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Full Paper

Tubular involutive surfaces with Frenet frame in Euclidean 3-space

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Abstract: In this study we first define the tubular involutive surfaces as a new surface form. We then investigate singularity, Gaussian curvature and mean curvatures of the tubular involutive surfaces and get an interesting relation between these curvatures as $2H = -(rK + 1/r)$. By calculating the Gaussian and mean curvatures of the tubular involutive surfaces, we find the necessary conditions of being flat or minimal of these surfaces. In addition, we analyse the necessary and sufficient conditions for parameter curves on the surface to be asymptotic, geodesic and line of curvature. Finally, we illustrate our method by presenting two examples.

Keywords: tubular surface, involute curve, singularity, Frenet frame

INTRODUCTION

Canal surfaces were first described in 1850 by the French mathematician Gaspard Monge [1]. The canal surface is defined as the envelope of a moving sphere with a variable radius. If the radius function of the movable sphere forming the canal surface is constant, the canal surface is called the tubular surface. These types of surfaces are used to represent pipes, ropes, poles and 3-dimensional castings that we encounter in daily life. Again, these surfaces are useful in planning the lines of motion of robots, showing long thin objects, human internal organs, surface modelling for computer-aided design and computer-aided manufacturing.

Maekawa et al. [2] investigated necessary and sufficient conditions for the nonsingularity of tubular surfaces. Blaga [3] considered tubular surfaces as swept surfaces and gave a parametric representation of the inverse of a canal surface. Dogan and Yayli [4] defined tubular surface with respect to the Bishop frame and gave some characterisations regarding special curves lying on it.

Dede [5] defined the tubular surfaces according to the Flc frame. Ates et al. [6] designed the tubular surface formed by the spherical indicators of any space curve for the alternate moving frame. Akyigit et al. [7] defined the tubular surfaces using the modified orthogonal frame to give researchers an alternative perspective. Tubular and canal surfaces have been handled by many authors so far [8-16]. Also, in order to synthesise interdisciplinary studies, it is useful to note some important studies on ruled surfaces in different spaces [17-37].

In the present paper we define tubular involutive surfaces according to the Frenet frame as a new surface form and examine their characteristic properties. In this sense, we first investigate singularity, Gaussian and mean curvatures of these surfaces. Then using the Gaussian and mean curvatures, we obtain necessary conditions of being flat or minimal of these surfaces. Also, we analyse the necessary and sufficient conditions for parameter curves on the surface to be asymptotic, geodesic and line of curvature. At the end of this article, we visualise the main idea by providing two examples.

PRELIMINARIES

The general concepts in this section are taken from Do Carmo [38]. Let $\gamma(s), s \in [0, L]$ be a regular 3D curve with curvature $\kappa(s)$ and torsion $\tau(s)$. In this paper $\gamma(s)'$ denotes the derivative of $\gamma(s)$ with respect to arc length parameter s. We assume that $\gamma(s)'' \neq 0$, which means that $\kappa(s) \neq 0$ and the Frenet frame $\{T(s), N(s), B(s)\}$ along $\gamma(s)$ is defined. If we assume that $\gamma(s)'' \neq 0$, then we can write $N(s) = \frac{\gamma(s)''}{\|\gamma(s)''\|}$ for the normal vector and $B(s) = T(s) \times N(s)$ for the binormal vector, where $T(s) = \gamma(s)'$ is the tangent vector. The Serret-Frenet equations are given by the following relations

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where $\kappa(s) = \|\gamma(s)''\|$ and $\tau(s) = \langle B'(s), N(s) \rangle$ are called the curvature and torsion of the curve $\gamma(s)$ respectively.

