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Full Paper

Difference approximation of positive linear operators on unit square

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Abstract: In this article we present quantitative estimates for the difference of certain positive linear operators defined on unit square. We prove some theorems for the difference of these operators. As an application to our main results, we estimate the difference of some well-known bivariate positive linear operators, namely Bernstein, Bernstein-Durrmeyer, Bernstein-Kantorovich and genuine Bernstein-Durrmeyer operators in terms of modulus of smoothness and K-functional.

Keywords: positive linear operators, estimation of differences of operators, linear difference operators, modulus of continuity

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INTRODUCTION

In 1995 while working on positive linear operators, Romanian mathematician Lupaş [1] was interested in finding a solution to the differences problem:

$$\mathcal{X}_n - \mathcal{Y}_n := \mathcal{B}_n \circ \overline{\mathcal{B}_n} - \overline{\mathcal{B}_n} \circ \mathcal{B}_n, \tag{1}$$

where $\mathcal{B}_n$ and $\overline{\mathcal{B}_n}$ are the Bernstein and Beta operators respectively. This problem encouraged many aspiring researchers to study the difference approximation of positive linear operators. In 2006, using Taylor's expansion, Gonska et al. [2] presented a solution of (1) along with some more general results regarding Lupaş problem. Continuing in the same manner, Gonska et al. [3, 4] and Gonska and Raşa [5] derived the difference estimates for positive linear operators having equal moments up to certain orders. Acu and Raşa [6] formulated some inequalities involving positive linear functional by using Taylor's formula. Based on these inequalities, they obtained differences of many operators in quantitative form. The researchers [7, 8] estimated the difference between positive linear operators acting on bounded or unbounded intervals and their respective derivatives, depending on first-order modulus of continuity. The difference approximation of discrete operators defined on the interval $[0, \infty)$ was thoroughly investigated by Aral et al. [9]. Their results for the differences of the operators involved appropriate weighted modulus of continuity and K-functional. Inspired by this study, Gupta and Tachev [10] calculated the difference of two arbitrary positive linear operators in terms of Păltănea weighted modulus of continuity. Gupta et al. [11] extended the work of Acu and Raşa [6] for operators having different basis functions and obtained more general results by relaxing a condition on the associated linear functional. Estimates regarding the differences of Lupaş-type operators were provided by Gupta [12].

Gupta [13-15] proposed general families of positive linear operators which contain several well-known operators and studied their differences. He estimated the quantitative difference of Miheşan operators $M_{n,\gamma}$ with Summation-integral type operators $\mathcal{A}_{n,\alpha}^{\beta,\rho}$, $V_{n,\alpha}^{\beta,\rho}$ and integral type operators $V_{n,\alpha,\beta}$ using moduli of continuity. The fundamental convergence results such as Korovkin-type theorems and Voronovskaja-type theorems have been studied for several operators [16-19]. Voronovskaja-type results for the difference of certain operators were established by Acu et al. [20]. They considered positive linear operators with the same basic functions but with different functionals and obtained the above-mentioned results in quantitative form. Acar et al. [21] presented Voronovskaja-type theorems for the difference of positive linear operators in polynomial weighted spaces. These results were verified by taking into account several well-known operators. A survey on approximation results for the difference of operators was provided [22]. In 2021 Acu et al. [23] obtained general estimations of difference between positive linear operators acting on simplices. These estimations were verified for Bernstein-type operators in two variables with the help of appropriate moduli of smoothness and K-functional.

Influenced by the notable work of Acu et al. [23], we begin our investigation on differences of positive linear operators defined on unit square. We prove some general results in quantitative form by means of first-order and second-order modulus of smoothness and K-functional. We also construct genuine Bernstein-Durrmeyer operators working on a unit square. As an application to our results, we derive quantitative estimates for the differences of Bernstein, Bernstein-Kantorovich, Bernstein-Durrmeyer and genuine Bernstein-Durrmeyer operators.

BASIC DEFINITIONS AND THEOREMS

Here, we establish some notations and definitions which we use throughout the article. Let $\mathcal{J} = [0,1]$; then $\mathcal{J}^2 = \mathcal{J} \times \mathcal{J} = \{(a,b): a,b \in \mathcal{J}\}$ is the unit square in $\mathbb{R}^2$. We denote the constant function by I_1 and the i^{th}coordinate functions by $pr_i (i = 1,2)$, defined as follows:

$$I_1 : \mathcal{J}^2 \to \mathbb{R},$$
$$I_1(x,y) = 1, \forall (x,y) \in \mathcal{J}^2$$

and

$$pr_i : \mathcal{J}^2 \to \mathbb{R}, i = 1,2,$$
$$pr_1(x,y) = x, \; pr_2(x,y) = y.$$

$E(\mathcal{J}^2)$ represents a space of real-value and continuous functions of two variables defined on $\mathcal{J}^2$, containing the polynomials. Suppose $F: E(\mathcal{J}^2) \to \mathbb{R}$ is a positive linear functional preserving the constant function I_1, i.e $F(I_1) = 1$. Let us assume

$$a_1^F = F(pr_1), \qquad a_2^F = F(pr_2),$$

and

$$\nu_{i,j}^F = F\big((pr_1 - a_1^F I_1)^i * (pr_2 - a_2^F I_1)^j\big),\ i,j \in \mathbb{N} \cup \{0\}.$$

Then for $i,j = 0,1,2$, we have

$$\begin{aligned} \nu_{1,0}^F &= 0, \nu_{2,0}^F = F(pr_1^2) - (a_1^F)^2 \geq 0, \\ \nu_{0,1}^F &= 0, \nu_{0,2}^F = F(pr_2^2) - (a_2^F)^2 \geq 0. \end{aligned} \tag{2}$$

Let $C^r(\mathcal{J}^2), r \in \mathbb{N}$ be the space of all functions $f: \mathcal{J}^2 \to \mathbb{R}$ such that
(i) f is a continuous function;
(ii) f is differentiable on interior of $\mathcal{J}^2$;
(iii) partial derivatives of f of order $\leq r$ can be continuously extended to $\mathcal{J}^2$.
Next, we prove an important inequality in Lemma 1.

