M_{k,n+1}^2 + M_{k,n+2}^2 + \cdots + M_{k,n+15}^2},$$

$$\mathrm{N}(\widehat{ML_{k,n}}) = \sqrt{ML_{k,n}^2 + ML_{k,n+1}^2 + ML_{k,n+2}^2 + \cdots + ML_{k,n+15}^2}.$$

Theorem 2. The recurrence relations of k-Mersenne sedenions and k-Mersenne-Lucas sedenions are given by the following equations:

1. $\widehat{M_{k,n+1}} = 3k\widehat{M_{k,n}} - 2\widehat{M_{k,n-1}}$,
2. $\widehat{ML_{k,n+1}} = 3k\widehat{ML_{k,n}} - 2\widehat{ML_{k,n-1}}$.

Proof:

1. Using Eq.(3) and (8), the recurrence relations of k-Mersemme sedenions are obtained as follows:

$$\begin{aligned}\widehat{M_{k,n+1}} &= M_{k,n+1} + M_{k,n+2}e_1 + M_{k,n+3}e_2 + \cdots + M_{k,n+16}e_{15} \\ &= 3kM_{k,n} - 2M_{k,n-1} + \left(3kM_{k,n+1} - 2M_{k,n}\right)e_1 + \left(3kM_{k,n+2} - 2M_{k,n+1}\right)e_2 \\ &\quad + \cdots + \left(3kM_{k,n+15} - 2M_{k,n+14}\right)e_{15} \\ &= 3k\left(M_{k,n} + M_{k,n+1}e_1 + M_{k,n+2}e_2 + \cdots + M_{k,n+15}e_{15}\right) \\ &\quad -2\left(M_{k,n-1} + M_{k,n}e_1 + M_{k,n+1}e_2 + \cdots + M_{k,n+14}e_{15}\right) \\ &= 3k\widehat{M_{k,n}} - 2\widehat{M_{k,n-1}}.\end{aligned}$$

2. Similarly, using Eq.(4) and (9), the recurrence relations of k-Mersenne-Lucas sedenions are obtained as follows:

$$\begin{aligned}\widehat{ML_{k,n+1}} &= ML_{k,n+1} + ML_{k,n+2}e_1 + ML_{k,n+3}e_2 + \cdots + ML_{k,n+16}e_{15} \\ &= 3kML_{k,n} - 2ML_{k,n-1} + \left(3kML_{k,n+1} - 2ML_{k,n}\right)e_1 \\ &\quad +\left(3kML_{k,n+2} - 2ML_{k,n+1}\right)e_2 + \cdots + \left(3kM_{k,n+15} - 2M_{k,n+14}\right)e_{15} \\ &= 3k\left(M_{k,n} + M_{k,n+1}e_1 + M_{k,n+2}e_2 + \cdots + M_{k,n+15}e_{15}\right) \\ &\quad -2\left(M_{k,n-1} + M_{k,n}e_1 + M_{k,n+1}e_2 + \cdots + M_{k,n+14}e_{15}\right) \\ &= 3k\widehat{M_{k,n}} - 2\widehat{M_{k,n-1}}.\end{aligned}$$

Theorem 3. For $n, s \in \mathbb{N}$, the Binet's formulas of k −Mersenne sedenions, k −Mersenne−Lucas sedenions and their conjugates and negative indices are given by the following equations respectively:

1. $\widehat{M_{k,n}} = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}$,
2. $\widehat{ML_{k,n}} = A\alpha^n + B\beta^n$,
3. $\overline{\widehat{M_{k,n}}} = \frac{\bar{A}\alpha^n - \bar{B}\beta^n}{\alpha - \beta}$,
4. $\overline{\widehat{ML_{k,n}}} = \bar{A}\alpha^n + \bar{B}\beta^n$,
5. $\widehat{M_{k,-n}} = -\frac{1}{2^n}\frac{B\alpha^n - A\beta^n}{\alpha - \beta}$,
6. $\widehat{ML_{k,-n}} = \frac{1}{2^n}\left(B\alpha^n + A\beta^n\right)$,

where $A = \sum_{s=0}^{15} \alpha^s e_s$, $B = \sum_{s=0}^{15} \beta^s e_s$, $\bar{A} = e_0 - \sum_{s=1}^{15} \alpha^s e_s$ and $\overline{B} = e_0 - \sum_{s=1}^{15} \beta^s e_s$.