Let $\gamma(s)$ and $\bar{\gamma}(s), s \in [0, L]$ be two curves such that $\bar{\gamma}(s)$ intersects the tangents of $\gamma(s)$ orthogonally. Then $\bar{\gamma}(s)$ is called an involute of $\gamma(s)$. An involute of a curve $\gamma(s)$ with arc length s is given by

$$\bar{\gamma}: \bar{\gamma}(s) = \gamma(s) + \mu T(s), \tag{1}$$

where $\mu = c - s$, c being a real constant, and $T(s)$ is the unit tangent vector of $\gamma: \gamma(s)$. For convenience, we suppose that $\mu \neq 0$. If $\bar{\gamma}$ is the involute of γ, then the relationship between the Frenet frames of involute-evolute curve pair $(\bar{\gamma}, \gamma)$ is given by

$$\begin{bmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \\ cos\theta & 0 & sin\theta \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2}$$

where θ is the angle between $T(s)$ and $B^*(s)$.

A canal surface is briefly called the envelope of a one-parameter family of spheres. More precisely, it can be defined as the envelope of a moving sphere of variable radius, with a curve (spine curve) $\gamma(s)$ as the orbit of its centre and a radius function $r(s)$. If $r(s)$ is a constant function, then the canal surface is called a tubular surface. For example, if $\gamma(s)$ is a circle, then the corresponding tubular surface is a torus. The tubular base surface $Q(s,\vartheta)$, which passes through a given 3D space curve $\gamma(s)$, can be expressed as

$$M: Q(s,\vartheta) = \gamma(s) + r\big(cos\vartheta N(s) + sin\vartheta B(s)\big), \tag{3}$$

where $N(s)$ and $B(s)$ are the principal normal and binormal vectors of $\gamma(s)$ respectively.

TUBULAR INVOLUTIVE SURFACE WITH FRENET FRAME

In this section we define a new type of tubular surfaces. We assume that the spine curve of the tubular surface is the involute curve $\bar{\gamma}$ and call this surface a tubular involutive surface $\bar{Q}(s,\vartheta)$. We can easily write the following equation according to the connection between the $\bar{Q}(s,\vartheta)$ and a family of spheres which are great circles C_λ of the unit spheres lying in the sub-space $Sp\{N^*(s), B^*(s)\}$ of the spine curve $\bar{\gamma}(s)$ (Figure 1):

$$\bar{M}: \bar{Q}(s,\vartheta) = \bar{\gamma}(s) + r\big(cos\vartheta N^*(s) + sin\vartheta B^*(s)\big). \tag{4}$$

Using equations (1) and (2) in equation (4), we get the equation of $\bar{M}$ as follows:

$$\bar{M}: \bar{Q}(s,\vartheta) = \gamma(s) + \mu T(s) + r\big(sin\psi T(s) + cos\psi B(s)\big), \tag{5}$$

where $\psi = \vartheta - \theta$ is the angle between B and the position vector $\vec{\bar{Q}}$ of the characteristic circles C_λ lying in the plane spanned by $\{N^*, B^*\}$ (Figure 2).

