Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

Maejo International Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

On zero-inflated mixed Poisson transmuted exponential distribution: Properties and applications to observation with excess zeros

Ademola Abiodun Adetunji [1, 2, *] and Shamsul Rijal Muhammed Sabri [1]

[1] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia
[2] Department of Statistics, Federal Polytechnic, 35101, Ile-Oluji, Nigeria
* Corresponding author, e-mail: adecap4u@gmail.com

Received: 26 October 2022 / Accepted: 17 April 2023 / Published: 21 April 2023

Abstract: Zero-inflated distributions are assumed when count observations are characterised by an excess frequency of zero. This study utilises the cubic rank transmutation map to extend the exponential distribution and obtain a new mixing distribution. The distribution is then used to obtain a new mixed Poisson distribution and its zero-inflated form. Different moment-based mathematical properties of the mixed Poisson distributions and their zero-inflated forms are presented. Five count data sets with varying percentages of zero counts are assessed with new propositions and with both Poisson and negative binomial distributions (along with their respective zero-inflated forms). Performance is compared using both *–2LL* and chi-square goodness of fit. The new proposition outperforms both Poisson and negative binomial distributions (and their zero-inflated forms). Results also reveal that zero-inflated forms of the new proposition are inferior to their classical form. In most cases the classical negative binomial distribution also provides a better fit than its zero-inflated form while the zero-inflated Poisson distribution outperforms the Poisson distribution. In conclusion, most mixed Poisson distributions exhibit the ability to effectively model the observations with excess zero and tend to provide a better fit to the count observations with excess zero than their zero-inflated forms.

Keywords: cubic rank transmutation, count observations, excess zero, mixed Poisson distribution, maximum likelihood estimation

---

INTRODUCTION

When analysing count data, the frequency of observations can be found to be characterised by an unusual frequency of zeros. This can give results that are not reliable when applying distributions in their classical forms. The case when there are too few zeros is termed zero-deflation while it is termed zero-inflation when there are too many zeros. The zero-deflated feature characteristics occur when the observed frequency of zeros in a data set is lower than the expected frequency. In the literature this scenario is less frequently reported [1, 2] than the situation where zero frequency is substantially higher [3].

In diverse areas of applied statistics, count data analysis takes the central stage. In many situations, the classical Poisson distribution is assumed but in many cases there are perceived reasons not to utilise the distribution. This usually occurs when the data sets are dispersed or there are too many (or too few) zero counts. Generally, zero-modified distributions are assumed when the frequency of zero counts in an observation is below (or beyond) expectation. The modification process involves paying special attention to the frequency of zeros in the observations. Among the common techniques for zero modifications are zero-hurdle (ZH), zero-truncated (ZT) and zero-inflated (ZI) distributions depending on the nature of zero counts.

The ZH modification is assumed when the random variable is presumed to have come from two segments [4]. In the first segment the variable assumes zero value (zero mass) while it assumes positive non-zero values (truncated counts) in the second part. There is a ZT distribution when we isolate the probability of having a zero count from the general observations.

The ZI distribution [5] is useful in modelling observations with several zeros by assigning an extra probability to the occurrence of zero counts. The distribution is applicable when excess zeros result from two different processes: one in which zeros occur by chance just like ones, twos, etc. (sampling zeros); and the other where some data are constrained to be zeros [5, 6]. An example of a phenomenon that best illustrates the application of the zero-inflation model is claim frequency. Policyholders may have no case of an accident in automobile insurance, depicting a true zero claim. There could also be situations when a policyholder is involved in a minor accident and may not report it for a claim, indicating a false zero. This is more prominent in systems that practice the Bonus Manus System of premium adjustment. A major difference between the ZH and ZI distributions is in their conceptualisations of zero counts [7]. Some researchers utilised both interchangeably [8, 9].

A major demerit in assuming the classical Poisson distribution for count data is the assumption of the equality of its mean and variance [10]. Count data are usually overdispersed [11 – 13]; hence several methods have been introduced for their modelling [14, 15]. The mixed Poisson distribution first utilised on the gamma mixing distribution [16] is among the most widely used techniques for modelling dispersed observations. The method assumes a mixing distribution for the parameter of the Poisson distribution. In obtaining mixed Poisson distribution, different mixing distributions have been assumed for the Poisson parameter. Given the probability function of a discrete random X with the Poisson distribution as $f(x|\lambda)$ and suppose parameter λ is assumed to follow a probability distribution function (PDF) given as $\pi(\lambda)$, then the mixed Poisson distribution is obtained by solving for the conditional distribution of X in equation (1).

$$P_x = \int_0^\infty f(x|\lambda)\,\pi(\lambda) d\lambda \qquad (1)$$

A detailed survey of different choices of $\pi(\lambda)$ is provided [10]. Also, different mathematical properties of the mixed Poisson distribution are provided [17].

When the exponential distribution is assumed as the mixing distribution, the geometric distribution is the resulting mixed Poisson distribution. To improve the flexibility of the mixing

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

distributions, this study utilises the cubic transmutation map [18] to extend the exponential distribution and obtain a new mixing distribution. Different forms of extended exponential distributions pervade the literature. These distributions have been applied in economics, reliability, industry, and engineering [19 – 21]. Most of these distributions have different forms of failure rate (unlike the classical exponential distribution with constant failure rate) making them more suitable for modelling observations with different shapes. In the mixed Poisson paradigm, the shapes of any mixing distribution mimic those of the resulting mixed Poisson distribution [17]. Also, the tail properties of the mixing distribution have a resemblance with those of the obtained mixed Poisson distribution [22, 23]. The choice of the transmuted exponential distribution in this study is to allow more flexibility inherent in the extended exponential distribution for the newly proposed mixed Poisson distribution. The zero-inflated form of the new proposition is also obtained. The performance of the new proposition is assessed on count observations with a higher frequency of zero and comparisons are made with the Poisson and the negative binomial distributions (along with their zero-inflated forms).