Proof:

1. From Eq.(6) and (8), we get the Binet's formula of k −Mersenne sedenions as

$$\widehat{M_{k,n}} = \sum_{s=0}^{15} M_{k,n+s}e_s = \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)e_0 + \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)e_1 + \cdots + \left(\frac{\alpha^{n+15}-\beta^{n+15}}{\alpha-\beta}\right)e_{15}$$
$$= \frac{\alpha^n}{\alpha-\beta}\underbrace{(e_0+\alpha e_1+\cdots+\alpha^{15}e_{15})}_{\text{A}} - \frac{\beta^n}{\alpha-\beta}\underbrace{(e_0+\beta e_1+\cdots+\beta^{15}e_{15})}_{\text{B}}$$
$$= \frac{A\alpha^n-B\beta^n}{\alpha-\beta}.$$

2. From Eq.(6) and (9), we get the Binet's formula of k −Mersenne −Lucas sedenions as

$$\widehat{ML_{k,n}} = \sum_{s=0}^{15} ML_{k,n+s}e_s = (\alpha^n+\beta^n)e_0 + (\alpha^n+\beta^n)e_1 + \cdots + (\alpha^n+\beta^n)e_{15}$$
$$= \alpha^n\underbrace{(e_0+\alpha e_1+\cdots+\alpha^{15}e_{15})}_{\text{A}} + \beta^n\underbrace{(e_0+\beta e_1+\cdots+\beta^{15}e_{15})}_{\text{B}}$$
$$= A\alpha^n + B\beta^n. \tag{15}$$

3. From Eq.(6) and (12), we get the Binet's formula of k −Mersenne sedenions' conjugate as

$$\overline{\widehat{M_{k,n}}} = M_{k,n} - \sum_{s=1}^{15} M_{k,n+s}e_s = \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)e_0 - \left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)e_1 - \cdots - \left(\frac{\alpha^{n+15}-\beta^{n+15}}{\alpha-\beta}\right)e_{15}$$
$$= \frac{\alpha^n}{\alpha-\beta}\underbrace{(e_0-\alpha e_1-\cdots-\alpha^{15}e_{15})}_{\bar{A}} - \frac{\beta^n}{\alpha-\beta}\underbrace{(e_0-\beta e_1-\cdots-\beta^{15}e_{15})}_{\bar{B}}$$
$$= \frac{\bar{A}\alpha^n-\bar{B}\beta^n}{\alpha-\beta}. \tag{16}$$

4. From Eq.(6) and (13), we get the Binet's formula of k − Mersenne −Lucas sedenions' conjugate as

$$\overline{\widehat{ML_{k,n}}} = ML_{k,n} - \sum_{s=1}^{15} ML_{k,n+s}e_s = (\alpha^n+\beta^n)e_0 - (\alpha^n+\beta^n)e_1 - \cdots - (\alpha^n+\beta^n)e_{15}$$
$$= \alpha^n\underbrace{(e_0-\alpha e_1-\cdots-\alpha^{15}e_{15})}_{\bar{A}} + \beta^n\underbrace{(e_0-\beta e_1-\cdots-\beta^{15}e_{15})}_{\bar{B}}$$
$$= \bar{A}\alpha^n + \bar{B}\beta^n. \tag{17}$$

5. From Eq.(6) and (10), we get the Binet's formula of negative version of k −Mersenne sedenions as

$$\widehat{M_{k,-n}} = -\frac{1}{2^n}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)e_0 - \frac{1}{2^{n-1}}\left(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\right)e_1 - \frac{1}{2^{n-2}}\left(\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha-\beta}\right)e_2$$
$$-\cdots-\frac{1}{2^{n-15}}\left(\frac{\alpha^{n-15}-\beta^{n-15}}{\alpha-\beta}\right)e_{15}$$
$$= -\frac{1}{2^n}\left(\frac{\alpha^n e_0+2\alpha^{n-1}e_1+2^2\alpha^{n-2}e_2+2^3\alpha^{n-3}e_3+\cdots+2^{15}\alpha^{n-15}e_{15}}{\alpha-\beta}\right.$$
$$\left.-\frac{\beta^n e_0+2\beta^{n-1}e_1+2^2\beta^{n-2}e_2+2^3\beta^{n-3}e_3+\cdots+2^{15}\beta^{n-15}e_{15}}{\alpha-\beta}\right)$$
$$= -\frac{1}{2^n}\left(\frac{\alpha^n(e_0+\beta e_1+\beta^2e_2+\beta^3e_3+\cdots+\beta^{15}e_{15})}{\alpha-\beta} - \frac{\beta^n(e_0+\alpha e_1+\alpha^2e_2+\alpha^3e_3+\cdots+\alpha^{15}e_{15})}{\alpha-\beta}\right)$$
$$= -\frac{1}{2^n}\frac{B\alpha^n-A\beta^n}{\alpha-\beta}. \tag{18}$$

