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Full Paper

Bernoulli-Padovan polynomials and Pado-Bernoulli matrices

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Abstract: In the present work we introduce Bernoulli-Padovan numbers and polynomials. We give their generating functions of the Bernoulli-Padovan numbers and polynomials. We establish various relations involving the Bernoulli-Padovan numbers and polynomials by considering the Pado-derivative. We describe Pado-Bernoulli matrices in terms of the Bernoulli-Padovan numbers and polynomials. We establish a factorisation of the Pado-Bernoulli matrix by using a generalised Pado-Pascal matrix, and obtain the inverse of the Pado-Bernoulli matrix. Also, we give a relationship between the Pado-Bernoulli matrix and the Pado-Pascal matrix.

Keywords: Padovan polynomials, Bernoulli numbers, Pascal matrices, exponential generating function

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INTRODUCTION

Bernoulli numbers were first introduced by Swiss mathematician Jacob Bernoulli (1654-11705) in his posthumously published book Ars Conjectandi in 1713 [1]. He discovered them while working on Faulhaber's formula for the sum of the first n positive integers' n^{th} powers [1]. The Bernoulli numbers can be found in Taylor series expansions of tangent, hyperbolic tangent, cotangent and hyperbolic cotangent functions, as well as Euler Mac-Laurent formula and expressions at certain values of Rieman Zeta function [2]. The Bernoulli numbers can be found in nearly every branch of mathematics. Furthermore, they appear in the proof of Fermat's last theorem by Kummer’s theorem [3].

Bernoulli numbers and polynomials are a very current topic that has been studied by many researchers and has several generalisations. The q-expansions of Bernoulli numbers are one of them. Ernst [4, 5] published two important studies under the umbral approximation which helped to reveal a variety of q-specific matrices such as q-Bernoulli, q-Euler, q-Pascal and q-Bernoulli matrices. Zhang's work [6] included information on the Bernoulli matrix and its algebraic

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properties. Al-Salam [7] and Carlitz [8] defined the q-Bernoulli numbers and polynomials. Kus et al. [9] have used the Fibonacci calculus to reveal the Bernoulli F-polynomials and Fibo-Bernoulli matrices.

The special numbers, polynomials and their generating functions have many applications in all branches of mathematics and applied science. Using the generating functions is a very useful method; by generating functions for special numbers and polynomials we can derive not only new properties of these special numbers and polynomials, we can also give a combinatorial interpretation in enumerative combinatorics [10]. The applications of special numbers and polynomials may be given in probability and statistics, mathematical physics, engineering and cryptology [11, 12].

The Bernoulli polynomials $B_n(x)$ are defined by the generating function as follows:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad n>0.$$

On the other hand, the Bernoulli polynomial $B_n(x)$ can be given by the explicit formula

$$B_n(x)=\sum_{r=0}^{n}\binom{n}{r}B_r x^{n-r},$$

where B_n is n^{th} Bernoulli number and defined by $B_n=B_n(0)$. Also, the Bernoulli numbers are defined by the generating function as

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\qquad (|t|<2\pi).$$

On the other hand, the Bernoulli numbers B_n can be given by the formula

$$\sum_{r=0}^{n}\binom{n}{r}B_r=B_n,\quad (n>1)$$

with $B_0=1$.

A few terms of the Bernoulli polynomials and numbers are given below:

$B_0(x)=1,$

$B_1(x)=x-\frac{1}{2},$

$B_2(x)=x^2-x+\frac{1}{6},$

$B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x,$

$B_4(x)=x^4-2x^3+x^2-\frac{1}{30},$

$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0.$

The Bernoulli polynomials and numbers obey the following relations:

$$B_n(x+1)-B_n(x)=nx^{n-1},$$

$$\frac{d}{dx}B_n(x)=n.B_{n-1}(x),$$

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$$\sum_{r=0}^{n-1}\binom{n}{r}B_r(x)=nx^{n-1},\quad n\geq 1,$$

$$\sum_{r=0}^{n-1}\binom{n}{r}B_r=0,\quad n\geq 2.$$

The Padovan numbers $\{P_n\}_{n=0}^{\infty}$ are defined by the third recurrence relation:

$$P_n=P_{n-2}+P_{n-3},\ (n>3)$$

with the initial conditions $P_0=P_1=P_2=1$. The Padovan sequence appears as sequence A000931 on the On-Line Encyclopedia of Integer Sequences [13]. Studies on generalised Padovan numbers are given by Soykan [14, 15].