CUBIC RANK TRANSMUTED EXPONENTIAL DISTRIBUTION

The exponential distribution plays an important role in the probability distribution theory with applications of its different extended forms in lifetime observation modelling and reliability analysis [20, 21, 24]. If a random variable λ follows an exponential distribution with parameter θ, then its distribution function (CDF) is

$$G(\lambda) = 1 - e^{-\theta\lambda}; \qquad \theta > 0,\ \lambda > 0. \tag{2}$$

If a baseline distribution has the distribution function as defined in equation (2), its cubic rank transmutation map [18] has the CDF of the form:

$$F(\lambda) = (1-p)G(\lambda) + 3p\big(G(\lambda)\big)^2 - 2p\big(G(\lambda)\big)^3. \tag{3}$$

The cubic transmuted exponential (CTE) distribution is obtained by inserting equation (2) into equation (3). Hence the CDF and PDF of the CTE distribution are respectively obtained as

$$F(\lambda) = 1 - e^{-\theta\lambda} + pe^{-\theta\lambda} + 2pe^{-3\theta\lambda} - 3pe^{-2\theta\lambda}, \tag{4}$$

$$f(\lambda) = \theta e^{-\theta\lambda}\big(1 - p + 6pe^{-\theta\lambda} - 6pe^{-2\theta\lambda}\big). \tag{5}$$

The shapes of the PDF for the CTE distribution for different values of λ are presented in Figure 1. The distribution is skewed and unimodal for different parameter combinations.

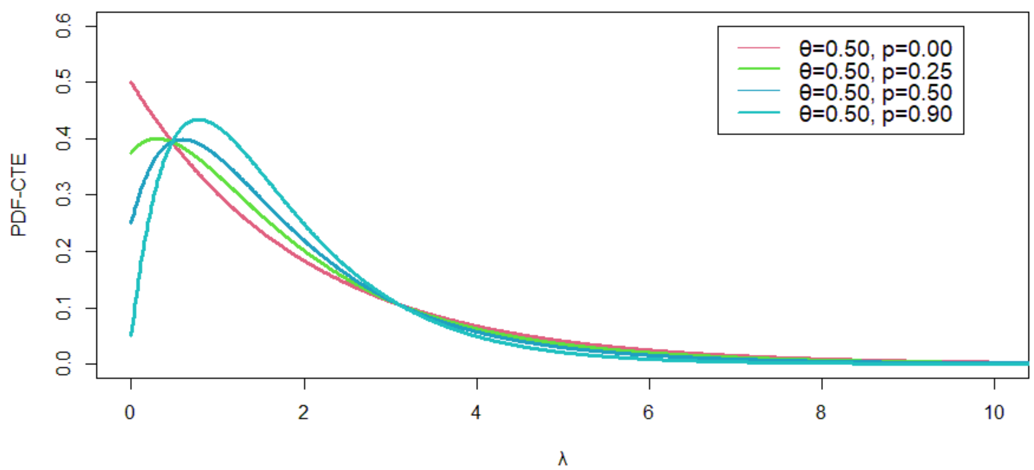

Figure 1. Shapes of PDF for CTE distribution

Moment and Moment Generating Function of CTE Distribution

Proposition 1. If a random variable λ has a CTE distribution as defined in equation (5), the rth moment is obtained as

$$E(\lambda^r) = \left(1 - p + \frac{3p}{2^r} - \frac{2p}{3^r}\right)\frac{r!}{\theta^r}. \tag{6}$$

Proof:

$$E(\lambda^r) = \int_0^\infty \lambda^r f(\lambda)\, d\lambda = \int_0^\infty \lambda^r \theta e^{-\theta\lambda}\left(1 - p + 6pe^{-\theta\lambda} - 6pe^{-2\theta\lambda}\right) d\lambda$$

$$= \theta\int_0^\infty \lambda^r e^{-\theta\lambda}\, d\lambda - p\theta\int_0^\infty \lambda^r e^{-\theta\lambda}\, d\lambda + 6p\theta\int_0^\infty \lambda^r e^{-2\theta\lambda}\, d\lambda - 6p\theta\int_0^\infty \lambda^r e^{-3\theta\lambda}\, d\lambda$$

$$= \frac{r!}{\theta^r} - \frac{pr!}{\theta^r} + \frac{3pr!}{(2\theta)^r} - \frac{2pr!}{(3\theta)^r} = \left(1 - p + \frac{3p}{2^r} - \frac{2p}{3^r}\right)\frac{r!}{\theta^r}.$$

Proposition 2. If a random variable λ has a CTE distribution, the moment generating function is obtained as

$$E\left(e^{t\lambda}\right) = \frac{\theta}{\theta - t} - \frac{p\theta}{\theta - t} + \frac{6p\theta}{2\theta - t} - \frac{6p\theta}{3\theta - t}. \tag{7}$$

Proof:

$$E\left(e^{t\lambda}\right) = \int_0^\infty e^{t\lambda} f(\lambda)\, d\lambda = \int_0^\infty e^{t\lambda}\,\theta e^{-\theta\lambda}\left(1 - p + 6pe^{-\theta\lambda} - 6pe^{-2\theta\lambda}\right) d\lambda$$

$$= \int_0^\infty \theta e^{-(\theta-t)\lambda} - p\theta e^{-(\theta-t)\lambda} + 6p\theta e^{-(2\theta-t)\lambda} - 6p\theta e^{-(3\theta-t)\lambda}\, d\lambda$$

$$= \theta\int_0^\infty e^{-(\theta-t)\lambda}\, d\lambda - p\theta\int_0^\infty e^{-(\theta-t)\lambda}\, d\lambda + 6p\theta\int_0^\infty e^{-(2\theta-t)\lambda}\, d\lambda - 6p\theta\int_0^\infty e^{-(3\theta-t)\lambda}\, d\lambda$$

$$= \frac{\theta}{\theta - t} - \frac{p\theta}{\theta - t} + \frac{6p\theta}{2\theta - t} - \frac{6p\theta}{3\theta - t}.$$