6. From Eq.(6) and (11), we get the Binet's formula of negative version of k −Mersenne−Lucas sedenions as

$$\widehat{ML_{k,-n}} = \frac{1}{2^n}(\alpha^n+\beta^n) + \frac{1}{2^{n-1}}(\alpha^{n-1}+\beta^{n-1})e_1 + \frac{1}{2^{n-2}}(\alpha^{n-2}+\beta^{n-2})e_2$$
$$+\cdots+\frac{1}{2^{n-15}}(\alpha^{n-15}+\beta^{n-15})e_{15}$$
$$= \frac{1}{2^n}[(\alpha^n+\beta^n) + 2(\alpha^{n-1}+\beta^{n-1})e_1 + 2^2(\alpha^{n-2}+\beta^{n-2})e_2$$
$$+\cdots+2^{15}(\alpha^{n-15}+\beta^{n-15})e_{15}]$$
$$= \frac{1}{2^n}[(\alpha^n + 2\alpha^{n-1}e_1 + 2^2\alpha^{n-2}e_2 + \cdots + 2^{15}\alpha^{n-15}e_{15})$$

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$$+(\beta^n + 2\beta^{n-1}e_1 + 2^2\beta^{n-2}e_2 + \cdots + 2^{15}\beta^{n-15}e_{15})]$$
$$= \frac{1}{2^n}[\alpha^n(1 + \beta e_1 + \beta^2 e_2 + \cdots + \beta^{15}e_{15})$$
$$+\beta^n(1 + \alpha e_1 + \alpha^2 e_2 + \cdots + \alpha^{15}e_{15})]$$
$$= \frac{1}{2^n}(B\alpha^n + A\beta^n). \qquad (19)$$

Theorem 4. For $r, n, s \epsilon \mathbb{N}$ and $s \geq r$, the generating functions of $k-$Mersenne sedenions and $k-$Mersenne$-$Lucas sedenions are given respectively as

1. $f(x) = \sum_{n=0}^{\infty} M_{k,n}x^n = \frac{\widehat{M_{k,0}} + (\widehat{M_{k,1}} - 3k\widehat{M_{k,0}})x}{1-3kx+2x^2}$,
2. $g(x) = \sum_{n=0}^{\infty} ML_{k,n}x^n = \frac{\widehat{ML_{k,0}} + (\widehat{ML_{k,1}} - 3k\widehat{ML_{k,0}})x}{1-3kx+2x^2}$.

Proof:

1. Let $f(x) = \sum_{n=0}^{\infty} M_{k,n}x^n$ be given. We can write

$$f(x) = \widehat{M_{k,0}} + \widehat{M_{k,1}}x + \widehat{M_{k,2}}x^2 + \cdots$$
$$-3kxf(x) = -\left(3kx\widehat{M_{k,0}} + 3kx^2\widehat{M_{k,1}} + 3kx^3\widehat{M_{k,2}} + \cdots\right)$$
$$2x^2f(x) = 2x^2\widehat{M_{k,0}} + 2x^3\widehat{M_{k,1}} + 2x^4\widehat{M_{k,2}} + \cdots.$$

Thus, we get the following result:

$$(1 - 3kx + 2x^2)f(x) = \widehat{M_{k,0}} + \left(\widehat{M_{k,1}} - 3k\widehat{M_{k,0}}\right)x$$
$$f(x) = \frac{\widehat{M_{k,0}} + \left(\widehat{M_{k,1}} - 3k\widehat{M_{k,0}}\right)x}{1 - 3kx + 2x^2}.$$