Lemma 1. If $g \in C^2(\mathcal{J}^2)$, then $|F(g) - g(a_1^F, a_2^F)| \leq N_g\{v_{2,0}^F + v_{0,2}^F\}$,
where $N_g = max\{\|g_{xx}\|, \|g_{xy}\|, \|g_{yy}\|\}$,
and $\|g\| = sup\{|g(x,y)|: (x,y) \in \mathcal{J}^2\} < \infty$. (3)

Proof. Let $(a_1^F, a_2^F), (t_1, t_2) \in \mathcal{J}^2$ and L be the line segment joining these points in the domain $\mathcal{J}^2$. By Taylor's formula, there exists a point (d_1, d_2) on L, different from (a_1^F, a_2^F) and (t_1, t_2) such that

$$g(t_1,t_2) = g(a_1^F, a_2^F) + g_x(a_1^F, a_2^F)(t_1 - a_1^F) + g_y(a_1^F, a_2^F)(t_2 - a_2^F) + \frac{1}{2}\{g_{xx}(d_1,d_2) \\ (t_1 - a_1^F)^2 + 2g_{xy}(d_1,d_2)(t_1 - a_1^F)(t_2 - a_2^F) + g_{yy}(d_1,d_2)(t_2 - a_2^F)^2\}.$$

Therefore, we can write

$$\begin{aligned} &|g - g(a_1^F, a_2^F)I_1 - g_x(a_1^F, a_2^F)(pr_1 - a_1^F I_1) - g_y(a_1^F, a_2^F)(pr_2 - a_2^F I_1)| \\ &\leq \frac{1}{2}\{\|g_{xx}\|(pr_1 - a_1^F I_1)^2 + 2\|g_{xy}\|(pr_1 - a_1^F I_1)(pr_2 - a_2^F I_1) + \|g_{yy}\|(pr_2 - a_2^F I_1)^2\} \\ &\leq \frac{1}{2}\{(\|g_{xx}\| + \|g_{xy}\|)(pr_1 - a_1^F I_1)^2 + (\|g_{xy}\| + \|g_{yy}\|)(pr_2 - a_2^F I_1)^2\} \\ &\leq N_g((pr_1 - a_1^F I_1)^2 + (pr_2 - a_2^F I_1)^2). \end{aligned}$$

Implementing the functional F, we get the required inequality.

DIFFERENCE ESTIMATES OF BIVARIATE OPERATORS

We start this section by assuming that the space $C(\mathcal{J}^2) = \{g: \mathcal{J}^2 \to \mathbb{R}: g\ is\ continuous\}$ has the norm $\|g\| := sup\{|g(x,y)| : (x,y) \in \mathcal{J}^2\}$. Let M be a set of non-negative integers and for any $m, n \in M$, the basic function $p_{m,n}$ has the following properties:

(i) $p_{m,n} \in C(\mathcal{J}^2)$; (ii) $p_{m,n} \geq 0$; (iii) $\sum_{m,n \in M} p_{m,n} = I_1$. .

It is assumed that $\mathcal{A}_{m,n}, \mathcal{B}_{m,n}: E(\mathcal{J}^2) \to \mathbb{R}, m,n \in M$ are positive linear functionals such that $\mathcal{A}_{m,n} = (I_1) = \mathcal{B}_{m,n}(I_1) = 1$,
and

$$Y(\mathcal{J}^2) = \left\{g \in E(\mathcal{J}^2) : \sum_{m,n \in M} \mathcal{A}_{m,n}(g) p_{m,n}, \sum_{m,n \in M} \mathcal{B}_{m,n}(g) p_{m,n} \in C(\mathcal{J}^2)\right\}.$$

Now we consider positive linear operators $\mathcal{A}, \mathcal{B}: Y(\mathcal{J}^2) \to C(\mathcal{J}^2)$ given by

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$$\mathcal{A}(g;(x,y)) = \sum_{m,n\in M} \mathcal{A}_{m,n}(g)\, p_{m,n}(x,y),$$

and

$$\mathcal{B}(g;(x,y)) = \sum_{m,n\in M} \mathcal{B}_{m,n}(g)\, p_{m,n}(x,y).$$

For further development we denote

$$\lambda = sup_{m,n\in M}\left\{\left|a_1^{\mathcal{A}_{m,n}} - a_1^{\mathcal{B}_{m,n}}\right| + \left|a_2^{\mathcal{A}_{m,n}} - a_2^{\mathcal{B}_{m,n}}\right|\right\}, \tag{4}$$

and

$$\gamma(x,y) = \sum_{m,n\in M}\left(\nu_{2,0}^{\mathcal{A}_{m,n}} + \nu_{2,0}^{\mathcal{B}_{m,n}} + \nu_{0,2}^{\mathcal{A}_{m,n}} + \nu_{0,2}^{\mathcal{B}_{m,n}}\right) p_{m,n}(x,y). \tag{5}$$

For functions belonging to the space $C^r(\mathcal{J}^2)$, we use the following norm [24]:

$$\|g\|_{C^r(\mathcal{J}^2)} = \|g\|_{C(\mathcal{J}^2)} + \sum_{\xi_1+\xi_2=r}\left\|\frac{\partial^r g}{\partial x^{\xi_1}\,\partial y^{\xi_2}}\right\|_{C(\mathcal{J}^2)};\ \xi_1,\xi_2 \geq 0. \tag{6}$$

To measure the differences of bivariate positive linear operators, we use the concept of bivariate modulus of smoothness and bivariate K-functional. The r^{th} order modulus of smoothness $\omega_r(g,\delta)$ and the r^{th} order K-functional $K_r(g,\delta)$ [25, 26] for any $g \in C(\mathcal{J}^2)$ and $\delta > 0$, are given by

$$\omega_r(g,\delta) = \sup\{\|\Delta_h^r g(x)\| : |h| \leq \delta, h \in \mathbb{R}^2;\ x, x+kh \in \mathcal{J}^2, k \geq 1\}, \tag{7}$$

where

$$\Delta_h^r g(x) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} g(x+kh),$$

and

$$K_r(g,\delta) = \inf\left\{\|g-f\| + \delta\|f\|_{C^r(\mathcal{J}^2)} : f \in C^r(\mathcal{J}^2)\right\}. \tag{8}$$

Moreover, given any $\delta > 0$, there exists $\tau_1, \tau_2 > 0$ [25, 26] such that

$$\tau_1 K_r(g,\delta^r) \leq \omega_r(g,\delta) \leq \tau_2 K_r(g,\delta^r). \tag{9}$$

Theorem 1. If $g \in Y(\mathcal{J}^2) \cap C^2(\mathcal{J}^2)$, then $|(\mathcal{A}-\mathcal{B})(g;(x,y))| \leq N_g\gamma(x,y) + \omega_1(g,\lambda)$, where N_g and $\gamma(x,y)$ are defined in (3) and (5) respectively.