MIXED POISSON TRANSMUTED EXPONENTIAL DISTRIBUTION

Proposition 3. Given that $X \sim Poisson(\lambda)$, where λ has the PDF as given in equation (5), the probability mass function (PMF) of the mixed Poisson transmuted exponential distribution (MPTED) is obtained as

$$P_x = \frac{\theta(1-p)}{(1+\theta)^{x+1}} + \frac{6p\theta}{(1+2\theta)^{x+1}} - \frac{6p\theta}{(1+3\theta)^{x+1}}. \tag{8}$$

Proof:

$$P_x = \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!}.f(\lambda)d\lambda = \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!}.\theta e^{-\theta\lambda}\left(1 - p + 6pe^{-\theta\lambda} - 6pe^{-2\theta\lambda}\right) d\lambda$$

$$= \frac{1}{x!}\int_0^\infty \lambda^x\left(\theta e^{-(1+\theta)\lambda} - p\theta e^{-(1+\theta)\lambda} + 6p\theta e^{-(1+2\theta)\lambda} - 6p\theta e^{-(1+3\theta)\lambda}\right) d\lambda$$

$$= \frac{1}{x!}\left(\frac{\theta x!}{(1+\theta)^{x+1}} - \frac{p\theta x!}{(1+\theta)^{x+1}} + \frac{6p\theta x!}{(1+2\theta)^{x+1}} - \frac{6p\theta x!}{(1+3\theta)^{x+1}}\right)$$

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

$$= \frac{\theta(1-p)}{(1+\theta)^{x+1}} + \frac{6p\theta}{(1+2\theta)^{x+1}} - \frac{6p\theta}{(1+3\theta)^{x+1}}.$$

The shapes of the PMF of the MPTED (Figure 2) show that for different values of the parameters of the distribution, it is unimodal, positively skewed, and can model observations with many zeros.

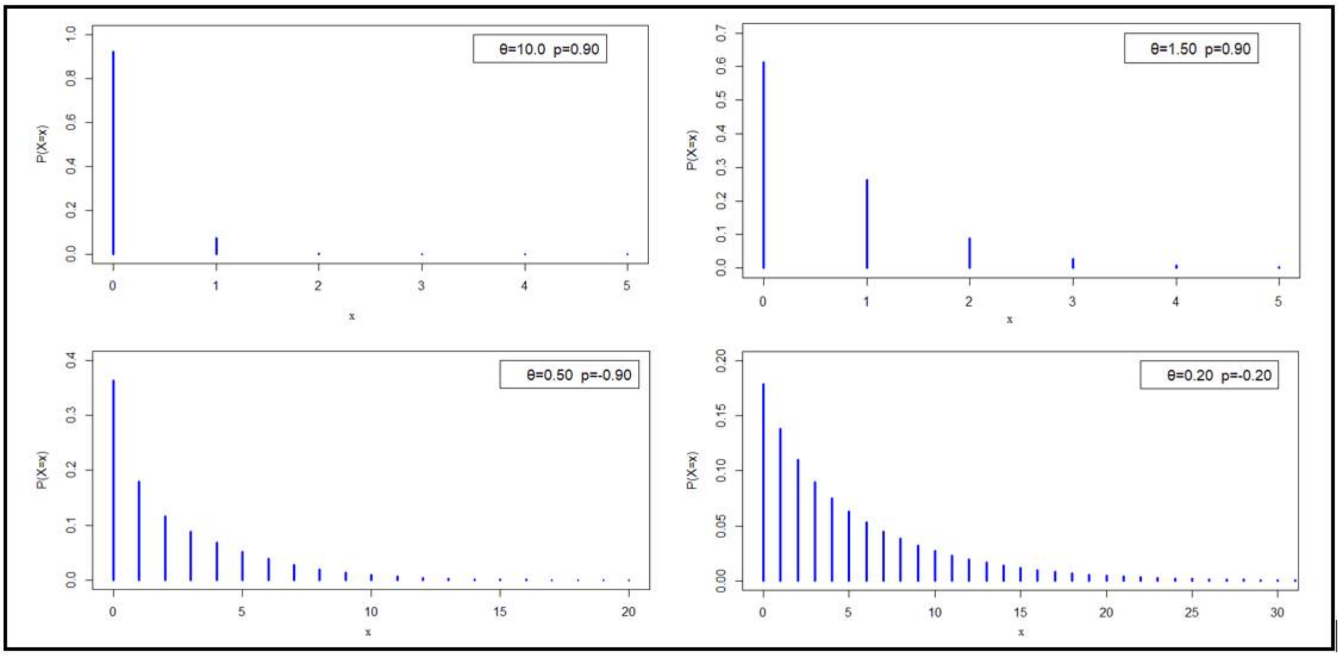

Figure 2. Shapes of PMF for MPTED

Proposition 4. Given that $f(\lambda)$ is the mixing distribution of a random variable *X* with the MPTED, the probability generating function (PGF) of mixed Poisson distribution is defined [25] as

$$P_x(z) = \int_0^\infty e^{\lambda(z-1)} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{\lambda(z-1)}\, \theta e^{-\theta\lambda}\left(1 - p + 6pe^{-\theta\lambda} - 6pe^{-2\theta\lambda}\right) d\lambda$$

$$= \theta \int_0^\infty \left(e^{-(1-z+\theta)\lambda} - pe^{-(1-z+\theta)\lambda} + 6pe^{-(1-z+2\theta)\lambda} - 6pe^{-(1-z+3\theta)\lambda}\right) d\lambda$$

$$= \theta \int_0^\infty \left(\frac{1}{(1+\theta-z)} e^{-v} - \frac{p}{(1+\theta-z)} e^{-v} + \frac{6p}{(1+2\theta-z)} e^{-v} - \frac{6p}{(1+3\theta-z)} e^{-v}\right) dv$$

$$= \frac{\theta}{(1+\theta-z)} - \frac{p\theta}{(1+\theta-z)} + \frac{6p\theta}{(1+2\theta-z)} - \frac{6p\theta}{(1+3\theta-z)}.$$