2. Similarly, let $g(x) = \sum_{n=0}^{\infty} ML_{k,n}x^n$ be given. Then

$$g(x) = \widehat{ML_{k,0}} + \widehat{ML_{k,1}}x + \widehat{ML_{k,2}}x^2 + \cdots$$
$$-3kxg(x) = -\left(3kx\widehat{ML_{k,0}} + 3kx^2\widehat{ML_{k,1}} + 3kx^3\widehat{ML_{k,2}} + \cdots\right)$$
$$2x^2g(x) = 2x^2\widehat{ML_{k,0}} + 2x^3\widehat{ML_{k,1}} + 2x^4\widehat{ML_{k,2}} + \cdots.$$

Thus, we get the following result:

$$(1 - 3kx + 2x^2)g(x) = \widehat{ML_{k,0}} + \left(\widehat{ML_{k,1}} - 3k\widehat{ML_{k,0}}\right)x$$
$$g(x) = \frac{\widehat{ML_{k,0}} + \left(\widehat{ML_{k,1}} - 3k\widehat{ML_{k,0}}\right)x}{1 - 3kx + 2x^2}.$$

Theorem 5. For $k \neq 1$, the finite sum formulas of these sedenions can be written as follows:

1. $\sum_{s=0}^{n} \widehat{M_{k,s}} = \frac{2\widehat{M_{k,n}} - \widehat{M_{k,n+1}} + \widehat{M_{k,1}} + (1-3k)\widehat{M_{k,0}}}{3(1-k)}$,
2. $\sum_{s=0}^{n} \widehat{ML_{k,s}} = \frac{2\widehat{ML_{k,n}} - \widehat{ML_{k,n+1}} + \widehat{M_{k,1}} + (1-3k)\widehat{ML_{k,0}}}{3(1-k)}$.

Proof:

1. Let $k \neq 1$. From Eq.(8) and (14), we get the finite sum formula of $k-$Mersenne sedenions as follows:

$$\sum_{s=0}^{n} \widehat{M_{k,s}} = \sum_{s=0}^{n}\left(\frac{A\alpha^s - B\beta^s}{\alpha-\beta}\right) = \left(\frac{A}{\alpha-\beta}\right)\sum_{s=0}^{n}\alpha^s - \left(\frac{B}{\alpha-\beta}\right)\sum_{s=0}^{n}\beta^s$$
$$= \frac{A}{\alpha-\beta}\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) - \frac{B}{\alpha-\beta}\left(\frac{1-\beta^{n+1}}{1-\beta}\right)$$
$$= \frac{A\left(1-\beta-\alpha^{n+1}-\alpha^{n+1}\beta\right) - B\left(1-\alpha-\beta^{n+1}-\beta^{n+1}\alpha\right)}{(\alpha-\beta)(1-(\alpha+\beta)+\alpha\beta)}$$

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$$= \frac{A-B-\alpha\beta\left(A\alpha^{-1}-B\beta^{-1}\right)-\left(A\alpha^{n+1}-B\beta^{n+1}\right)+\alpha\beta\left(A\alpha^{n}-B\beta^{n}\right)}{(\alpha-\beta)3(1-k)}$$
$$= \frac{\widehat{M_{k,0}}-2\widehat{M_{k,-1}}-\widehat{M_{k,n+1}}+2\widehat{M_{k,n}}}{3(1-k)}$$
$$= \frac{2\widehat{M_{k,n}}-\widehat{M_{k,n+1}}+\widehat{M_{k,1}}+(1-3k)\widehat{M_{k,0}}}{3(1-k)}. \tag{20}$$