BERNOULLI P-NUMBERS AND P-POLYNOMIALS

Krot [16] introduced the finite Fibonomial calculus, which is a special case of ψ – extended Rota's finite operator calculus given by Kwaśniewski [17]. In the present work we consider similar calculus, which is called Padonomial calculus by using Padovan numbers.

The main concepts of Padonomial calculus, P-factorial and P-binomial coefficients, are defined by

$$P_n!=P_nP_{n-1}P_{n-2}\ldots P_1P_0,\quad P_0!=1.$$

For $n\geq k\geq 1$,

$$\binom{n}{k}_P=\frac{P_n!}{P_{n-k}!P_k!},$$

$$\binom{n}{0}_P=1,\text{ and } n<k \text{ için } \binom{n}{k}_P=0.$$

It is clear that the following equalities hold:

$$\binom{n}{k}_P=\binom{n}{n-k}_P,\quad \binom{n}{k}_P\binom{k}{j}_P=\binom{n}{j}_P\binom{n-j}{k-j}_P.$$

The Padonomial's theorem (P-analog of binomial theorem) can be given as

$$(x+_P y)^n=\sum_{k=0}^{n}\binom{n}{k}_P x^k y^{n-k}.$$

The Pado-exponantial function (P-analog of exponantial) is defined by

$$e_P^t=\sum_{n=0}^{\infty}\frac{t^n}{P_n!}.$$

Hence we write

$$e_P^{tx}=\sum_{n=0}^{\infty}\frac{(tx)^n}{P_n!}.$$

The linear operator $D_P^x:\mathrm{P}\rightarrow\mathrm{P}$ such that $D_P^x(x^n)=P_nx^{n-1},\ n\geq 0$, is called Pado-derivative. Here P denotes the vector space of polynomials over the field of real or complex numbers. According to this definition we have

$$D_P^x\left(e_P^{tx}\right)=te_P^{tx}. \tag{1}$$

Definition 1. Let $\binom{n}{k}_P$ be the Padonomial coefficients and P_n be the n^{th} Padovan number. The Bernoulli $P-$numbers $B_{n,P}$ are defined as

$$B_{n,P}=\sum_{k=0}^{n}\frac{1}{P_{k+1}}\binom{n}{k}_P.$$

A few elements of Bernoulli $P-$numbers $B_{n,P}$ are

$B_{0,P}=1,\ B_{1,P}=2,\ B_{2,P}=\frac{5}{2},\ B_{3,P}=\frac{9}{2},\ B_{4,P}=\frac{19}{3},\ B_{5,P}=\frac{45}{4}.$

Theorem 1. The exponential generating function of the Bernoulli $P-$numbers $B_{n,P}$ is

$$g(x)=\frac{\left(e_P^t-1\right)e_P^t}{t}.$$

Proof. Let $g(x)=\sum_{n=0}^{\infty}B_{n,P}\frac{t^n}{P_n!}$. By the definition of $B_{n,P}$, we write

$$\begin{aligned}
\sum_{n=0}^{\infty}B_{n,P}\frac{t^n}{P_n!}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{P_{k+1}}\binom{n}{k}_P\right)\frac{t^n}{P_n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{P_{k+1}}\frac{P_n!}{P_{n-k}!P_k!}\frac{1}{P_n!}\right)t^n\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{P_{k+1}!}\frac{1}{P_{n-k}!}\right)t^n\\
&=\left(\sum_{n=0}^{\infty}\frac{t^n}{P_{n+1}!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{P_n!}\right)\\
&=\frac{1}{t}\left(\sum_{n=0}^{\infty}\frac{t^{n+1}}{P_{n+1}!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{P_n!}\right)\\
&=\frac{1}{t}\left(\sum_{n=0}^{\infty}\frac{t^n}{P_n!}-1\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{P_n!}\right)
\end{aligned}$$