Hence the PGF of the MPTED is

$$P_x(z) = \frac{\theta(1-p)}{(1+\theta-z)} + \frac{6p\theta}{1+2\theta-z} - \frac{6p\theta}{1+3\theta-z}. \qquad (9)$$

The moment generating function (MGF) for the distribution is obtained by replacing *z* with e^t in equation (9). Hence the MGF is obtained as

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

$$M_X(t)=\frac{\theta(1-p)}{(1+\theta-e^t)}+\frac{6p\theta}{1+2\theta-e^t}-\frac{6p\theta}{1+3\theta-e^t}. \quad (10)$$

Using (10), the first four moments for the MPTED are presented in equations (11) to (14):

$$E(X)=\frac{6-p}{6\theta}. \quad (11)$$

$$E(X^2)=\frac{36-17p-3p\theta+18\theta}{18\theta^2} \quad (12)$$

$$E(X^3)=\frac{216-151p-6p\theta^2+36\theta^2-102p\theta+216\theta}{36\theta^3} \quad (13)$$

$$E(X^4)=\frac{1296-9p\theta^3-357p\theta^2+54\theta^3-1359p\theta+756\theta^2-1085p+1944\theta}{54\theta^4} \quad (14)$$

The skewness and kurtosis for the MPTED are obtained using the moment-based relationships [26] respectively as

$$S_k=\frac{E(X^3)-3E(X^2)E(X)+2\big(E(X)\big)^3}{\big(Var(X)\big)^{\frac{3}{2}}}$$

$$Ku=\frac{E(X^4)-4E(X^3)E(X)+6E(X^2)\big(E(X)\big)^2-3\big(E(X)\big)^4}{\big(Var(X)\big)^2}.$$

The degree of dispersion in observations is measured with the dispersion index (DI). If the DI is greater than 1, it implies an over-dispersion; when it is less than 1, it indicates an under-dispersion; and there is an equi-dispersion when it is equal to 1. The measure is obtained from the ratio of the variance and the mean, i.e. $\frac{Var(N)}{E(N)}$. Table 1 shows simulation results of some parameters of the MPTED to assess the behaviour of its skewness, kurtosis and DI.

Table 1. Skewness, kurtosis and DI for some parameters of MPTED

	Skewness			*Kurtosis*			*DI*		
	$\theta=0.2$	$\theta=0.5$	$\theta=8.0$	$\theta=0.2$	$\theta=0.5$	$\theta=8.0$	$\theta=0.2$	$\theta=0.5$	$\theta=8.0$
$p=-0.8$	1.730	1.852	3.237	6.939	7.543	15.480	7.490	3.596	1.162
$p=-0.5$	1.838	1.932	3.278	7.649	8.138	15.775	6.994	3.397	1.150
$p=0.0$	2.008	2.041	3.333	9.033	9.167	16.111	6.000	3.000	1.125
$p=0.5$	2.081	2.040	3.364	10.253	9.743	16.069	4.750	2.500	1.094
$p=0.8$	1.913	1.869	3.361	9.620	8.879	15.706	3.846	2.138	1.071

Remarks

i. When p is fixed, both skewness and kurtosis increase as parameter θ increases.
ii. When p is fixed, the DI decreases as θ increases.
iii. When θ is fixed, the skewness and the kurtosis increase as parameter p increases.
iv. When θ is fixed, the DI decreases as p increases.

Maximum Likelihood Estimation of MPTED

Assuming $x_1, x_2, \ldots, x_n$ are random samples of size *n* drawn from the MPTED (p,θ), the log-likelihood function for the distribution is obtained as

$$P_x=\frac{\theta(1-p)}{(1+\theta)^{x+1}}+\frac{6p\theta}{(1+2\theta)^{x+1}}-\frac{6p\theta}{(1+3\theta)^{x+1}}$$

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

$$\mathcal{L}=\prod_{i=1}^{n}P_{(x_i)}=\prod_{i=1}^{n}\left(\frac{\theta(1-p)}{(1+\theta)^{x+1}}+\frac{6p\theta}{(1+2\theta)^{x+1}}-\frac{6p\theta}{(1+3\theta)^{x+1}}\right)$$

$$\ell=log\,\mathcal{L}=\sum_{i=1}^{n}log\left(\frac{\theta(1-p)}{(1+\theta)^{x+1}}+\frac{6p\theta}{(1+2\theta)^{x+1}}-\frac{6p\theta}{(1+3\theta)^{x+1}}\right).$$

The estimators for (p,θ), denoted with $(\hat{p},\hat{\theta})$, are the solutions of the function using the *optimr* package [27] in the *R-language* [28].