2. Let $k \neq 1$. From Eq.(9) and (15), we get the finite sum formula of k −Mersenne−Lucas sedenions as follows:

$$\sum_{s=0}^{n} \widehat{ML_{k,s}} = \sum_{s=0}^{n}(A\alpha^s + B\beta^s) = A\sum_{s=0}^{n}\alpha^s + B\sum_{s=0}^{n}\beta^s$$
$$= A\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) + B\left(\frac{1-\beta^{n+1}}{1-\beta}\right)$$
$$= \frac{A\left(1-\beta-\alpha^{n+1}-\alpha^{n+1}\beta\right)+B\left(1-\alpha-\beta^{n+1}-\beta^{n+1}\alpha\right)}{(1-(\alpha+\beta)+\alpha\beta)}$$
$$= \frac{A+B-\alpha\beta\left(A\alpha^{-1}+B\beta^{-1}\right)-\left(A\alpha^{n+1}+B\beta^{n+1}\right)+\alpha\beta\left(A\alpha^{n}+B\beta^{n}\right)}{3(1-k)}$$
$$= \frac{\widehat{ML_{k,0}}-2\widehat{ML_{k,-1}}-\widehat{ML_{k,n+1}}+2\widehat{ML_{k,n}}}{3(1-k)}$$
$$= \frac{2\widehat{ML_{k,n}}-\widehat{ML_{k,n+1}}+\widehat{ML_{k,1}}+(1-3k)\widehat{ML_{k,0}}}{3(1-k)}. \tag{21}$$

Lemma 1. Let $\mathrm{A} = \sum_{s=0}^{15}\alpha^s e_s$, $\mathrm{B} = \sum_{s=0}^{15}\beta^s e_s$ be given. AB and BA can be calculated by

$$\begin{aligned} AB &= \left(\sum_{s=0}^{15}\alpha^s e_s\right)\left(\sum_{s=0}^{15}\beta^s e_s\right) \\ &= -65533e_0 + (14477\beta - 11469\alpha)e_1 + (11475\alpha - 11473\beta)e_2 \\ &\quad +\left(3825\alpha^3 - 3823\beta^3 - 7650(\alpha-\beta)\right)e_3 + (2083\beta^4 - 3825\alpha^4)e_4 \\ &\quad +\left(1277\beta^5 - 1275\alpha^5 + 2550(\alpha^3-\beta^3)\right)e_5 \\ &\quad +\left(255\alpha^6 - 253\beta^6 - 1020(\alpha^2-\beta^2)\right)e_6 \\ &\quad +\left(257\beta^7 - 253\alpha^7 - 510(\alpha^5-\beta^5) + 1020(\alpha^3-\beta^3) + 2040(\alpha-\beta)\right)e_7 \\ &\quad +(255\alpha^8 - 253\beta^8)e_8 + \left(45\beta^9 - 43\alpha^9 + 86(\alpha^7-\beta^7)\right)e_9 \\ &\quad +\left(47\alpha^{10} - 45\beta^{10} - 86(\alpha^6-\beta^6)\right)e_{10} \\ &\quad +\left(17\alpha^{11} - 15\beta^{11} - 30(\alpha^9-\beta^9) + 60(\alpha^7-\beta^7) - 136(\alpha^5-\beta^5)\right)e_{11} \\ &\quad +\left(15\beta^{12} - 13\alpha^{12} + 208(\alpha^4-\beta^4)\right)e_{12} \\ &\quad +\left(5\beta^{13} - 3\alpha^{13} + 10(\alpha^{11}-\beta^{11}) - 80(\alpha^5-\beta^5) + 96(\alpha^3-\beta^3)\right)e_{13} \\ &\quad +\left(\beta^{14} + 3\alpha^{14} + 2\beta^{13} - 4(\alpha^{10}-\beta^{10}) + 16(\alpha^6-\beta^6) - 192(\alpha^2-\beta^2)\right)e_{14} \\ &\quad +(\beta^{15} + \alpha^{15} - 2\beta^{14} - 2\alpha^{13} + 4(\alpha^{11}-\beta^{11}) - 8(\alpha^9+\beta^9) - 16\alpha^7 - \beta^7) \\ &\quad -32(\alpha^5-\beta^5) + 64(\alpha^3-\beta^3) - 128(\alpha-\beta)\big)e_{15}. \end{aligned} \tag{22}$$