By definition of the Pado-exponential function e_P^t, we obtain the desired result, i.e.

$$g(x)=\frac{\left(e_P^t-1\right)e_P^t}{t}.$$

Definition 2. Let $\binom{n}{k}_P$ be the Padonomial coefficients and P_n be the n^{th} Padovan numbers. The Bernoulli $P-$polynomials are defined as

$$B_{n,P}(x)=\sum_{k=0}^{n}\frac{1}{P_{k+1}}\binom{n}{k}_P x^{n-k}.$$

A few elements of Bernoulli $P-$polynomials are

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$B_{0,P}(x)=1,\qquad B_{1,P}(x)=x+1,\qquad B_{2,P}(x)=x^2+x+\frac{1}{2},\qquad B_{3,P}(x)=x^3+2x^2+x+\frac{1}{2},$

$B_{0,P}(x)=x^4+2x^3+2x^2+x+\frac{1}{3}.$

Theorem 2. The exponential generating function of the Bernoulli $P-$polynomials $B_{n,P}(x)$ is

$$h(x)=\frac{\left(e_P^t-1\right)e_P^{xt}}{t}.$$

Proof. The proof similar to that of Theorem 1 can be given.

BERNOULLI-PADOVAN NUMBERS AND BERNOULLI-PADOVAN POLYNOMIALS

It is well known that Krot defined and investigated the Bernoulli-Fibonacci numbers and Bernoulli-Fibonacci polynomial [16]. Kus et al. [9] have studied the Bernoulli *F*-polynomials and Fibo-Bernoulli matrices. In the present work we consider similar investigations by using the Padovan numbers.

Definition 3. The Bernoulli-Padovan polynomials $B_n^P(x)$ are defined by the exponential generating function as

$$\frac{te_P^{tx}}{e_P^t-1}=\sum_{n=0}^{\infty}B_n^P(x)\frac{t^n}{P_n!}. \tag{2}$$

Also, the Bernoulli-Padovan polynomials can be given by the explicit formula

$$B_n^P(x)=\sum_{r=0}^{n}\binom{n}{r}_P B_r^P x^{n-r},$$

with $B_0^P(x)=1$. In fact, we can write

$$\frac{te_P^{tx}}{e_P^t-1}=\left(\sum_{n=0}^{\infty}B_n^P\frac{t^n}{P_n!}\right)\left(\sum_{n=0}^{\infty}\frac{t^nx^n}{P_n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}B_r^P\frac{t^r}{P_r!}\cdot\frac{t^{n-r}x^{n-r}}{P_{n-r}!}\right).$$

Hence we have

$$\sum_{n=0}^{\infty}B_n^P(x)\frac{t^n}{P_n!}=\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}B_r^P\frac{x^{n-r}}{P_{n-r}!P_r!}\right)t^n.$$

The Bernoulli-Padovan numbers B_n^P are special values of the Bernoulli-Padovan polynomials. The following definition shows the exponential generating function of the Bernoulli-Padovan numbers.