Proof. Assuming $g \in Y(\mathcal{J}^2) \cap C^2(\mathcal{J}^2)$, by Lemma 1, for any $(x,y) \in \mathcal{J}^2$, we have

$$\begin{aligned}
|(\mathcal{A}-\mathcal{B})(g;(x,y))| &= \left|\sum_{m,n\in M}\left(\mathcal{A}_{m,n}(g) - \mathcal{B}_{m,n}(g)\right) p_{m,n}(x,y)\right| \\
&\leq \sum_{m,n\in M}\left|\left(\mathcal{A}_{m,n}(g) - \mathcal{B}_{m,n}(g)\right)\right| p_{m,n}(x,y) \\
&\leq \sum_{m,n\in M} p_{m,n}(x,y)\left\{\left|\mathcal{A}_{m,n}(g) - g\left(a_1^{\mathcal{A}_{m,n}}, a_2^{\mathcal{A}_{m,n}}\right)\right|\right. \\
&\quad \left. + \left|\mathcal{B}_{m,n}(g) - g\left(a_1^{\mathcal{B}_{m,n}}, a_2^{\mathcal{B}_{m,n}}\right)\right| + \left|g\left(a_1^{\mathcal{A}_{m,n}}, a_2^{\mathcal{A}_{m,n}}\right) - g\left(a_1^{\mathcal{B}_{m,n}}, a_2^{\mathcal{B}_{m,n}}\right)\right|\right\} \\
&\leq N_g \sum_{m,n\in M} p_{m,n}(x,y)\left[\nu_{2,0}^{\mathcal{A}_{m,n}} + \nu_{0,2}^{\mathcal{A}_{m,n}} + \nu_{2,0}^{\mathcal{B}_{m,n}} + \nu_{0,2}^{\mathcal{B}_{m,n}}\right]
\end{aligned}$$

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$$+\omega_1\left(g,\left|\left(a_1^{\mathcal{A}_{m,n}},a_2^{\mathcal{A}_{m,n}}\right)-\left(a_1^{\mathcal{B}_{m,n}},a_2^{\mathcal{B}_{m,n}}\right)\right|\right)$$
$$\leq N_g\gamma(x,y)+\omega_1(g,\lambda).$$

Theorem 2. If g is any continuous function defined on $\mathcal{J}^2$, it follows that

$$\left|(\mathcal{A}-\mathcal{B})\big(g;(x,y)\big)\right| \leq \alpha_1\omega_1(g,\lambda)+\alpha_2\omega_2\left(g,\sqrt{\gamma(x,y)}\right),$$

where $\alpha_1,\alpha_2>0$, λ and $\gamma(x,y)$ are defined in (4) and (5) respectively.

Proof. Applying Theorem 1 for $h\in Y(\mathcal{J}^2)\cap C^2(\mathcal{J}^2)$, we have

$$\left|(\mathcal{A}-\mathcal{B})\big(g;(x,y)\big)\right| \leq \left|\mathcal{A}\big(g-h;(x,y)\big)\right|+\left|\mathcal{B}\big(h-g;(x,y)\big)\right|+\left|(\mathcal{A}-\mathcal{B})\big(h;(x,y)\big)\right|$$
$$\leq 2\|g-h\|+N_h\gamma(x,y)+\left|h(a_1^{\mathcal{A}_{m,n}},a_2^{\mathcal{A}_{m,n}})-h(a_1^{B_{m,n}},a_2^{B_{m,n}})\right|. \quad (10)$$

For an upper bound of $\left|h\left(a_1^{\mathcal{A}_{m,n}},a_2^{\mathcal{A}_{m,n}}\right)-h\left(a_1^{\mathcal{B}_{m,n}},a_2^{\mathcal{B}_{m,n}}\right)\right|$, consider any line segment L joining the points $\left(a_1^{\mathcal{A}_{m,n}},a_2^{\mathcal{A}_{m,n}}\right)$ and $\left(a_1^{\mathcal{B}_{m,n}},a_2^{\mathcal{B}_{m,n}}\right)$ in the domain $\mathcal{J}^2$. Since $h_x(x,y),h_y(x,y)$ exist for all $(x,y)\in L$, therefore by mean value theorem we have a point (k_1,k_2) on L such that

$$h\left(a_1^{\mathcal{A}_{m,n}},a_2^{\mathcal{A}_{m,n}}\right)-h\left(a_1^{\mathcal{B}_{m,n}},a_2^{\mathcal{B}_{m,n}}\right)=h_x(k_1,k_2)\left(a_1^{\mathcal{A}_{m,n}}-a_1^{\mathcal{B}_{m,n}}\right)$$
$$+h_y(k_1,k_2)\left(a_2^{\mathcal{A}_{m,n}}-a_2^{\mathcal{B}_{m,n}}\right). \quad (11)$$

Also, from (6) we get

$$\left|h\left(a_1^{\mathcal{A}_{m,n}},a_2^{\mathcal{A}_{m,n}}\right)-h\left(a_1^{\mathcal{B}_{m,n}},a_2^{\mathcal{B}_{m,n}}\right)\right|\leq\|h_x\|\left|\left(a_1^{\mathcal{A}_{m,n}}-a_1^{\mathcal{B}_{m,n}}\right)\right|+\|h_y\|\left|\left(a_2^{\mathcal{A}_{m,n}}-a_2^{\mathcal{B}_{m,n}}\right)\right|.$$