Zero-Inflated MPTED

Proposition 5. If a random variable *X* has an MPTED with PMF as given in equation (8), and if the inflation parameter is denoted with a, then a discrete random variable X_Z has a zero-inflated MPTED (ZI-MPTED) if its PMF is defined as

$$P_x^Z=\begin{cases}a+(1-a)P_0, & x=0\\ (1-a)P_x, & x=1,2,3,\ldots\end{cases}$$

This is obtained as

$$P_x^Z=\begin{cases}a+(1-a)\left(\frac{\theta(1-p)}{(1+\theta)}+\frac{6p\theta}{(1+2\theta)}-\frac{6p\theta}{(1+3\theta)}\right), & x=0\\ (1-a)\left(\frac{\theta(1-p)}{(1+\theta)^{x+1}}+\frac{6p\theta}{(1+2\theta)^{x+1}}-\frac{6p\theta}{(1+3\theta)^{x+1}}\right), & x=1,2,3,\ldots\end{cases} \quad (15)$$

Proof:

Since the PMF of the MPTED is given as $P_x=\frac{\theta(1-p)}{(1+\theta)^{x+1}}+\frac{6p\theta}{(1+2\theta)^{x+1}}-\frac{6p\theta}{(1+3\theta)^{x+1}}$ and its realisation at $x=0$ is obtained as $P_0=\frac{\theta(1-p)}{(1+\theta)}+\frac{6p\theta}{(1+2\theta)}-\frac{6p\theta}{(1+3\theta)}$. Hence the result.

Mathematical Properties of ZI-MPTED

Recall that the PGF of the MPTED is given in equation (9) as $P_x(z)$. Therefore, the PGF of the ZI-MPTED, denoted by $P_x^Z(z)$, is obtained using $P_n^Z(z)=(1-a)P_x(z)$:

$$P_x^Z(z)=(1-a)\left(\frac{\theta(1-p)}{(1+\theta-z)}+\frac{6p\theta}{1+2\theta-z}-\frac{6p\theta}{1+3\theta-z}\right). \quad (16)$$

The corresponding MGF is expressed as

$$M_x^Z(t)=(1-a)\left(\frac{\theta(1-p)}{1+\theta-e^t}+\frac{6p\theta}{1+2\theta-e^t}-\frac{6p\theta}{1+3\theta-e^t}\right). \quad (17)$$

The first four moments of the ZI-MPTED are obtained as

$$m_1=(1-a)\frac{6-p}{6\theta} \quad (18)$$

$$m_2=(1-a)\frac{36-17p-3p\theta+18\theta}{18\theta^2} \quad (19)$$

$$m_3=(1-a)\frac{216-151p-6p\theta^2+36\theta^2-102p\theta+216\theta}{36\theta^3} \quad (20)$$

$$m_4=(1-a)\frac{1296-9p\theta^3-357p\theta^2+54\theta^3-1359p\theta+756\theta^2-1085p+1944\theta}{54\theta^4}. \quad (21)$$

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

Simulated skewness, kurtosis and DI for some combinations of parameters of the ZI-MPTED are presented in Tables 2 and 3.

Table 2. Skewness, kurtosis and DI for some parameters of ZI-MPTED when $a = 0.2$

	Skewness			*Kurtosis*			*DI*		
	$\theta = 0.2$	$\theta = 0.5$	$\theta = 8.0$	$\theta = 0.2$	$\theta = 0.5$	$\theta = 8.0$	$\theta = 0.2$	$\theta = 0.5$	$\theta = 8.0$
$p = -0.8$	1.966	2.123	3.711	7.906	8.786	19.554	8.624	4.049	1.191
$p = -0.5$	2.057	2.195	3.756	8.614	9.407	19.931	8.077	3.831	1.177
$p = 0.0$	2.187	2.285	3.817	9.893	10.419	20.370	7.000	3.400	1.150
$p = 0.5$	2.194	2.257	3.852	10.730	10.812	20.346	5.667	2.867	1.117
$p = 0.8$	1.984	2.072	3.849	9.697	9.708	19.919	4.713	2.485	1.093

Table 3. Skewness, kurtosis and DI for some parameters of ZI-MPTED when $a = 0.9$

	Skewness			*Kurtosis*			*DI*		
	$\theta = 0.2$	$\theta = 0.5$	$\theta = 8.0$	$\theta = 0.2$	$\theta = 0.5$	$\theta = 8.0$	$\theta = 0.2$	$\theta = 0.5$	$\theta = 8.0$
$p = -0.8$	6.605	7.021	11.390	58.073	65.795	163.641	12.590	5.636	1.290
$p = -0.5$	6.729	7.135	11.509	61.213	68.862	166.822	11.869	5.347	1.272
$p = 0.0$	6.858	7.246	11.677	65.762	73.042	170.737	10.500	4.800	1.238
$p = 0.5$	6.706	7.104	11.774	65.633	72.398	171.232	8.875	4.150	1.197
$p = 0.8$	6.240	6.709	11.773	57.297	64.552	168.457	7.746	3.698	1.169

Remarks

i. The behaviour of the simulations is similar to the one obtained for the MPTED in Table 1.
ii. The ZI-MPTED has a higher value of all statistics when compared with the MPTED.
iii. Increase in the value of the zero-inflation parameter increases all statistics.

Maximum Likelihood Estimation of ZI-MPTED

Given a random sample of size n $(x_1, x_2, \ldots, x_n)$ from the ZI-MPTED with PMF P_x indexed with (θ, p, a) where a is the zero-inflation parameter, then parameters of the distribution can be estimated using the maximum likelihood estimation (MLE) method. The likelihood function is defined as

$$\mathcal{L}(\theta, p, a) = \prod_{n_0} (a + (1-a)P_0) \prod_{n_1} \big((1-a)P_x\big),$$

where n_0 is the frequency of zero in the observation; n_1 is the frequency of non-zero observations; $n = (n_0 + n_1)$; P_0 is the realisation of P_x at $x = 0$. The log-likelihood function is obtained as

$$\ell = \log \mathcal{L}(\theta, p, a) = n_0 \ln(a + (1-a)P_0) + n_1 \ln(1-a) + \left(\sum_{n_1} \ln(P_x)\right)$$