Definition 4. The exponential generating function of the Bernoulli-Padovan numbers is defined as

$$\frac{t}{e_P^t-1}=\sum_{n=0}^{\infty}B_n^P\frac{t^n}{P_n!}.$$

In other words, the Bernoulli-Padovan numbers B_n^P can be given by the explicit formula

$$B_n^P=\sum_{r=0}^{n}\binom{n}{r}_P B_r^P,\ (n>1),$$

with $B_0^P=1$. A few elements of the Bernoulli-Padovan polynomials are

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$B_0^P(x) = 1,$

$B_1^P(x) = x - 1,$

$B_2^P(x) = x^2 - x + \frac{1}{2},$

$B_3^P(x) = x^3 - 2x^2 + x - \frac{1}{2},$

$B_4^P(x) = x^4 - 2x^3 + 2x^2 - x + \frac{2}{3},$

$B_5^P(x) = x^5 - 3x^4 + 3x^3 - 3x^2 + 2x - \frac{5}{4}.$

A few elements of the Bernoulli–Padovan numbers are

$B_0^P = 1,\ B_1^P = -1,\ B_2^P = \frac{1}{2},\ B_3^P = -\frac{1}{2},\ B_4^P = \frac{2}{3},\ B_5^P = -\frac{5}{4}.$

Now we give the relationship between the first few Bernoulli-Padovan polynomials $B_n^P(x)$, the Bernoulli P-polynomials $B_{n,P}(x)$ and the classical Bernoulli polynomials by means of graphs in Figure 1.