Since $N_h\leq\|h\|_{C^2(\mathcal{J}^2)}$, therefore from (10) we conclude that

$$\left|(\mathcal{A}-\mathcal{B})\big(g;(x,y)\big)\right|\leq 2\|g-h\|+\lambda\,\|h\|_{C^1(\mathcal{J}^2)}+\gamma(x,y)\|h\|_{C^2(\mathcal{J}^2)}$$
$$\leq K_1(g,\lambda)+K_2\big(g,\gamma(x,y)\big).$$

Finally, using (9) we get

$$\left|(\mathcal{A}-\mathcal{B})\big(g;(x,y)\big)\right|\leq \alpha_1\omega_1(g,\lambda)+\alpha_2\omega_2\left(g,\sqrt{\gamma(x,y)}\right).$$

APPLICATIONS

To demonstrate our main results (Theorems 1 and 2), we consider the bivariate form of some classical operators for estimation. We give a brief review of these bivariate operators and estimate their differences by means of modulus of smoothness and K-functional.

Bivariate Bernstein Operators

Kingsley [27] presented the Bernstein operators in two variables for any continuous function g defined on $\mathcal{J}^2$ and any $x,y\in\mathcal{J}$, as follows:

$$\mathcal{B}_{m,n}\big(g;(x,y)\big)=\sum_{k=0}^{m}\sum_{j=0}^{n}B_{m,n}^{k,j}(g)\,p_{m,n}^{k,j}(x,y),$$

where

$$B_{m,n}^{k,j}(g)=g\left(\frac{k}{m},\frac{j}{n}\right),$$

and the bivariate Bernstein fundamental function is given by

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$$p_{m,n}^{k,j}(x,y) := \lambda_{m,k}(x)\lambda_{n,j}(y) = \binom{m}{k} x^k (1-x)^{m-k} \binom{n}{j} y^j (1-y)^{n-j}.$$

Bivariate Bernstein-Durrmeyer Operators

For $g \in L^1(\mathcal{J}^2)$, where $L^1(\mathcal{J}^2) = \{g : \mathcal{J}^2 \to \mathbb{R} : g \text{ is Lebesgue integrable}\}$ and $(x,y) \in \mathcal{J}^2$, the bivariate version $\mathcal{D}_{m,n}$ of Bernstein-Durrmeyer operators are defined [28] by

$$\mathcal{D}_{m,n}: L_1(\mathcal{J}^2) \to C(\mathcal{J}^2),$$

$$\mathcal{D}_{m,n}\big(g;(x,y)\big) = \sum_{k=0}^{m}\sum_{j=0}^{n} D_{m,n}^{k,j}(g)\, p_{m,n}^{k,j}(x,y),$$

where

$$D_{m,n}^{k,j}(g) = (m+1)(n+1)\int_0^1\int_0^1 \lambda_{m,k}(t)\lambda_{n,j}(s) g(t,s)\,dt\,ds.$$

Proposition 1. The Bernstein and Bernstein-Durrmeyer operators in two variables assert the following properties:

(i) If $g \in C^2(\mathcal{J}^2)$, then

$$\left|(\mathcal{B}_{m,n} - \mathcal{D}_{m,n})\big(g;(x,y)\big)\right| \le N_g \gamma(x,y) + \omega_1\left(g, \frac{1}{m+2} + \frac{1}{n+2}\right),$$

where

$$\gamma(x,y) = \frac{m^2 x(1-x) + m(1-x+x^2) + 1}{(m+2)^2(m+3)} + \frac{n^2 y(1-y) + n(1-y+y^2) + 1}{(n+2)^2(n+3)}$$
$$\le \frac{1}{m+3} + \frac{1}{n+3}.$$

(ii) If $g \in C(\mathcal{J}^2)$, then

$$\left|(\mathcal{B}_{m,n} - \mathcal{D}_{m,n})\big(g;(x,y)\big)\right| \le \alpha_1\omega_1\left(g, \frac{1}{m+2} + \frac{1}{n+2}\right) + \alpha_2\omega_2\left(g, \sqrt{\gamma(x,y)}\right).$$

Proof. In order to find both the estimates for the differences of operators $\mathcal{B}_{m,n}$ and $\mathcal{D}_{m,n}$, we first compute λ and $\gamma(x,y)$. From (2), we observe that

$$a_1^{B_{m,n}^{k,j}} = \frac{k}{m}, \qquad a_1^{D_{m,n}^{k,j}} = \frac{k+1}{m+2},$$
$$a_2^{B_{m,n}^{k,j}} = \frac{j}{n}, \qquad a_2^{D_{m,n}^{k,j}} = \frac{j+1}{n+2}.$$

Substituting above values in (4), we have

$$\lambda = \max_{\substack{k,j \\ k=0,1,\ldots,m. \\ j=0,1,\ldots,n.}} \left\{\left|\frac{k}{m} - \frac{k+1}{m+2}\right| + \left|\frac{j}{n} - \frac{j+1}{n+2}\right|\right\} = \frac{1}{m+2} + \frac{1}{n+2}.$$

Again from (2), we get

$$\nu_{2,0}^{B_{m,n}^{k,j}} = 0, \nu_{2,0}^{D_{m,n}^{k,j}} = \frac{(k+1)(k+2)}{(m+2)(m+3)} - \left(\frac{k+1}{m+2}\right)^2,$$
$$\nu_{0,2}^{B_{m,n}^{k,j}} = 0, \nu_{0,2}^{D_{m,n}^{k,j}} = \frac{(j+1)(j+2)}{(n+2)(n+3)} - \left(\frac{j+1}{n+2}\right)^2.$$

Then (5) gives

$$\begin{aligned}\gamma(x,y) &= \sum_{k=0}^{m}\sum_{j=0}^{n}\left[\frac{(k+1)(k+2)}{(m+2)(m+3)}-\left(\frac{k+1}{m+2}\right)^2+\frac{(j+1)(j+2)}{(n+2)(n+3)}-\left(\frac{j+1}{n+2}\right)^2\right]p_{m,n}^{k,j}(x,y)\\ &= \frac{m^2x(1-x)+m(1-x+x^2)+1}{(m+2)^2(m+3)}+\frac{n^2y(1-y)+n(1-y+y^2)+1}{(n+2)^2(n+3)}\\ &\le \frac{1}{m+3}+\frac{1}{n+3}.\end{aligned}$$

Applying Theorems 1 and 2 for the above values of λ and $\gamma(x,y)$, we get the required estimates.