$$= n_0 \ln\left(a + (1-a)\left(\frac{\theta(1-p)}{(1+\theta)} + \frac{6p\theta}{(1+2\theta)} - \frac{6p\theta}{(1+3\theta)}\right)\right) + n_1 \ln(1-a) + \sum_{n_1} \ln\left(\left(\frac{\theta(1-p)}{(1+\theta)^{x+1}} + \frac{6p\theta}{(1+2\theta)^{x+1}} - \frac{6p\theta}{(1+3\theta)^{x+1}}\right)\right).$$

$$\frac{\partial \ell}{\partial a} = \frac{n_0(1-P_0)}{a + (1-a)P_0} - \frac{n_1}{(1-a)}$$

$$\hat{a} = \frac{n_0}{n_1} - \frac{n_1}{n}\left(\frac{P_0}{1-P_0}\right).$$

The MLE for parameter space (θ, p) are obtained numerically by solving $\frac{\partial \ell}{\partial \theta} = 0$ and $\frac{\partial \ell}{\partial p} = 0$. Solutions for these estimates contain non-linear equations. This makes it difficult to obtain analytical solutions. Using the optimr package [27] in the R-language [28], non-linear algorithms are used to obtain the estimates.

APPLICATIONS

The new propositions are examined on different count data sets. The first set of data is the frequency of one-year automobile insurance policies for Australian vehicle owners [26]. The second examined set of data is the 63,299 insurance count data from Belgium in 1993. The data have been previously utilised [29]. The third data set is the claim frequency of policyholders of a Turkish insurance company between 2012 – 2014. The data have been used on the Poisson-Shanker distribution [30] and on the zero-inflated modelling of claim frequency [31]. The fourth data set has been assessed on different Poisson-related distributions [32]. The fifth data set is the distribution of mistakes in copying groups of random digits [33]. The data set has also been used on the Poisson-Lindley and its different generalisations [34, 35].

The five data sets are assessed on the MPTED, ZI-MPTED, Poisson distribution, ZI Poisson (ZIP) distribution, negative binomial (NB) distribution, and ZI negative binomial (ZINB) distribution. Both *–2LL* and the chi-square goodness of fit are used for model comparison.

RESULTS

Tables 4 to 8 show results of applications of both mixed Poisson distribution and its zero-inflated form (along with the Poisson and NB distributions) on five different data sets with varying percentages of zeros. From Table 4, the MPTED provides the best fit for the first data set, while its zero-inflated form has the worst fit. It is also observed that the NB distribution performs better than the ZINB distribution. The ZIP distribution, however, performs better than the classical Poisson distribution.

For the second data set (Table 5), the MPTED has the least *–2LL*, followed by the NB distribution. The ZI-MPTED does not provide a good fit with the highest chi-square and *–2LL*. It is also noted that the ZIP distribution performs better than the Poisson distribution.

For Tables 6 to 8 (data sets III, IV and V), it is generally observed that

(i) the MPTED gives the best fit to the data sets;
(ii) the ZI-MPTED provides the worst fit to the data set;
(iii) the classical NB distribution outperforms the ZINB distribution in four out of the five assessed data sets;
(iv) the ZIP distribution outperforms the classical Poisson distribution in all cases.

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

Table 4. Parameter estimates on Australian claim frequency

Observation	Frequency	MPTED	ZI–MPTED	Poisson	ZIP	NB	ZINB
0	63232	63230.71	63231.79	63091.61	63230.49	63230.60	63317.89
1	4333	4332.75	4600.87	4593.07	4325.83	4330.57	4252.49
2	271	273.75	23.24	167.19	286.59	276.48	261.98
3	18	17.54	0.10	4.06	12.66	17.22	21.45
4	2	1.16	0.00	0.07	0.42	1.06	1.97
Estimate		$\hat{\theta}$=12.815 $\hat{p}$=0.405	$\hat{\theta}$=273.314 $\hat{p}$=-1.593 $\hat{a}$=-13.792	$\hat{\theta}$=0.073	$\hat{\theta}$=0.133 $\hat{p}$=0.451	$\hat{\theta}$=1.157 $\hat{p}$=0.941	$\hat{\theta}$=0.007 $\hat{p}$=0.878 $\hat{a}$=-76.060
-2 LL		36099.12	37714.42	36203.00	36104.40	36099.36	36211.16
χ^2		0.64	15499.49	177.66	9.07	0.98	2.51

Table 5. Parameter estimates on Belgian claim frequency

Observation	Frequency	MPTED	ZI–MPTED	Poisson	ZIP	NB	ZINB
0	57178	57182.90	57178.05	56949.763	57177.48	57188.34	57249.63
1	5617	5591.26	6061.41	6019.590	5584.80	5581.31	5558.90
2	446	479.55	59.03	318.135	504.87	485.28	438.37
3	50	41.21	0.50	11.209	30.43	40.47	45.91
4	8	3.70	0.00	0.296	1.38	3.30	5.40
Estimate		$\hat{\theta}$=8.548 $\hat{p}$=0.579	$\hat{\theta}$=140.070 $\hat{p}$=-1.530 $\hat{a}$=-9.899	$\hat{\theta}$=0.106	$\hat{\theta}$=0.181 $\hat{p}$=0.415	$\hat{\theta}$=1.279 $\hat{p}$=0.924	$\hat{\theta}$=0.008 $\hat{p}$=0.844 $\hat{a}$=-71.130
-2 LL		44126.50	46666.92	44301.08	44150.60	44128.62	44273.14
χ^2		2.34	23673.56	413.84	51.55	12.33	2.44