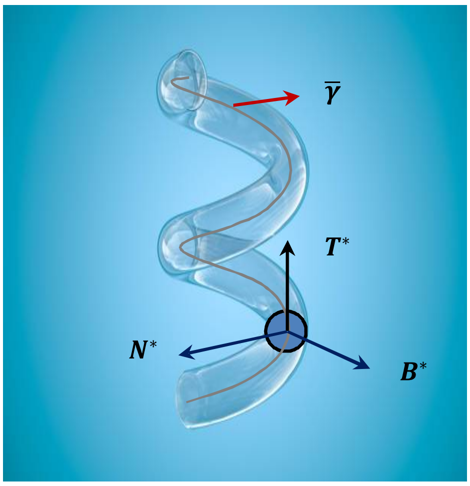

Figure 1. Representation of the surface $\bar{M}$

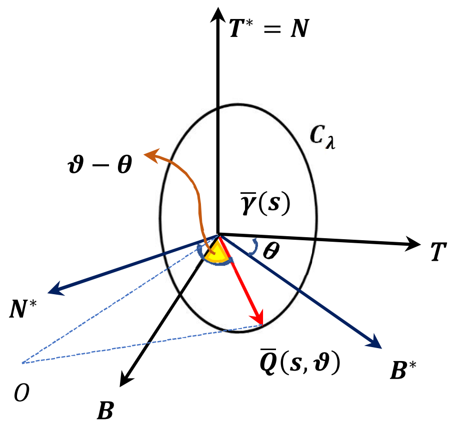

Figure 2. Frenet frame of $(\bar{\gamma},\gamma)$ and C_λ

Properties of Tubular Involutive Surfaces

In this section the geometric properties of the tubular involutive surfaces with Frenet frame is examined and the conditions of being minimal or flat are determined. Afterward, the conditions for parameter curves on the surface of being geodesic, asymptotic and line of curvature are investigated.

The surface $\bar{M}$ at a distance r $(r > 0)$ from the spine curve $\bar{\gamma}$ is represented by equation (5). If we take the partial derivatives of $\bar{M}$ with respect to s and ϑ, we get the following tangent vectors of $\bar{M}$

$$\begin{cases} \bar{Q}_s = -r\theta' cos\psi T + \delta N + r\theta' sin\psi B \\ \bar{Q}_\vartheta = rcos\psi T - rsin\psi B\,, \end{cases} \tag{6}$$

where

$$\delta : \delta(s,\vartheta) = \mu\kappa + r(\kappa sin\psi - \tau cos\psi). \tag{7}$$

By taking the vector product of $\bar{Q}_s$ and $\bar{Q}_\vartheta$, we get

$$\bar{Q}_s \times \bar{Q}_\vartheta = -r\delta(sin\psi T + cos\psi B). \tag{8}$$

Thus,

$$\|\bar{Q}_s \times \bar{Q}_\vartheta\| = r|\delta|. \tag{9}$$

From equations (8) and (9), we can get the unit normal vector of $\bar{M}$:

$$\xi(s,\vartheta) = \frac{\bar{Q}_s \times \bar{Q}_\vartheta}{\|\bar{Q}_s \times \bar{Q}_\vartheta\|} = \pm(sin\psi T + cos\psi B). \tag{10}$$

We know well that the condition $\bar{Q}_s \times \bar{Q}_\vartheta = 0$ indicates singular points for a surface. Then using equation (8), we can give the following result for the surface $\bar{M}$.

Corollary 1. The tubular involutive surface $\bar{M}$ has singular points if and only if the conditions $\delta = 0$ is satisfied.

Using equation (6), the components of the first fundamental form are obtained by

$$\begin{cases} E = \langle \bar{Q}_s, \bar{Q}_s \rangle = r^2\theta'^2 + \delta^2, \\ F = \langle \bar{Q}_s, \bar{Q}_\vartheta \rangle = -r^2\theta', \\ G = \langle \bar{Q}_\vartheta, \bar{Q}_\vartheta \rangle = r^2. \end{cases} \tag{11}$$

Theorem 1. Tubular involutive surface $\bar{M}$ is a regular surface if and only if δ is not equal to zero.

Proof: If the surface $\bar{M}$ is a regular surface, then we know that $EG - F^2 \neq 0$. By using equation (11), we can write

$$EG - F^2 = r^2\delta^2$$

for the surface $\bar{M}$. Since $r > 0$, we get

$$\delta \neq 0.$$

Conversely, if δ is not equal to zero, then we can easily see that the surface is regular. Thus, we give the following corollary.

Corollary 2. $\bar{M}$ is a regular surface if and only if $r \neq \frac{\mu\kappa}{\tau cos\psi - \kappa sin\psi}$.