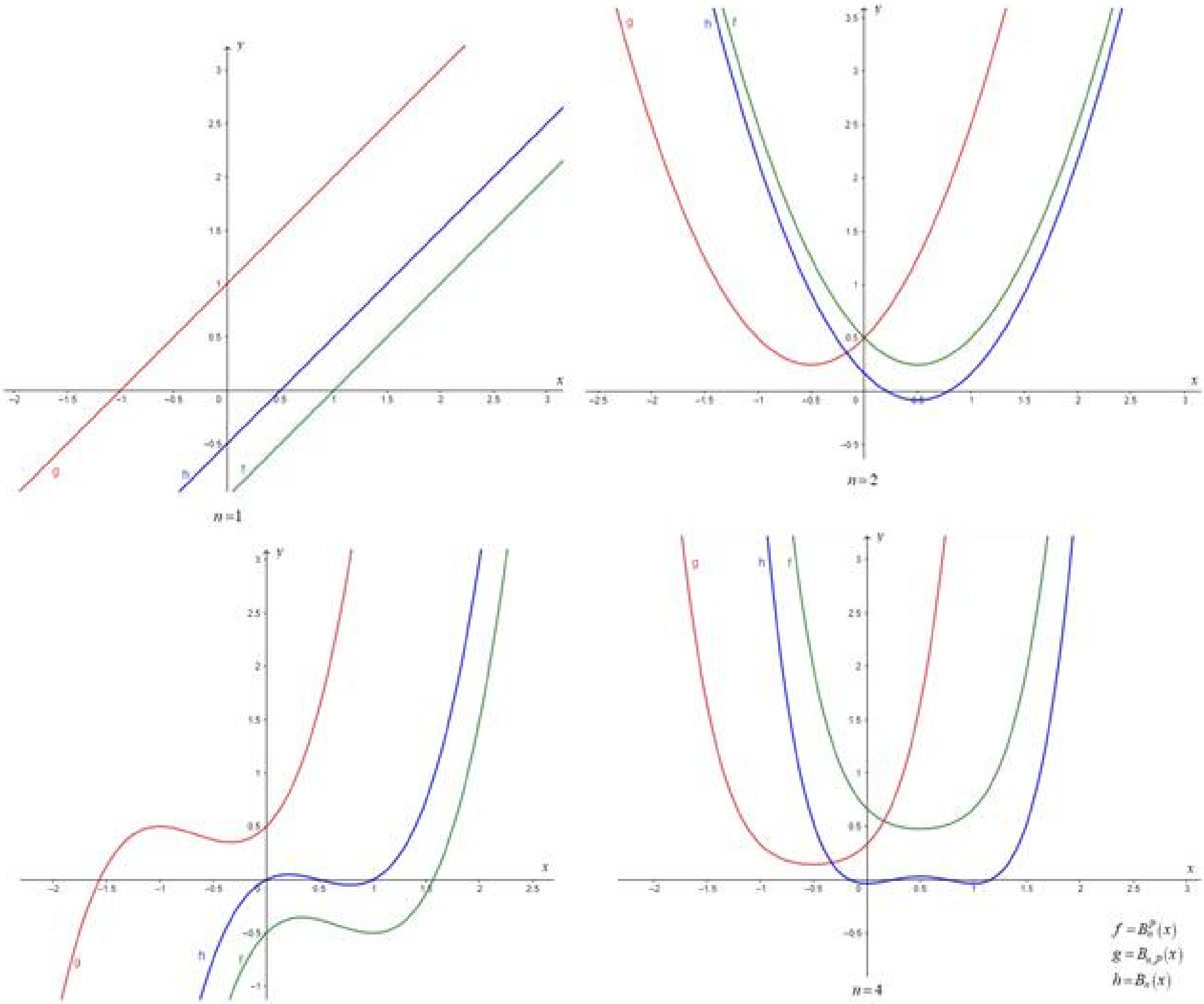

Figure 1. Graphs of $f = B_n^P(x)$, $g = B_{n,P}(x)$ and $h = B_n(x)$ for $n = 1,2,3,4$

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Proposition 1. The Pado-derivative application for the Bernoulli-Padovan polynomials $B_n^P(x)$ is given as follows:

$$D_P^x\left(B_n^P(x)\right)=P_nB_{n-1}^P(x).$$

Proof. Taking the Pado-derivative of both sides in equality (2), we have

$$D_P^x\left(\frac{te_P^{tx}}{e_P^t-1}\right)=D_P^x\left(\sum_{n=0}^{\infty}B_n^P(x)\frac{t^n}{P_n!}\right)$$

$$\frac{tD_P^x\left(e_P^{tx}\right)}{e_P^t-1}=D_P^x\left(B_0^P(x)+B_1^P(x)\frac{t}{P_1!}+B_2^P(x)\frac{t^2}{P_2!}+\cdots\right).$$

The Pado-derivative for the left side can be calculated by using the equality (1). For the right side, it is clear that

$$D_P^x\left(B_0^P(x)\right)=D_P^x(1)=0\,.$$

Then

$$t\frac{te_P^{tx}}{e_P^t-1}=\sum_{k=1}^{\infty}D_P^x\left(B_k^P(x)\right)\frac{t^k}{P_k!}$$

$$t\sum_{n=0}^{\infty}B_n^P(x)\frac{t^n}{P_n!}=\sum_{k=1}^{\infty}D_P^x\left(B_k^P(x)\right)\frac{t^k}{P_k!}\quad .$$

$$\sum_{n=0}^{\infty}B_n^P(x)\frac{t^{n+1}}{P_n!}=\sum_{k=1}^{\infty}D_P^x\left(B_k^P(x)\right)\frac{t^k}{P_k!}$$

By using the following relations,

$$\sum_{n=0}^{\infty}B_n^P(x)\frac{t^{n+1}}{P_n!}=\sum_{n=0}^{\infty}D_P^x\left(B_{n+1}^P(x)\right)\frac{t^{n+1}}{P_{n+1}!}\quad ,$$

$$\sum_{n=0}^{\infty}B_n^P(x)\frac{t^{n+1}}{P_n!}=\sum_{n=0}^{\infty}D_P^x\left(B_{n+1}^P(x)\right)\frac{1}{P_{n+1}}\frac{t^{n+1}}{P_n!}$$

we obtain the desired result:

$$D_P^x\left(B_n^P(x)\right)=P_nB_{n-1}^P(x).$$

Proposition 2. The Bernoulli-Padovan polynomials $B_k^P(x)$ are calculated by the recurrence relation for $n\geq 1$ as

$$\sum_{k=0}^{n-1}\binom{n}{k}_P B_k^P(x)=P_nx^{n-1}\,.$$

Proof. By multiplying all sides of the equality (2) by e_P^t, the following equality is obtained:

$$\frac{te_P^{tx}e_P^t}{e_P^t-1}=\sum_{n=0}^{\infty}B_n^P(x)e_P^t\frac{t^n}{P_n!}\,.$$