Bivariate Bernstein-Kantorovich Operators

Cabulea and Aldea [29] constructed the bivariate Bernstein-Kantorovich operators $\mathcal{K}_{m,n}$ for Lebesgue integrable function g defined on $\mathcal{J}^2$, which are given by

$$\mathcal{K}_{m,n}\big(g;(x,y)\big)=\sum_{k=0}^{m}\sum_{j=0}^{n}K_{m,n}^{k,j}(g)\,p_{m,n}^{k,j}(x,y),$$

where

$$K_{m,n}^{k,j}(g)=(m+1)(n+1)\int_{\frac{j}{(n+1)}}^{\frac{(j+1)}{(n+1)}}\int_{\frac{k}{(n+1)}}^{\frac{(k+1)}{(n+1)}}g(t,s)dt\,ds.$$

Proposition 2. The bivariate operators $\mathcal{B}_{m,n}$ and $\mathcal{K}_{m,n}$ verify the following results:

(i) If $g\in C^2(\mathcal{J}^2)$, then

$$\left|\big(\mathcal{B}_{m,n}-\mathcal{K}_{m,n}\big)\big(g;(x,y)\big)\right|\le N_g\gamma(x,y)+\omega_1\left(g,\frac{1}{2}\left(\frac{1}{m+1}+\frac{1}{n+1}\right)\right),$$

where

$$\gamma(x,y)=\frac{1}{12}\left(\frac{1}{(m+1)^2}+\frac{1}{(n+1)^2}\right)\le\frac{1}{m+1}+\frac{1}{n+1}.$$

(ii) If $g\in C(\mathcal{J}^2)$, then

$$\left|\big(\mathcal{B}_{m,n}-\mathcal{K}_{m,n}\big)\big(g;(x,y)\big)\right|\le \alpha_1\omega_1\left(g,\frac{1}{2}\left(\frac{1}{m+1}+\frac{1}{n+1}\right)\right)+\alpha_2\omega_2\left(g,\sqrt{\gamma(x,y)}\right).$$

Proof. Firstly, we wish to find λ and $\gamma(x,y)$ for the operators $\mathcal{B}_{m,n}$ and $\mathcal{K}_{m,n}$. By using (2), we have

$$\begin{aligned} a_1^{B_{m,n}^{k,j}}&=\frac{k}{m}, & a_1^{K_{m,n}^{k,j}}&=\frac{2k+1}{2(m+2)},\\ a_2^{B_{m,n}^{k,j}}&=\frac{j}{n}, & a_2^{K_{m,n}^{k,j}}&=\frac{2j+1}{2(n+2)},\end{aligned}\tag{12}$$

and

$$\begin{aligned} \nu_{2,0}^{B_{m,n}^{k,j}}&=0, & \nu_{2,0}^{K_{m,n}^{k,j}}&=\frac{1}{12(m+1)^2},\\ \nu_{0,2}^{B_{m,n}^{k,j}}&=0, & \nu_{0,2}^{K_{m,n}^{k,j}}&=\frac{1}{12(n+1)^2}.\end{aligned}\tag{13}$$

Substituting (12) and (13) in (4) and (5) respectively, we have

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$$\lambda = \max_{\substack{k,j \\ k=0,1,\dots,m. \\ j=0,1,\dots,n.}} \left\{ \left| \frac{k+1}{m+2} - \frac{k}{m} \right| + \left| \frac{j+1}{n+2} - \frac{j}{n} \right| \right\} = \frac{1}{m+2} + \frac{1}{n+2}$$

and

$$\gamma(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} \left[\frac{1}{12(m+1)^2} + \frac{1}{12(n+1)^2} \right] p_{m,n}^{k,j}(x,y)$$
$$= \frac{1}{12}\left(\frac{1}{(m+1)^2} + \frac{1}{(n+1)^2} \right) \le \frac{1}{m+1} + \frac{1}{n+1}.$$

By implementing Theorems 1 and 2, we obtain the desired estimates.

Proposition 3. The bivariate form $\mathcal{D}_{m,n}$ of Bernstein-Durrmeyer operators and $\mathcal{K}_{m,n}$ of Bernstein-Kantorovich operators verify the following properties:

(i) If $g \in C^2(\mathcal{J}^2)$, then

$$\left|\left(\mathcal{D}_{m,n} - \mathcal{K}_{m,n}\right)\left(g;(x,y)\right)\right| \le N_g \gamma(x,y) + \omega_1\left(g, \frac{1}{2}\left(\frac{m}{(m+1)(m+2)} + \frac{n}{(n+1)(n+2)} \right) \right),$$

where

$$\gamma(x,y) = \frac{m^2 x(1-x) + m(1-x+x^2) + 1}{(m+2)^2(m+3)} + \frac{n^2 y(1-y) + n(1-y+y^2) + 1}{(n+2)^2(n+3)}$$
$$+ \frac{1}{12}\left(\frac{1}{(m+1)^2} + \frac{1}{(n+1)^2} \right)$$
$$\le 2\left(\frac{1}{m+1} + \frac{1}{n+1} \right).$$

(ii) If $g \in C(\mathcal{J}^2)$, then

$$\left|\left(\mathcal{D}_{m,n} - \mathcal{K}_{m,n}\right)\left(g;(x,y)\right)\right| \le \alpha_1 \omega_1\left(g, \frac{1}{2}\left(\frac{m}{(m+1)(m+2)} + \frac{n}{(n+1)(n+2)} \right) \right)$$
$$+ \alpha_2 \omega_2\left(g, \sqrt{\gamma(x,y)} \right).$$