To obtain the components of the second fundamental form of $\bar{M}$, we must calculate the following:

$$\begin{cases} \bar{Q}_{ss} = -\left(r\theta'' cos\psi + r\theta'^2 sin\psi + \delta\kappa\right)T + (\delta_s - r\theta'\kappa cos\psi - r\theta'\tau sin\psi)N \\ \qquad + \left(\delta\tau + r\theta'' sin\psi - r\theta'^2 cos\psi\right)B, \\ \bar{Q}_{s\vartheta} = r\theta' sin\psi T + \delta_\vartheta N + r\theta' cos\psi B, \\ \bar{Q}_{\vartheta\vartheta} = -r sin\psi T - r cos\psi B, \end{cases} \tag{12}$$

where

$$\delta_s = -\kappa + \mu\kappa' + r sin\psi(\kappa' - \theta'\tau) - r cos\psi(\tau' + \theta'\kappa), \;\; \delta_\vartheta = r(\kappa cos\psi + \tau sin\psi).$$

Thus we can give the following theorem.

Theorem 2. The Gaussian and mean curvatures of the tubular involutive surface $\bar{M}$ are respectively

$$\begin{cases} K = \frac{\kappa sin\psi - \tau cos\psi}{r\delta}, \\ H = \frac{r(\tau cos\psi - \kappa sin\psi) - \delta}{2r\delta}. \end{cases} \tag{13}$$

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Proof: From equation (12) we can compute the components $L = \langle \bar{Q}_{ss}, \xi \rangle, M = \langle \bar{Q}_{s\vartheta}, \xi \rangle$ and $N = \langle \bar{Q}_{\vartheta\vartheta}, \xi \rangle$ of the second fundamental form as

$$\begin{cases} L = -r\theta'^2 + \delta(\tau cos\psi - \kappa sin\psi), \\ M = r\theta', \\ N = -r. \end{cases} \tag{14}$$

We know well that the Gaussian and mean curvatures of a surface are given by

$$K = k_1 k_2 = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{1}{2}(k_1 + k_2) = \frac{EN + GL - 2FM}{2(EG - F^2)}, \tag{15}$$

where k_1, k_2 are the principal curvatures of the surface $\bar{M}$. By using equations (11), (14) in (15), the Gaussian and mean curvatures of the tubular involutive surface $\bar{M}$ are obtained by

$$K = \frac{\kappa sin\psi - \tau cos\psi}{r\delta}$$

and

$$H = \frac{r(\tau cos\psi - \kappa sin\psi) - \delta}{2r\delta}.$$

Remark 1. Observe that using equation (7), the Gaussian and mean curvatures of the tubular involutive surface $\bar{M}$ can be expressed as

$$\begin{cases} K = \dfrac{\delta - \mu\kappa}{r^2\delta}, \\ H = \dfrac{\mu\kappa - 2\delta}{2r\delta}, \end{cases}$$

where $\mu = c - s$ and c is a real constant.

Theorem 3. The Gaussian curvature K and the mean curvature H of the tubular involutive surface $\bar{M}$ satisfy

$$H = -\frac{1}{2}\left(rK + \frac{1}{r}\right).$$

Proof: Using equation (13) or Remark 1, the theorem is easily proved.

Corollary 3. The principal curvatures of the surface $\bar{M}$ are given by

$$\begin{cases} k_1 = -rK, \\ k_2 = -\dfrac{1}{r}. \end{cases}$$

Proof: We know that the roots of the quadratic equation

$$k^2 - 2kH + K = 0$$

for the surface $\bar{M}$ are the principal curvatures of

$$k_1 = H + \sqrt{H^2 - K} \quad \text{and} \quad k_2 = H - \sqrt{H^2 - K}.$$

Thus, from equations (7), (13) and the last equalities, we can give the principal curvatures of $\bar{M}$ as follows:

$$k_1 = \frac{\mu\kappa - \delta}{r\delta} = -r\frac{\delta - \mu\kappa}{r^2\delta} = -rK \quad \text{and} \quad k_2 = -\frac{1}{r}.$$

Theorem 4. Let $\bar{M}$ be a regular tubular involutive surface in $\mathbb{E}^3$. $\bar{M}$ is a flat surface if and only if

$$\psi = arctan\left(\frac{\tau}{\kappa}\right), \tag{16}$$

where $\psi = \vartheta - \theta$ is the angle between B and the position vector of the circles C_λ.