Table 6. Parameter estimates on Turkish claim frequency

Observation	Frequency	MPTED	ZI–MPTED	Poisson	ZIP	NB	ZINB
0	8544	8547.09	8543.99	8292.42	8544.19	8543.47	8561.78
1	1796	1789.79	2138.11	2201.64	1759.23	1795.62	1807.66
2	370	376.79	125.13	292.27	430.75	375.71	331.89
3	81	79.27	6.45	25.87	70.31	78.50	81.03
4	23	16.65	0.30	1.72	8.61	16.39	22.23
Estimate		$\hat{\theta}$=3.782 $\hat{p}$=-0.024	$\hat{\theta}$=22.455 $\hat{p}$=-1.492 $\hat{a}$=-3.006	$\hat{\theta}$=0.266	$\hat{\theta}$=0.490 $\hat{p}$=0.458	$\hat{\theta}$=1.009 $\hat{p}$=0.792	$\hat{\theta}$=0.006 $\hat{p}$=0.635 $\hat{a}$=-82.430
-2 LL		14059.43	15752.81	14306.32	14077.82	14059.43	14114.12
χ^2		2.61	3086.72	484.41	35.02	2.84	4.51

Table 7. Parameter estimates on Yeast cell count per square

Observation	Frequency	MPTED	ZI–MPTED	Poisson	ZIP	NB	ZINB
0	128	127.39	128.51	118.06	128.00	126.73	180.68
1	37	38.72	50.00	54.30	38.35	42.08	4.51
2	18	16.03	7.44	12.49	15.49	12.84	1.15
3	3	3.92	0.94	1.91	4.17	3.80	0.39
4	1	0.77	0.11	0.22	0.84	1.11	0.15
Estimate		$\hat{\theta}$=5.245 $\hat{p}$=-8.556	$\hat{\theta}$=9.116 $\hat{p}$=-2.083 $\hat{a}$=-1.467	$\hat{\theta}$=0.460	$\hat{\theta}$=0.808 $\hat{p}$=0.431	$\hat{\theta}$=1.195 $\hat{p}$=0.722	$\hat{\theta}$=0.004 $\hat{p}$=0.490 $\hat{a}$=-11.720
-2 LL		337.40	391.17	347.66	337.60	340.05	344.10
χ^2		0.60	30.52	12.16	0.81	2.88	517.31

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

Table 8. Parameter estimates on distribution of mistakes in copying groups of random digits

Observation	Frequency	MPTED	ZI–MPTED	Poisson	ZIP	NB	ZINB
0	35	34.96	35.00	27.41	35.00	33.95	35.08
1	11	10.95	16.51	21.47	11.25	14.49	14.75
2	8	8.61	6.07	8.41	8.04	6.39	5.07
3	4	3.70	1.81	2.20	3.83	2.85	2.32
4	2	1.25	0.47	0.43	1.37	1.28	1.19
	Estimate	$\hat{\theta}$=2.814 $\hat{p}$=-7.254	$\hat{\theta}$=3.609 $\hat{p}$=-2.709 $\hat{a}$=-0.521	$\hat{\theta}$=0.783	$\hat{\theta}$=1.430 $\hat{p}$=0.452	$\hat{\theta}$=0.938 $\hat{p}$=0.545	$\hat{\theta}$=0.003 $\hat{p}$=0.315 $\hat{a}$=-128.10
	-2 LL	144.05	172.62	155.09	143.84	146.74	149.91
	χ^2	0.52	10.14	14.44	0.30	2.15	4.43

DISCUSSION AND CONCLUSIONS

Using the cubic rank transmutation map [18] on the classical exponential distribution to obtain a new mixing distribution, this study proposes a new mixed Poisson distribution (MPTED) and its zero-inflated form. Different moment-based mathematical properties of the new propositions are presented. Comparisons are made with five data sets with varying percentages of zero counts.

Since the data sets assessed are plagued with relatively higher percentages of zero observations, it is assumed that the zero-inflated distribution will give a better fit. Results obtained, however, show that the MPTED gives a better fit to the data set than its zero-inflated forms. Results also reveal that the classical NB distribution also provides a better fit than its zero-inflated form in most cases while the zero-inflated Poisson distribution outperforms the classical Poisson distribution.

An assessment of the shapes of the MPTED shows that in its classical form, the distribution can efficiently be utilised to model observations with excess zeros. Thus, introducing a special form that gives special attention to excess zeros will negate the already zero-modified form of the distribution. Also, since the NB distribution is a mixed Poisson distribution (when the gamma distribution is assumed as the mixing distribution), it tends to effectively model observations with excess zeros, and thus, its zero-inflated form may also give unnecessary attention to the already zero-modified natural form of the distribution.

Unlike the mixed Poisson distribution, the classical Poisson distribution does not give special attention to zero observations and it makes sense to modify it when observations are plagued with excess zeros. This is noticed in the results where the ZIP distribution gives a better fit than the classical Poisson distribution when applied to the five count data sets.

Results from this study have shown that most mixed Poisson distributions are naturally in zero-inflated forms. They tend to provide a good fit to count observations with excess zeros. Obtaining their zero-inflated forms may be counter-productive.

REFERENCES

1. J. F. Angers and A. Biswas, "A Bayesian of zero-inflated generalised Poisson model", *Comput. Stat. Data Anal.*, **2003**, *42*, 37-46.
2. E. Dietz and D. Böhning, "On estimation of the Poisson parameter in zero-modified Poisson models", *Comput. Stat. Data Anal.*, **2000**, *34*, 441-459.