Proof. For the operators $\mathcal{D}_{m,n}$ and $\mathcal{K}_{m,n}$, from (2) we have

$$a_1^{D_{m,n}^{k,j}} = \frac{k+1}{m+2}, \qquad a_1^{K_{m,n}^{k,j}} = \frac{2k+1}{2(m+1)},$$
$$a_2^{D_{m,n}^{k,j}} = \frac{j+1}{n+2}, \qquad a_2^{K_{m,n}^{k,j}} = \frac{2j+1}{2(n+1)}. \tag{14}$$

A simple calculation using (14) shows that

$$\nu_{2,0}^{D_{m,n}^{k,j}} = \frac{(k+1)(k+2)}{(m+2)(m+3)} - \left(\frac{k+1}{m+2} \right)^2, \quad \nu_{2,0}^{K_{m,n}^{k,j}} = \frac{1}{12(m+1)^2},$$
$$\nu_{0,2}^{D_{m,n}^{k,j}} = \frac{(j+1)(j+2)}{(n+2)(n+3)} - \left(\frac{j+1}{n+2} \right)^2, \quad \nu_{0,2}^{K_{m,n}^{k,j}} = \frac{1}{12(n+1)^2}. \tag{15}$$

Putting (14) in (4) and (15) in (5), we can write

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$$\lambda = \max_{\substack{k,j \\ k=0,1,\dots,m. \\ j=0,1,\dots,n.}} \left\{ \left| \frac{k+1}{m+2} - \frac{2k+1}{2(m+1)} \right| + \left| \frac{j+1}{n+2} - \frac{2j+1}{2(n+1)} \right| \right\}$$
$$= \frac{1}{2}\left(\frac{m}{(m+1)(m+2)} + \frac{n}{(n+1)(n+2)} \right)$$

and

$$\gamma(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n}\left[\frac{(k+1)(k+2)}{(m+2)(m+3)} - \left(\frac{k+1}{m+2}\right)^2 + \frac{1}{12(m+1)^2} + \frac{(j+1)(j+2)}{(n+2)(n+3)} - \left(\frac{j+1}{n+2}\right)^2 + \frac{1}{12(n+1)^2} \right] p_{m,n}^{k,j}(x,y)$$
$$= \frac{m^2x(1-x)+m(1-x+x^2)+1}{(m+2)^2(m+3)} + \frac{n^2y(1-y)+n(1-y+y^2)+1}{(n+2)^2(n+3)} + \frac{1}{12}\left(\frac{1}{(m+1)^2} + \frac{1}{(n+1)^2} \right)$$
$$\leq 2\left(\frac{1}{m+1} + \frac{1}{n+1} \right).$$

The remaining part of the proof is completed by implementing Theorems 1 and 2.

Bivariate Genuine Bernstein-Durrmeyer Operators

The genuine Bernstein-Durrmeyer operators were proposed by Chen [30] and Goodman and Sharma [31] for $g \in C(\mathcal{J}^2)$ and $x \in \mathcal{J}$ and were defined as

$$\mathcal{U}_{n,1}(g;x) = g(0)p_{n,0}(x) + g(1)p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x)(n-1)\int_0^1 p_{n-2,k-1}(t)g(t)dt,$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}, 0 \leq k \leq n.$$

The multivariate genuine Bernstein-Durrmeyer operators acting on a simplex were introduced and studied by Goodman and Sharma [32].

Now we construct the bivariate form of the operators $\mathcal{U}_{n,1}$ defined on a unit square as follows. Let $(x,y) \in \mathcal{J}^2$and $g \in L^1(\mathcal{J}^2)$. We define

$$\mathcal{U}_{m,n}\big(g;(x,y)\big) = g(0,0)(1-x)^m(1-y)^n + g(0,1)(1-x)^m y^n$$
$$+ g(1,0)x^m(1-y)^n + g(1,1)x^m y^n$$
$$+ (n-1)\sum_{j=1}^{n-1} p_{m,n}^{0,j}(x,y)\int_0^1 p_{n-2,j-1}(t)\, g(0,t)dt$$
$$+ (m-1)\sum_{k=1}^{m-1} p_{m,n}^{k,0}(x,y)\int_0^1 p_{m-2,k-1}(t)\, g(s,0)ds$$
$$+ (n-1)\sum_{j=1}^{n-1} p_{m,n}^{m,j}(x,y)\int_0^1 p_{n-2,j-1}(t)\, g(1,t)dt$$

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$$+(m-1)\sum_{k=1}^{m-1} p_{m,n}^{k,n}(x,y)\int_0^1 p_{m-2,k-1}(t)\,g(s,1)ds$$
$$+(m-1)(n-1)\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m,n}^{k,j}(x,y)\int_0^1\int_0^1 p_{m-2,n-2}^{k-1,j-1}(s,t)\,g(s,t)ds\,dt.$$

For the sake of simplicity, we write the operators $\mathcal{U}_{m,n}$ in a compact form. Let us denote

$$U_{m,n}^{k,j}(g) = g\left(\frac{k}{m},\frac{j}{n}\right), 0 \le k \le m, 0 \le j \le n,$$

and

$$U_{m,n}^{0,0}(g) = g(0,0), k = 0, j = 0,\ \ U_{m,n}^{0,n}(g) = g(0,1), k = 0, j = n,$$
$$U_{m,n}^{m,0}(g) = g(1,0), k = m, j = 0,\ U_{m,n}^{m,n}(g) = g(1,1), k = m, j = n,$$
$$U_{m,n}^{0,j}(g) = (n-1)\int_0^1 p_{n-2,j-1}(t)\,g(0,t)dt, k = 0, 1 \le j \le n-1,$$
$$U_{m,n}^{k,0}(g) = (m-1)\int_0^1 p_{m-2,k-1}(t)\,g(s,0)ds, 1 \le k \le m-1, j = 0,$$
$$U_{m,n}^{m,j}(g) = (n-1)\int_0^1 p_{n-2,j-1}(t)\,g(1,t)dt, k = m, 1 \le j \le n-1,$$
$$U_{m,n}^{k,n}(g) = (m-1)\int_0^1 p_{m-2,k-1}(s)\,g(s,1)ds, 1 \le k \le m-1, j = n,$$
$$U_{m,n}^{k,j}(g) = (m-1)(n-1)\int_0^1\int_0^1 p_{m-2,n-2}^{k-1,j-1}(s,t)\,g(s,t)dsdt,$$
$$1 \le k \le m-1, 1 \le \mathrm{j} \le \mathrm{n}-1.$$