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Proof: We know that if the surface $\bar{M}$ is a flat surface, then the Gaussian curvature of $\bar{M}$ vanishes, Thus, from equation (13) we can write

$$K = 0 \Rightarrow \kappa sin\psi - \tau cos\psi = 0.$$

Direct computation gives

$$\psi = arctan\left(\frac{\tau}{\kappa}\right).$$

In particular, a similar result is valid if the parameter curves of $\bar{M}$ is a line of curvature.

Theorem 5. Let $\bar{M}$ be a regular tubular involutive surface in $\mathbb{E}^3$. $\bar{M}$ is a minimal surface if and only if

$$r = \frac{\delta}{\tau cos\psi - \kappa sin\psi}. \tag{17}$$

Proof: If the surface $\bar{M}$ is a minimal, then $H = 0$. From equation (13), the proof is clear.

Theorem 6. Let $\bar{M}$ be a regular tubular involutive surface in $\mathbb{E}^3$. The parameter curves of $\bar{M}$ have the following properties:

(ii) The s parameter curve of the surface $\bar{M}$ is an asymptotic curve if and only if

$$r = \frac{\delta(\tau cos\psi - \kappa sin\psi)}{\theta'^2}, \quad (\theta' \neq 0) \tag{18}$$

or

$$\theta = \frac{1}{\sqrt{r}} \int \sqrt{\delta(\tau cos\psi - \kappa sin\psi)} ds. \tag{19}$$

(ii) The ϑ parameter curve of the surface $\bar{M}$ cannot be an asymptotic curve.

Proof: (i) The s parameter curves are asymptotic curves on $\bar{M}$ if and only if

$$\langle \xi, \bar{Q}_{ss} \rangle = 0 \Leftrightarrow -r\theta'^2 + \delta(\tau cos\psi - \kappa sin\psi) = 0.$$

Using this last equation, we can easily obtain equations (18) and (19) for the surface $\bar{M}$.

(ii) Since the equality $\langle \xi, \bar{Q}_{\vartheta\vartheta} \rangle = -r \neq 0$ holds, the ϑ parameter curve cannot be an asymptotic curve.

Theorem 7. Let $\bar{M}$ be a regular tubular involutive surface in $\mathbb{E}^3$. The parameter curves of $\bar{M}$ have the following properties:

(i) The s parameter curve of the surface $\bar{M}$ is a geodesic curve if and only if

$$\begin{cases} (\delta_s + r\theta'\kappa cos^2\psi - r\theta'\tau sin\psi)(cos\psi - sin\psi) = 0 \\ r\theta'' + \delta\kappa cos\psi + \delta\tau sin\psi = 0\,. \end{cases} \tag{20}$$

(ii) The ϑ parameter curve of the surface $\bar{M}$ is always a geodesic curve.

Proof: (i) For the s parameter curve of the surface $\bar{M}$ to be geodesic curves, a necessary and sufficient condition is that $\xi \times \bar{Q}_{ss} = 0$. In this case we obtain the following relations for the s parameter curve:

$$\begin{aligned} \xi \times \bar{Q}_{ss} = &(\delta_s cos\psi - r\theta'\kappa cos^2\psi - r\theta'\tau sin\psi cos\psi)T \\ &+(r\theta'' + \delta\kappa cos\psi + \delta\tau sin\psi)N \\ &+(\delta_s sin\psi - r\theta'\kappa cos\psi sin\psi - r\theta'\tau sin^2\psi)B = 0. \end{aligned}$$

Because $\{T, N, B\}$ are linearly independent, we have the following equalities:

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$$\begin{cases} \delta_s cos\psi - r\theta' \kappa cos^2\psi - r\theta' \tau sin\psi cos\psi = 0, \\ r\theta'' + \delta\kappa cos\psi + \delta\tau sin\psi = 0, \\ \delta_s sin\psi - r\theta' \kappa cos\psi sin\psi - r\theta' \tau sin^2\psi = 0. \end{cases} \tag{21}$$

If the first and last equation are arranged, we have

$$(\delta_s + r\theta'\kappa cos^2\psi - r\theta'\tau sin\psi)(cos\psi - sin\psi) = 0.$$

Thus, this last equation and the second equality in equation (21) together give the desired result for the s parameter curve.