3. K. S. Conceição, F. Louzada, M. G. Andrade and E. S. Helou, "Zero-modified power series distribution and its hurdle distribution version", *J. Stat. Comput. Simul.*, **2017**, *87*, 1842-1862.
4. J. Mullahy, "Specification and testing of some modified count data models", *J. Econom.*, **1986**, *33*, 341-365.
5. D. Lambert, "Zero-inflated Poisson regression, with an application to defects in manufacturing", *Technometr.*, **1992**, *34*, 1-14.
6. M. Shahmandi, P. Wilson and M. Thelwall, "A new algorithm for zero-modified models applied to citation counts", *Scientometr.*, **2020**, *125*, 993-1010.
7. C. X. Feng, "A comparison of zero-inflated and hurdle models for modeling zero-inflated count data", *J. Stat. Distrib. Appl.*, **2021**, *8*, Art.no.8.
8. L. Xu, A. D. Paterson, W. Turpin and W. Xu, "Assessment and selection of competing models for zero-inflated microbiome data", *PLoS One*, **2015**, *10*, Art.no.e0129606.
9. F. Tüzen, S. Erbaş and H. Olmuş, "A simulation study for count data models under varying degrees of outliers and zeros", *Commun. Stat. Simul. Comput.*, **2020**, *49*, 1078-1088.
10. S. H. Ong, Y. C. Low and K. K. Toh, "Recent developments in mixed Poisson distributions", *ASM Sci. J.*, **2021**, *14*, 1-10.
11. C. O. Omari, S. G. Nyambura and J. M. W. Mwangi, "Modeling the frequency and severity of auto insurance claims using statistical distributions", *J. Math. Finance*, **2018**, *8*, 137-160.
12. A. A. Ademola and S. R. M. Sabri, "Modelling claim frequency in insurance using count models", *Asian J. Prob. Stat.*, **2021**, *14*, 14-20.
13. N. M. Khan and M. H. M. Khan, "Model for analysing counts with over-,equi-and under-dispersion in actuarial statistics", *J. Math. Stat.*, **2010**, *6*, 92-95.
14. H. Zakerzadeh and A. Dolati, "Generalised Lindley distribution", *J. Math. Ext.*, **2009**, *3*, 13-25.
15. K. K. Das, I. Ahmed and S. Bhattacharjee, "A new three-parameter Poisson-Lindley distribution for modelling over-dispersed count data", *Int. J. Appl. Eng. Res.*, **2018**, *13*, 16468-16477.
16. M. Greenwood and G. U. Yule, "An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents", *J. R. Stat. Soc.*, **1920**, *83*, 255-279.
17. D. Karlis and E. Xekalaki, "Mixed Poisson distributions", *Int. Stat. Rev.*, **2005**, *73*, 35-58.
18. M. M. Rahman, B. Al-Zahrani, S. H. Shahbaz and M. Q. Shahbaz, "Cubic transmuted uniform distribution: An alternative to beta and Kumaraswamy distributions", *Eur. J. Pure Appl. Math.*, **2019**, *12*, 1106-1121.
19. Y. A. Mohammed, B. Yatim and S. Ismail, "Mixture model of the exponential, gamma and Weibull distributions to analyse heterogeneous survival data", *J. Sci. Res. Rep.*, **2015**, *5*, 132-139.
20. G. A. S. Aguilar, F. A. Moala and G. M. Cordeiro, "Zero-truncated Poisson exponentiated gamma distribution : Application and estimation methods", *J. Stat. Theory Pract.*, **2019**, *13*, Art.no.57.
21. M. Rasekhi, M. Alizadeh, E. Altun, G. G. Hamedani, A. Z. Afify and M. Ahmad, "The modified exponential distribution with applications", *Pak. J. Stat.*, **2017**, *33*, 383-398.
22. B. Rémillard and R. Theodorescu, "Inference based on the empirical probability generating function for mixtures of Poisson distributions", *Stat. Decis.*, **2000**, *18*, 349-366.

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 68-80

23. G. E. Willmot, "Asymptotic tail behaviour of Poisson mixtures with applications", *Adv. Appl. Probab.*, **1990**, *22*, 147-159.
24. Y. Yang, W. Tian and T. Tong, "Generalised mixtures of exponential distribution and associated inference", *Math.*, **2021**, *9*, Art.no.1371.
25. G. E. Willmot, "Mixed compound Poisson distributions", *ASTIN Bull.*, **1986**, *16*, 59-79.
26. P. De Jong and G. Z. Heller, "Generalised linear models for insurance data", 1st Edn., Cambridge University Press, Cambridge, **2008**, pp.29-30.
27. J. C. Nash, R. Varadhan and G. Grothendieck, "Package 'optimr'": A replacement and extension of the 'optim' function", **2022**, cran.r-project.org/web/packages/optimr/optimr.pdf (Accessed: October 2022).
28. R-Core Team, "R: A language and environment for statistical computing", R Foundation for Statistical Computing, Vienna, Austria, **2020**, https://www.r-project.org/ (Accessed: June 2022).
29. H. Zamani, N. Ismail and P. Faroughi, "Poisson-weighted exponential univariate version and regression model with applications", *J. Math. Stat.*, **2014**, *10*, 148-154.
30. A. Meytrianti, S. Nurrohmah and M. Novita, "An alternative distribution for modelling overdispersion count data: Poisson-Shanker distribution", *ICSA Int. Conf. Stat. Anal.*, **2019**, *1*, 108-120.
31. L. S. Sarul and S. Sahin, "An application of claim frequency data using zero-inflated and hurdle models in general insurance", *J. Bus. Econ. Finance*, **2015**, *4*, 732-743.
32. R. Shanker and H. Fesshave, "On Poisson-Lindley distribution and its applications to biological sciences", *Biometrics Biostat. Int. J.*, **2015**, *2*, 103-107.
33. C. D. Kemp and A. W. Kemp, "Some properties of the 'Hermite' distribution", *Biometrika*, **1965,** *52*, 381-394.
34. B. K. Sah and A. Mishra, "A generalised exponential-Lindley mixture of Poisson distribution", *Nepal. J. Stat.*, **2019**, *3*, 11-20.
35. S. Samutwachirawong, "Poisson-exponential and gamma distribution: Properties and applications", *J. Appl. Stat. Inform. Technol.*, **2021**, *6*, 17-24.