Hence we have

$$\frac{te_P^{tx}}{e_P^t-1}\left(e_P^t-1\right)=\sum_{n=0}^{\infty}\left(B_n^P(x)e_P^t-B_n^P(x)\right)\frac{t^n}{P_n!}$$

$$te_P^{tx}=\sum_{n=0}^{\infty}\left(B_n^P(x)e_P^t-B_n^P(x)\right)\frac{t^n}{P_n!}$$

$$D_P^x\left(e_P^{tx}\right)=\sum_{n=0}^{\infty}\left(B_n^P(x)e_P^t-B_n^P(x)\right)\frac{t^n}{P_n!}$$

$$D_P^x\left(\sum_{n=0}^{\infty}\frac{(tx)^n}{P_n!}\right)=\sum_{n=0}^{\infty}\left(B_n^P(x)e_P^t-B_n^P(x)\right)\frac{t^n}{P_n!}$$

$$\sum_{n=0}^{\infty}\frac{t^nD_P^x(x)^n}{P_n!}=\sum_{n=0}^{\infty}\left(B_n^P(x)e_P^t-B_n^P(x)\right)\frac{t^n}{P_n!}$$

$$\sum_{n=0}^{\infty}\frac{t^nP_nx^{n-1}}{P_n!}=\sum_{k=0}^{\infty}B_k^P(x)e_P^t\frac{t^k}{P_k!}-\sum_{n=0}^{\infty}B_n^P(x)\frac{t^n}{P_n!}.$$

On the right side of this equality,

$$\begin{aligned}\sum_{k=0}^{\infty}B_k^P(x)e_P^t\frac{t^k}{P_k!}&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}B_k^P(x)\frac{t^l}{P_l!}.\frac{t^k}{P_k!}\\&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}B_k^P(x)\frac{t^{l+k}}{P_l!P_k!}\\&=\sum_{n=0}^{\infty}\frac{1}{P_n!}\sum_{k=0}^{\infty}B_k^P(x)\frac{t^nP_n!}{P_{n-k}!P_k!}\\&=\sum_{n=0}^{\infty}\frac{t^n}{P_n!}\left(\sum_{k=0}^{n}\binom{n}{k}B_k^P(x)\right).\end{aligned}$$

So we get

$$\sum_{n=0}^{\infty}\frac{t^nP_nx^{n-1}}{P_n!}=\sum_{n=0}^{\infty}\frac{t^n}{P_n!}\left(\sum_{k=0}^{n}\binom{n}{k}_P B_k^P(x)\right)-\sum_{n=0}^{\infty}B_n^P(x)\frac{t^n}{P_n!}.$$

Then we obtain

$$\sum_{k=0}^{n}\binom{n}{k}_P B_k^P(x)-B_n^P(x)=P_nx^{n-1}.$$

By the relation

$$\sum_{k=0}^{n-1}\binom{n}{k}_P B_k^P(x)+\binom{n}{n}_P B_n^P(x)-B_n^P(x)=P_nx^{n-1},$$

we get the desired equality,

$$\sum_{k=0}^{n-1}\binom{n}{k}_P B_k^P(x)=P_nx^{n-1}.$$

PADO-BERNOULLI MATRICES

Using a generalised Pado-Pascal matrix, we create a factorisation of the Pado-Bernoulli matrix in this section. The Bernoulli P-polynomials are then used to create an interesting matrix. In addition, the inverse of the Pado-Bernoulli matrix is obtained. We also show that the Pado-Bernoulli matrix and the Pado-Pascal matrix have a link.