(ii) Since the equality $\xi \times \bar{Q}_{\vartheta\vartheta} = 0$ holds, the ϑ parameter curve is always a geodesic curve.

Theorem 8. Let $\bar{M}$ be a regular tubular involutive surface in $\mathbb{E}^3$. The parameter curves of $\bar{M}$ is a line of curvature if and only if θ is a constant angle.

Proof: We know that the parameter curves on the surface $\bar{M}$ are also a line of curvature if and only if $F = M$. Thus, we have from equations (11) and (14)

$$-r^2\theta' = 0 \text{ and } r\theta' = 0.$$

The common solution of the above two relations is that θ is a constant angle.

Using this last result and equations (11) and (14), we can give the following result.

Corollary 4. If we assume that the parameter curves of the surface $\bar{M}$ are its lines of curvature, then the components of the first and the second fundamental forms on the surface $\bar{M}$ are given by

$$E = \delta^2, \;\; F = 0, \;\; G = r^2 \;\; \text{and} \;\; L = \delta(\tau cos\psi - \kappa sin\psi), \;\; M = 0, \;\; N = -r.$$

EXAMPLES

Example 1. Let $\alpha: \alpha(s) = (coss, sins, 0)$ be a unit speed circle. We can compute the Frenet apparatus of α as follows:

$$\begin{cases} T = (-sins, coss, 0) \\ N = (-coss, -sins, 0) \\ B = (0,0,1) \\ \kappa = 1, \tau = 0. \end{cases}$$

The involute curve $\bar{\alpha}$

$$\bar{\alpha}: \bar{\alpha}(s) = (coss + ssins, sins - scoss, 0) \tag{22}$$

can be obtained by taking $c = 0$ in equation (1). Using equations (3) and (5), the tubular surface R around the curve α and tubular involutive surface $\bar{R}$ around the involute curve $\bar{\alpha}$ are given by the equation

$$R: R(s,\vartheta) = (coss, sins, 0) + r(-cosscos\vartheta, -sinscos\vartheta, sin\vartheta) \tag{23}$$

and

$$\bar{R}: \bar{R}(s,\vartheta) = (coss + ssins, sins - scoss, 0) + r(-sinssin\psi, cosssin\psi, cos\psi) \tag{24}$$

where $\psi = \vartheta - \theta$. The graphs of the surfaces R and $\bar{R}$ are shown in Figures 3 and 4; $\theta = \frac{\pi}{6}, r = 1$ and $-5 \leq s, \vartheta \leq 5$.

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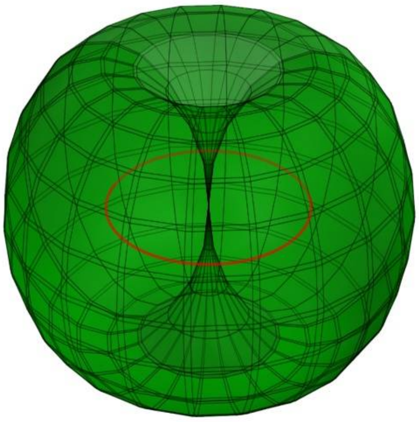

Figure 3. Tubular surface R and α (red)

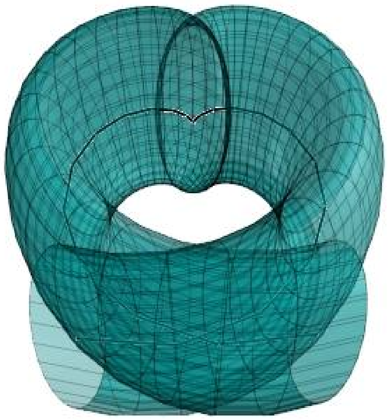

Figure 4. Tubular involutive surface $\bar{R}$ and $\bar{\alpha}$ (black)