Then the genuine Bernstein-Durrmeyer operators in two variables have the following compact form:

$$\mathcal{U}_{m,n}\big(g;(x,y)\big) = \sum_{k=0}^{m}\sum_{j=0}^{n} U_{m,n}^{k,j}(g)\,p_{m,n}^{k,j}(x,y).$$

Proposition 4. The Bernstein operators $\mathcal{B}_{m,n}$ and genuine Bernstein-Durrmeyer operators $\mathcal{U}_{m,n}$ in two variables assert the following properties:

(i) If $g \in C^2(\mathcal{J}^2)$, then $\left|\left(\mathcal{B}_{m,n} - \mathcal{U}_{m,n}\right)(g;(x,y))\right| \le N_g\gamma(x,y)$,
where

$$\gamma(x,y) = \frac{(m-1)x(1-x)}{m(m+1)} + \frac{(n-1)y(1-y)}{n(n+1)} \le \frac{1}{m+1} + \frac{1}{n+1}.$$

(ii) If $g \in \mathrm{C}(\mathcal{J}^2)$, then $\left|\left(\mathcal{B}_{m,n} - \mathcal{U}_{m,n}\right)(g;(x,y))\right| \le \alpha_2\omega_2\left(g,\sqrt{\gamma(x,y)}\right)$.

Proof. To prove both parts of proposition 4, we need to calculate λ and $\gamma(x,y)$. From (2), we observe that

$$a_1^{B_{m,n}^{k,j}} = \frac{k}{m}, \qquad a_1^{U_{m,n}^{k,j}} = \frac{k}{m},$$
$$a_2^{B_{m,n}^{k,j}} = \frac{j}{n}, \qquad a_2^{U_{m,n}^{k,j}} = \frac{j}{n},$$

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and

$$\nu_{2,0}^{B_{m,n}^{k,j}} = 0, \quad \nu_{2,0}^{U_{m,n}^{k,j}} = \frac{k(k+1)}{m(m+1)} - \left(\frac{k}{m}\right)^2,$$
$$\nu_{0,2}^{B_{m,n}^{k,j}} = 0, \quad \nu_{0,2}^{U_{m,n}^{k,j}} = \frac{j(j+1)}{n(n+1)} - \left(\frac{j}{n}\right)^2.$$

Consequently, from (4) and (5), we obtain $\lambda = 0$, and

$$\begin{aligned}\gamma(x,y) &= \sum_{k=0}^{m}\sum_{j=0}^{n}\left[\frac{k(k+1)}{m(m+1)} - \left(\frac{k}{m}\right)^2 + \frac{j(j+1)}{n(n+1)} - \left(\frac{j}{n}\right)^2\right] p_{m,n}^{k,j}(x,y) \\ &= \frac{(m-1)x(1-x)}{m(m+1)} + \frac{(n-1)y(1-y)}{n(n+1)} \le \frac{1}{m+1} + \frac{1}{n+1}.\end{aligned}$$

Now by applying Theorems 1 and 2, we get the required estimates for the difference of operators $\mathcal{B}_{m,n}$ and $\mathcal{U}_{m,n}$.

Proposition 5. The bivariate operators $\mathcal{D}_{m,n}$ and $\mathcal{U}_{m,n}$ verify the following inequalities:

(i) If $g \in C^2(\mathcal{J}^2)$, then

$$\left|\left(\mathcal{D}_{m,n} - \mathcal{U}_{m,n}\right)\left(g;(x,y)\right)\right| \le N_g \gamma(x,y) + \omega_1\left(g, \frac{1}{m+2} + \frac{1}{n+2}\right),$$

where

$$\begin{aligned}\gamma(x,y) &= \frac{m^2x(1-x) + m(1-x+x^2) + 1}{(m+2)^2(m+3)} + \frac{n^2y(1-y) + n(1-y+y^2) + 1}{(n+2)^2(n+3)} \\ &\quad + \frac{(m-1)x(1-x)}{m(m+1)} + \frac{(n-1)y(1-y)}{n(n+1)} \\ &\le 2\left(\frac{1}{m+1} + \frac{1}{n+1}\right).\end{aligned}$$

(ii) If $g \in C(\mathcal{J}^2)$, then

$$\left|\left(\mathcal{D}_{m,n} - \mathcal{U}_{m,n}\right)\left(g;(x,y)\right)\right| \le \alpha_1\omega_1\left(g, \frac{1}{m+2} + \frac{1}{n+2}\right) + \alpha_2\omega_2\left(g, \sqrt{\gamma(x,y)}\right).$$

Proof. To compute the desired estimations, it is enough to find λ and $\gamma(x,y)$ for the operators $\mathcal{D}_{m,n}$ and $\mathcal{U}_{m,n}$.

By (2), we obtain

$$a_1^{D_{m,n}^{k,j}} = \frac{k+1}{m+2}, \quad a_1^{U_{m,n}^{k,j}} = \frac{k}{m},$$
$$a_2^{D_{m,n}^{k,j}} = \frac{j+1}{n+2}, \quad a_2^{U_{m,n}^{k,j}} = \frac{j}{n},$$

and therefore we get

$$\nu_{2,0}^{D_{m,n}^{k,j}} = \frac{(k+1)(k+2)}{(m+2)(m+3)} - \left(\frac{k+1}{m+2}\right)^2, \quad \nu_{2,0}^{U_{m,n}^{k,j}} = \frac{k(k+1)}{m(m+1)} - \left(\frac{k}{m}\right)^2,$$
$$\nu_{0,2}^{D_{m,n}^{k,j}} = \frac{(j+1)(j+2)}{(n+2)(n+3)} - \left(\frac{j+1}{n+2}\right)^2, \quad \nu_{0,2}^{U_{m,n}^{k,j}} = \frac{j(j+1)}{n(n+1)} - \left(\frac{j}{n}\right)^2.$$