The $n\times n$ Pascal matrix $PC_n=(c_{ij})$ is defined [18, 19] as

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$$c_{ij}=\begin{cases}\binom{i-1}{j-1}, & if\ i\ge j,\\ 0, & if\ i<j.\end{cases}$$

Definition 5. For the integers i, j and n, $1\le i, j\le n$, the generalised $n\times n$ Pado-Pascal matrix $PP_n[x]=(PP_n(x;i,j))$ is defined as

$$PP_n(x;i,j)=\begin{cases}\binom{i-1}{j-1}_P x^{i-j}, & if\ \ i\ge j,\\ 0, & if\ \ i<j.\end{cases}$$

For example

$$PP_5[x]=\begin{bmatrix}1&0&0&0&0\\ x&1&0&0&0\\ x^2&x&1&0&0\\ x^3&2x^2&2x&1&0\\ x^4&2x^3&4x^2&2x&1\end{bmatrix}.$$

Definition 6. For $n\ge 2$, the inverse of the generalised Pado-Pascal matrix $PP_n^{-1}[x]=(PP_n^{-1}(x;i,j))$ is defined as

$$PP_n^{-1}(x;i,j)=\begin{cases}b_{i-j+1}\binom{i-1}{j-1}_P x^{i-j}, & if\ i\ge j,\\ 0, & if\ i<j,\end{cases}$$

where $b_1=1$ and $b_n=-\sum_{k=1}^{n-1}b_k\binom{n}{k}_P$. For example

$$PP_5^{-1}[x]=\begin{bmatrix}1&0&0&0&0\\ -x&1&0&0&0\\ 0&-x&1&0&0\\ 2x^3&0&-2x&1&0\\ -3x^4&4x^3&0&-2x&1\end{bmatrix}.$$

Definition 7. For the integers i, j and n, $1\le i, j\le n$, the Pado-Bernoulli matrix $PB_n[x]=(PB_n(x;i,j))$ is defined as

$$PB_n(x;i,j)=\begin{cases}\binom{i-1}{j-1}_P B_{i-j,P}(x), & if\ i\ge j,\\ 0, & if\ i<j,\end{cases}$$

where $B_{n,P}(x)$ is the n^{th} Bernoulli P-polynomial. For example

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$$PB_5[x]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x+1 & 1 & 0 & 0 & 0 \\ x^2+x+\frac{1}{2} & x+1 & 1 & 0 & 0 \\ x^3+2x^2+x+\frac{1}{2} & 2x^2+2x+1 & 2x+2 & 1 & 0 \\ x^4+2x^3+2x^2+x+\frac{1}{3} & 2x^3+4x^2+2x+1 & 4x^2+4x+2 & 2x+2 & 1 \end{bmatrix}.$$

Using the Padonomial coefficients, we now create a special matrix. The factorised Pado-Bernoulli matrix is then obtained using the extended Pado-Pascal matrix.

Definition 8. Let P_n be the n^{th} Padovan number. For the integers i, j and n, and $1 \le i, j \le n$, the $Q_n(P)=\left[p_{ij}\right]_{n\times n}$ matrix is defined as

$$p_{ij}=\begin{cases} \dfrac{1}{P_{i-j+1}}\dbinom{i-1}{j-1}_P, & \text{if } i\ge j, \\ 0, & \text{if } i<j. \end{cases}$$

For example

$$Q_5(P)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 2 & 1 & 0 \\ \frac{1}{3} & 1 & 2 & 2 & 1 \end{bmatrix}.$$

Proposition 3. For every positive integer n,

$$\sum_{k=0}^{n}\binom{n}{k}_P B_{n-k}^P\frac{1}{P_{k+1}}=P_n!\,\delta_{n,0}$$

is true, where $\delta_{n,m}$ is the Kronecker delta symbol.