$$\begin{aligned} BA &= \left(\sum_{s=0}^{15}\beta^s e_s\right)\left(\sum_{s=0}^{15}\alpha^s e_s\right) \\ &= -65533e_0 + (14477\alpha - 11469\beta)e_1 + (11475\beta - 11473\alpha)e_2 \\ &\quad +\left(3825\beta^3 - 3823\alpha^3 - 7650(\beta-\alpha)\right)e_3 + (2083\alpha^4 - 3825\beta^4)e_4 \\ &\quad +\left(1277\alpha^5 - 1275\beta^5 + 2550(\beta^3-\alpha^3)\right)e_5 \\ &\quad +\left(255\beta^6 - 253\alpha^6 - 1020(\beta^2-\alpha^2)\right)e_6 \\ &\quad +\left(257\alpha^7 - 253\beta^7 - 510(\beta^5-\alpha^5) + 1020(\beta^3-\alpha^3) + 2040(\beta-\alpha)\right)e_7 \\ &\quad +(255\beta^8 - 253\alpha^8)e_8 + \left(45\alpha^9 - 43\beta^9 + 86(\beta^7-\alpha^7)\right)e_9 \\ &\quad +\left(47\alpha^{10} - 45\beta^{10} - 86(\beta^6-\alpha^6)\right)e_{10} \\ &\quad +\left(17\beta^{11} - 15\alpha^{11} - 30(\beta^9-\alpha^9) + 60(\beta^7-\alpha^7) - 136(\beta^5-\alpha^5)\right)e_{11} \end{aligned}$$

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$$+\left(15\alpha^{12} - 13\beta^{12} + 208(\beta^4 - \alpha^4)\right)e_{12}$$
$$+\left(5\alpha^{13} - 3\beta^{13} + 10(\beta^{11} - \alpha^{11}) - 80(\beta^5 - \alpha^5) + 96(\beta^3 - \alpha^3)\right)e_{13}$$
$$+\left(\alpha^{14} + 3\beta^{14} + 2\alpha^{13} - 4(\beta^{10} - \alpha^{10}) + 16(\beta^6 - \alpha^6) - 192(\beta^2 - \alpha^2)\right)e_{14}$$
$$+\left(\alpha^{15} + \beta^{15} - 2\alpha^{14} - 2\beta^{13} + 4(\beta^{11} - \beta^{11}) - 8(\beta^9 + \alpha^9) - 16(\beta^7 - \alpha^7)\right)$$
$$-32(\beta^5 - \alpha^5) + 64(\beta^3 - \alpha^3) - 128(\beta - \alpha)\Big)e_{15}. \tag{23}$$

SOME RELATIONS BETWEEN SPECIAL IDENTITIES WITH $k-$MERSENNE SEDENIONS AND $k-$MERSENNE−LUCAS SEDENIONS

In this section some special identities in number theory are examined. Interesting relations between these identities and *k*-Mersenne sedenions and *k*-Mersenne-Lucas sedenions are obtained by using definitions given in the previous section.

Theorem 6 (Catalan's Identity). For $n, r \in \mathbb{N}^+$ such that $n \geq r$, Catalan's identities for $k-$Mersenne sedenions and $k-$Mersenne−Lucas sedenions can be given respectively as

1. $\widehat{M_{k,n-r}}\widehat{M_{k,n+r}} - \widehat{M_{k,n}}^2 = 2^{n-r} M_{k,r} \frac{(AB\beta^r - BA\alpha^r)}{\sqrt{9k^2-8}}$,
2. $\widehat{ML_{k,n-r}}\widehat{ML_{k,n+r}} - \widehat{ML_{k,n}}^2 = 2^{n-r} M_{k,r}\sqrt{9k^2-8}(BA\alpha^r - AB\beta^r)$.

Proof:

1. If we use Eq.(14) and Lemma 1, we can write the following equation:

$$\begin{aligned}
\widehat{M_{k,n-r}}\widehat{M_{k,n+r}} - \widehat{M_{k,n}}^2 &= \left(\frac{A\alpha^{n-r} - B\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{A\alpha^{n+r} - B\beta^{n+r}}{\alpha-\beta}\right) - \left(\frac{A\alpha^n - B\beta^n}{\alpha-\beta}\right)^2 \\
&= \frac{1}{(\alpha-\beta)^2}\left[AB\alpha^n\beta^n\left(1 - \frac{\beta^r}{\alpha^r}\right) + BA\alpha^n\beta^n\left(1 - \frac{\alpha^r}{\beta^r}\right)\right] \\
&= \frac{\alpha^n\beta^n(\alpha^r - \beta^r)}{(\alpha-\beta)^2} \cdot \frac{AB\beta^r - BA\alpha^r}{\alpha^r\beta^r} \\
&= (\alpha\beta)^{n-r} M_{k,r} \frac{(AB\beta^r - BA\alpha^r)}{\alpha-\beta} \\
&= 2^{n-r} M_{k,r} \frac{(AB\beta^r - BA\alpha^r)}{\sqrt{9k^2-8}}.
\end{aligned} \tag{24}$$