Substituting these values in (4) and (5), we may write

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$$\lambda = \max_{\substack{k,j \\ k=0,1,\dots,m. \\ j=0,1,\dots,n.}} \left\{ \left| \frac{k+1}{m+2} - \frac{k}{m} \right| + \left| \frac{j+1}{n+2} - \frac{j}{n} \right| \right\} = \frac{1}{m+2} + \frac{1}{n+2},$$

and

$$\begin{aligned}\gamma(x,y) &= \sum_{k=0}^{m}\sum_{j=0}^{n}\left[\frac{(k+1)(k+2)}{(m+2)(m+3)} - \left(\frac{k+1}{m+2}\right)^2 + \frac{k(k+1)}{m(m+1)} - \left(\frac{k}{m}\right)^2 + \frac{(j+1)(j+2)}{(n+2)(n+3)}\right. \\ &\quad \left. - \left(\frac{j+1}{n+2}\right)^2 + \frac{j(j+1)}{n(n+1)} - \left(\frac{j}{n}\right)^2\right] p_{m,n}^{k,j}(x,y) \\ &= \frac{m^2x(1-x) + m(1-x+x^2)+1}{(m+2)^2(m+3)} + \frac{(m-1)x(1-x)}{m(m+1)} \\ &\quad + \frac{n^2y(1-y)+n(1-y+y^2)+1}{(n+2)^2(n+3)} + \frac{(n-1)y(1-y)}{n(n+1)} \\ &\leq 2\left(\frac{1}{m+1} + \frac{1}{n+1}\right).\end{aligned}$$

Finally, applying Theorems 1 and 2, the required estimates are obtained.

Proposition 6. The bivariate form $\mathcal{K}_{m,n}$ of Bernstein-Kantorovich operators and $\mathcal{U}_{m,n}$ of genuine Bernstein-Durrmeyer operators satisfy the following inequalities:

(i) If $g \in C^2(\mathcal{J}^2)$, then

$$\left|\left(\mathcal{K}_{m,n} - \mathcal{U}_{m,n}\right)\left(g;(x,y)\right)\right| \leq N_g\gamma(x,y) + \omega_1\left(g, \frac{1}{2}\left(\frac{1}{m+2} + \frac{1}{n+2}\right)\right),$$

where

$$\begin{aligned}\gamma(x,y) &= \frac{(m-1)x(1-x)}{m(m+1)} + \frac{(n-1)y(1-y)}{n(n+1)} + \frac{1}{12}\left(\frac{1}{(n+1)^2} + \frac{1}{(m+1)^2}\right) \\ &\leq 2\left(\frac{1}{m+1} + \frac{1}{n+1}\right).\end{aligned}$$

(ii) If $g \in C(\mathcal{J}^2)$, then

$$\left|\left(\mathcal{K}_{m,n} - \mathcal{U}_{m,n}\right)\left(g;(x,y)\right)\right| \leq \alpha_1\omega_1\left(g, \frac{1}{2}\left(\frac{1}{m+2} + \frac{1}{n+2}\right)\right) + \alpha_2\omega_2\left(g, \sqrt{\gamma(x,y)}\right).$$

Proof. First we calculate $\gamma(x,y)$ and λ. From (2) we observe

$$a_1^{K_{m,n}^{k,j}} = \frac{2k+1}{2(m+1)}, \qquad a_1^{U_{m,n}^{k,j}} = \frac{k}{m},$$
$$a_2^{K_{m,n}^{k,j}} = \frac{2j+1}{2(n+1)}, \qquad a_2^{U_{m,n}^{k,j}} = \frac{j}{n}.$$

A direct calculation shows that

$$\nu_{2,0}^{K_{m,n}^{k,j}} = \frac{1}{12(m+1)^2}, \qquad \nu_{2,0}^{U_{m,n}^{k,j}} = \frac{k(k+1)}{m(m+1)} - \left(\frac{k}{m}\right)^2,$$
$$\nu_{0,2}^{K_{m,n}^{k,j}} = \frac{1}{12(n+1)^2}, \qquad \nu_{0,2}^{U_{m,n}^{k,j}} = \frac{j(j+1)}{n(n+1)} - \left(\frac{j}{n}\right)^2.$$

Thus, from (4) and (5), we have

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$$\lambda = \max_{\substack{k,j \\ k=0,1,\ldots,m. \\ j=0,1,\ldots,n.}} \left\{ \left| \frac{2k+1}{2(m+1)} - \frac{k}{m} \right| + \left| \frac{2j+1}{2(n+1)} - \frac{j}{n} \right| \right\} = \frac{1}{2}\left(\frac{1}{m+2} + \frac{1}{n+2} \right),$$

and

$$\begin{aligned} &\gamma(x,y) \\ &= \sum_{k=0}^{m} \sum_{j=0}^{n} \left[\frac{1}{12(m+1)^2} + \frac{k(k+1)}{m(m+1)} - \left(\frac{k}{m}\right)^2 + \frac{j(j+1)}{n(n+1)} - \left(\frac{j}{n}\right)^2 + \frac{1}{12(n+1)^2} \right] p_{m,n}^{k,j}(x,y) \\ &= \frac{(m-1)x(1-x)}{m(m+1)} + \frac{(n-1)y(1-y)}{n(n+1)} + \frac{1}{12}\left(\frac{1}{(n+1)^2} + \frac{1}{(m+1)^2} \right) \\ &\leq \ 2\left(\frac{1}{m+1} + \frac{1}{n+1} \right). \end{aligned}$$

The required estimates follow immediately by Theorems 1 and 2.

CONCLUSIONS

Finding estimates for the differences of bivariate positive linear operators is a recently developed topic in approximation theory. In this direction we have proved some general results regarding the differences of bivariate operators. Our results involve the most extensively used measuring tools like modulus of smoothness and K-functional in two variables. Considering suitable operators, one can find their quantitative difference estimates. This approach can be helpful in determining the difference estimates of other operators.

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