Example 2. Let $\gamma:\gamma(s)=\left(-\frac{1}{\sqrt{2}}coss,-\frac{1}{\sqrt{2}}sins,\frac{s}{\sqrt{2}}\right)$ be a unit speed circular helix. Then it is easy to show that

$$\begin{cases} T=\left(\frac{1}{\sqrt{2}}sins,-\frac{1}{\sqrt{2}}coss,\frac{1}{\sqrt{2}}\right) \\ N=(coss,sins,0) \\ B=\left(-\frac{1}{\sqrt{2}}sins,\frac{1}{\sqrt{2}}coss,\frac{1}{\sqrt{2}}\right) \\ \kappa=\tau=\frac{1}{\sqrt{2}} \end{cases}.$$

From equation (1), the involute curve of $\bar{\gamma}$ can be given as

$$\bar{\gamma}:\bar{\gamma}(s)=\frac{1}{\sqrt{2}}(-coss+\mu sins,-sins-\mu coss,c) \tag{25}$$

where $\mu=c-s$, c being a real constant. From equations (3) and (5), the tubular surface S around the curve γ and tubular involutive surface $\bar{S}$ around the involute curve $\bar{\gamma}$ are given by the equation

$$\begin{aligned} S:S(s,\vartheta)=&\left(-\frac{1}{\sqrt{2}}coss,-\frac{1}{\sqrt{2}}sins,\frac{s}{\sqrt{2}}\right)\\ &+r\left(cos\vartheta coss-\frac{1}{\sqrt{2}}sin\vartheta sins,cos\vartheta sins+\frac{1}{\sqrt{2}}sin\vartheta coss,\frac{1}{\sqrt{2}}sin\vartheta\right) \end{aligned} \tag{26}$$

and

$$\begin{aligned} \bar{S}:\bar{S}(s,\vartheta)=&\frac{1}{\sqrt{2}}(-coss+\mu sins,-sins-\mu coss,c)\\ &+\frac{1}{\sqrt{2}}r(sins(sin\psi-cos\psi),coss(cos\psi-sin\psi),sin\psi+cos\psi), \end{aligned} \tag{27}$$

where $\psi=\vartheta-\theta$. The graphs of the surfaces S and $\bar{S}$ are shown in Figures 5 and 6; $\theta=\frac{\pi}{4}, r=0.5$, $c=5$ and $-5\leq s,\vartheta\leq 5$.

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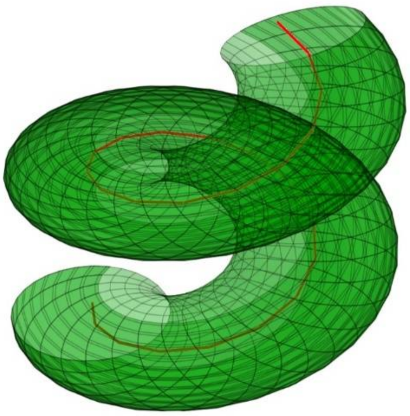

Figure 5. Tubular surface S and γ (red)

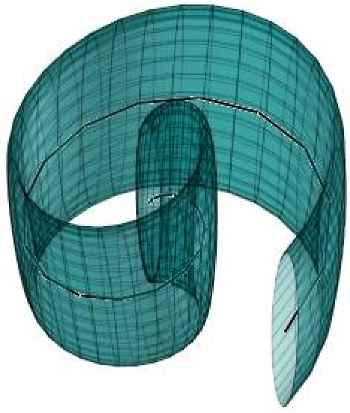

Figure 6. Tubular involutive surface $\bar{S}$ and $\bar{\gamma}$ (black)

CONCLUSIONS

We have presented a new approach to constructing tubular surfaces by using the involute curve of a 3D space curve. This study may open new horizons for studying tubular and canal surface construction derived from special curves and alternative frames.

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