2. If we use Eq.(15) and Lemma 1, we can write the following equation:

$$\begin{aligned}
\widehat{ML_{k,n-r}}\widehat{ML_{k,n+r}} - \widehat{ML_{k,n}}^2 &= (A\alpha^{n-r} + B\beta^{n-r})(A\alpha^{n+r} + B\beta^{n+r}) - (A\alpha^n + B\beta^n)^2 \\
&= AB\alpha^n\beta^n\left(\frac{\beta^r}{\alpha^r} - 1\right) + BA\alpha^n\beta^n\left(\frac{\alpha^r}{\beta^r} - 1\right) \\
&= \alpha^n\beta^n\left[BA\left(\frac{\alpha^r}{\beta^r} - 1\right) - AB\left(1 - \frac{\beta^r}{\alpha^r}\right)\right] \\
&= \alpha^n\beta^n\left[\frac{BA\alpha^r(\alpha^r - \beta^r) - AB\beta^r(\alpha^r - \beta^r)}{\alpha^r\beta^r}\right] \\
&= 2^{n-r}(\alpha^r - \beta^r)[BA\alpha^r - AB\beta^r] \\
&= 2^{n-r} M_{k,r}\sqrt{9k^2-8}(BA\alpha^r - AB\beta^r).
\end{aligned} \tag{25}$$

Theorem 7 (Cassini's Identity). For $n \geq 1$, Cassini's identities for $k-$Mersenne sedenions and $k-$Mersenne−Lucas sedenions can be given respectively as

1. $\widehat{M_{k,n-1}}\widehat{M_{k,n+1}} - \widehat{M_{k,n}}^2 = 2^{n-1} \frac{(AB\beta - BA\alpha)}{\sqrt{9k^2-8}}$,
2. $\widehat{ML_{k,n-1}}\widehat{ML_{k,n+1}} - \widehat{ML_{k,n}}^2 = 2^{n-1}\sqrt{9k^2-8}(BA\alpha - AB\beta)$.

Proof: By substituting $r = 1$ in the Catalan's identity, we obtain required results.

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Theorem 8 (D'Ocagne's Identity). For $n, r \geq 0$, D'Ocagne's identities for $k-$Mersenne sedenions and $k-$Mersenne$-$Lucas sedenions can be given respectively as

1. $\widehat{M_{k,r}}\widehat{M_{k,n+1}} - \widehat{M_{k,r+1}}\widehat{M_{k,n}} = \frac{2^n(AB\alpha^{r-n}-BA\beta^{r-n})}{\sqrt{9k^2-8}}$,
2. $\widehat{ML_{k,r}}\widehat{ML_{k,n+1}} - \widehat{ML_{k,r+1}}\widehat{ML_{k,n}} = 2^n\sqrt{9k^2-8}(BA\beta^{r-n} - AB\alpha^{r-n})$.

Proof:

1. It can be written using Eq.(14) and Lemma 1 as follows:

$$\begin{aligned}\widehat{M_{k,r}}\widehat{M_{k,n+1}} - \widehat{M_{k,r+1}}\widehat{M_{k,n}} &= \left(\frac{A\alpha^r-B\beta^r}{\alpha-\beta}\right)\left(\frac{A\alpha^{n+1}-B\beta^{n+1}}{\alpha-\beta}\right) - \left(\frac{A\alpha^{r+1}-B\beta^{r+1}}{\alpha-\beta}\right)\left(\frac{A\alpha^n-B\beta^n}{\alpha-\beta}\right)\\ &= \frac{1}{(\alpha-\beta)^2}(-AB\alpha^r\beta^{n+1} - BA\beta^r\alpha^{n+1} + AB\alpha^{r+1}\beta^n + BA\beta^{r+1}\alpha^n)\\ &= \frac{1}{(\alpha-\beta)^2}\big(AB\alpha^r\beta^n(\alpha-\beta) - BA\beta^r\alpha^n(\alpha-\beta)\big)\\ &= \frac{2^n(AB\alpha^{r-n}-BA\beta^{r-n})}{\sqrt{9k^2-8}}. \qquad (26)\end{aligned}$$