Theorem 3. Let B_n^P be the n^{th} Bernoulli-Padovan numbers. $Q_n^{-1}(P)=\left[q_{ij}\right]_{n\times n}$. The inverse of the $Q(P)=\left[p_{ij}\right]_{n\times n}$ matrix is

$$q_{ij}=\begin{cases} \dbinom{i-1}{j-1}_P B_{i-j}^P, & \text{if } i\ge j, \\ 0, & \text{if } i<j. \end{cases}$$

Proof.

$$\left(Q_n^{-1}(P)Q_n(P)\right)_{ij}=\sum_{k=j}^{i}q_{ik}p_{kj}$$

$$=\sum_{k=j}^{i}\binom{i-1}{k-1}_P B_{i-k}^P\frac{1}{P_{k-j+1}}\binom{k-1}{j-1}_P$$

$$=\sum_{k=j}^{i}\binom{i-1}{j-1}_P\binom{i-j}{k-j}_P B_{i-k}^P\frac{1}{P_{k-j+1}}$$

$$=\binom{i-1}{j-1}_P\sum_{k=0}^{i-j}\binom{i-j}{k}_P B_{i-j-k}^P\frac{1}{P_{k+1}}$$

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$$=\binom{i-1}{j-1}_P \sum_{k=0}^{n}\binom{n}{k}_P B_{n-k}^{P}\frac{1}{P_{k+1}}$$
$$=\binom{i-1}{j-1}_P P_n!\delta_{n,0},$$

where for $i=j$, $\left(Q_n^{-1}(P)Q_n(P)\right)_{ij}=1$ and for $i\neq j$, $\left(Q_n^{-1}(P)Q_n(P)\right)_{ij}=0$. For example

$$Q_5^{-1}(P)=\begin{bmatrix}1&0&0&0&0\\-1&1&0&0&0\\\frac{1}{2}&-1&1&0&0\\-\frac{1}{2}&1&-2&1&0\\\frac{2}{3}&-1&2&-2&1\end{bmatrix}.$$

Theorem 4. Let $PB_n[x]$ be the Pado-Bernoulli matrix and $PP_n[x]$ be a generalised Pado-Pascal matrix. Then

$$PB_n[x]=PP_n[x]Q_n(P).$$

Proof.

$$\left(PP_n[x]Q_n(P)\right)_{ij}=\sum_{k=j}^{i}t_{ik}p_{kj}$$
$$=\sum_{k=j}^{i}\binom{i-1}{k-1}_P x^{i-k}\frac{1}{P_{k-j+1}}\binom{k-1}{j-1}_P$$
$$=\binom{i-1}{j-1}_P\sum_{k=0}^{i-j}\frac{1}{P_{k-j+1}}\binom{i-j}{k}_P x^{i-j-k}$$
$$=\binom{i-1}{j-1}_P B_{i-j,P}(x)$$
$$=\left(PB_n[x]\right)_{ij}.$$

For example

$$PP_5[x]Q_5(P)=\begin{bmatrix}1&0&0&0&0\\x&1&0&0&0\\x^2&x&1&0&0\\x^3&2x^2&2x&1&0\\x^4&2x^3&4x^2&2x&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\\frac{1}{2}&1&1&0&0\\\frac{1}{2}&1&2&1&0\\\frac{1}{3}&1&2&2&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&0&0\\x+1&1&0&0&0\\x^2+x+\frac{1}{2}&x+1&1&0&0\\x^3+2x^2+x+\frac{1}{2}&2x^2+2x+1&2x+2&1&0\\x^4+2x^3+2x^2+x+\frac{1}{3}&2x^3+4x^2+2x+1&4x^2+4x+2&2x+2&1\end{bmatrix}=PB_5[x].$$

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CONCLUSIONS

In this study the relations of the Padovan numbers and polynomials with Bernoulli numbers have been established and the Bernoulli-Padovan numbers and polynomials have been obtained. The various equalities of the Bernoulli-Padovan numbers and polynomials have been given. The Pado-derivative is used to prove the stated equalities. Inspired by the definition of Pascal matrix, the Pado-Pascal matrix is defined. The Pado-Bernoulli matrix has been obtained by using the Pado-Pascal matrix. Finally, a relationship between the Pado-Pascal matrix and the Pado-Bernoulli matrix has been established. The Pado-Bernoulli matrix can be used in cryptology by interpreting the terms on finite fields.

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