2. If we use Eq.(15) and Lemma 1, then it can be written as follows:

$$\begin{aligned}\widehat{ML_{k,r}}\widehat{ML_{k,n+1}} - \widehat{ML_{k,r+1}}\widehat{ML_{k,n}} &= (A\alpha^r + B\beta^r)(A\alpha^{n+1} + B\beta^{n+1})\\ &\quad -(A\alpha^{r+1} + B\beta^{r+1})(A\alpha^n + B\beta^n)\\ &= AB\alpha^r\beta^n(\beta-\alpha) + BA\beta^r\alpha^n(\beta-\alpha)\\ &= 2^n\sqrt{9k^2-8}(BA\beta^{r-n} - AB\alpha^{r-n}). \qquad (27)\end{aligned}$$

Theorem 9 (Vajda's Identity). For $r, t, n \in \mathbb{N}^+$, Vajda's identities for $k-$Mersenne sedenions and $k-$Mersenne$-$Lucas sedenions can be given respectively by the following equations:

1. $\widehat{M_{k,n+r}}\widehat{M_{k,n+t}} - \widehat{M_{k,n}}\widehat{M_{k,n+r+t}} = \frac{-2^n M_{k,r}(AB\beta^k-BA\alpha^k)}{\sqrt{9k^2-8}}$,
2. $\widehat{ML_{k,n+r}}\widehat{ML_{k,n+t}} - \widehat{ML_{k,n}}\widehat{ML_{k,n+r+t}} = 2^n M_{k,r}\sqrt{9k^2-8}(AB\beta^k + BA\alpha^k)$.

Proof:

1. If we use Eq.(14) and Lemma 1, then it can be written as follows:

$$\begin{aligned}\widehat{M_{k,n+r}}\widehat{M_{k,n+t}} - \widehat{M_{k,n}}\widehat{M_{k,n+r+t}} &= \left(\frac{A\alpha^{n+r}-B\beta^{n+r}}{\alpha-\beta}\right)\left(\frac{A\alpha^{n+t}-B\beta^{n+t}}{\alpha-\beta}\right)\\ &\quad -\left(\frac{A\alpha^n-B\beta^n}{\alpha-\beta}\right)\left(\frac{A\alpha^{n+r+t}-B\beta^{n+r+t}}{\alpha-\beta}\right)\\ &= \frac{1}{(\alpha-\beta)^2}[AB\alpha^n\beta^{n+k}(\beta^r-\alpha^r) - BA\alpha^{n+k}\beta^n(\beta^r-\alpha^r)]\\ &= \frac{-2^n M_{k,r}(AB\beta^k-BA\alpha^k)}{\sqrt{9k^2-8}}. \qquad (28)\end{aligned}$$

2. If we use Eq.(15) and Lemma 1, then it can be written as follows:

$$\begin{aligned}\widehat{ML_{k,n+r}}\widehat{ML_{k,n+t}} - \widehat{ML_{k,n}}\widehat{ML_{k,n+r+t}} &= (A\alpha^{n+r} + B\beta^{n+r})(A\alpha^{n+t} + B\beta^{n+t})\\ &\quad -(A\alpha^n + B\beta^n)(A\alpha^{n+r+t} + B\beta^{n+r+t})\\ &= AB\alpha^n\beta^{n+t}(\alpha^r-\beta^r) + BA\alpha^{n+t}\beta^n(\alpha^r-\beta^r)\\ &= \alpha^n\beta^n(\alpha^r-\beta^r)[AB\beta^t + BA\alpha^t]\\ &= M_{k,r}2^n\sqrt{9k^2-8}(AB\beta^t + BA\alpha^t). \qquad (29)\end{aligned}$$

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CONCLUSIONS

In this article we first provided the general information about sedenions, $k-$Mersenne numbers and $k-$Mersenne$-$Lucas numbers. Then we defined the $k-$Mersenne sedenions and $k-$Mersenne$-$Lucas sedenions. Subsequently, we obtained Binet's formula, norm, conjugate and generating functions of these sedenion sequences. Following that, we got the finite sum formula of the $k-$Mersenne and $k-$Mersenne $-$Lucas sedenions. Finally, some information is given about special identities in number theory. Based on this information and new definitions, we established relations between Catalan's, Cassini's, D'Ocagne's and Vajda's identities and $k-$ Mersenne sedenions and $k-$Mersenne$-$Lucas sedenions.

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