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Full Paper

Some properties of fuzzy sedenion numbers and fuzzy sedenion valued series

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Abstract: The analytic properties of fuzzy sedenion numbers and sedenion valued series are studied. For this aim, we first introduce fuzzy sedenion numbers. Then we present infimum, supremum, distance, limit of sequences and other analytical properties of fuzzy sedenion numbers. We also define interval sedenion numbers and fuzzy sedenion valued series. Finally, we obtain comparison criteria of the fuzzy sedenion valued series.

Keywords: fuzzy real number, fuzzy sedenion number, interval sedenion number, fuzzy sedenion valued series

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INTRODUCTION

Sedenions are non-commutative, non-associative, non-alternative, non-normed division, and non-compositional, but power associative algebras over real numbers [1-4]. They are obtained by applying the Cayley-Dickson construction to the octonions [1]. The sedenions have been used in algebraic applications.

The studies of sedenions can be summarised as follows. Imaeda and Imaeda [4] studied sedenionic algebra and analysis. Bektas [5] introduced $\mathbb{C}, \mathbb{H}, \mathbb{O}$ coefficient sedenions and their matrix representations. Perrin sedenions and Tribonacci sedenions were given [6, 7]. Cimen et al. [8] gave Horadam sedenions. A new generalisation of Fibonacci and Lucas type sedenions were examined by Kizilates and Kirlak [9].

Fuzzy numbers were introduced by Chang and Zadeh [11] and Dubois and Prade [12]. In 1965 Zadeh [10] defined the notion of a fuzzy set. Zhang et al. [13] proposed a generalised version

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

of fuzzy numbers. Patra [14] introduced a new technique for ranking generalised trapezoidal fuzzy numbers. Raj et al. [15] gave a novel approach to arithmetic operations on trapezoidal fuzzy numbers.

The mathematical analysis results concerning fuzzy real, complex, quaternion, and octonion numbers have been studied [17-21]. The necessary and sufficient conditions for convergence of fuzzy complex valued series, comparison criterion, and some elementary properties on convergence were obtained by Guijun and Shumin [22].

Considering fuzzy numbers and their analytical properties, it is important to examine the analytical properties of fuzzy sedenion numbers. Therefore, in this article the mathematical analysis of fuzzy sedenion numbers as a generalisation of fuzzy numbers (real, complex, quaternion, octonion) is introduced with well-known basic concepts (limit, infimum, supremum, distance, etc.). In addition, interval sedenion numbers and the operations on these numbers are also defined and their properties are examined. Finally, the concept of fuzzy sedenion series is defined using fuzzy sedenion numbers and comparison criteria are given on these series. Thus, in this paper a new generalisation covering the entire fuzzy number system has been made.

PRELIMINARIES

Sedenions

A sedenion is constructed over real numbers. A set of base elements of the sedenion is denoted by E_{16} as follows:

$$E_{16} = \{e_i \in \mathbb{S} \mid i = 0,1,2,\dots,15\},$$

where $e_0 = 1$ is multiplicative scalar element and e_i's $(i = 1,2,\dots,15)$ are imaginary units.

Sedenionic units satisfy the following properties:

1. $e_0 = 1$ and $e_0 e_i = e_i\, e_0 = e_i,\ (i \neq 0)$
2. $e_i\, e_i = (e_i)^2 = -1, \qquad (i \neq 0)$
3. $e_i\, e_j = -e_j\, e_i, \qquad (i,j \neq 0)\ (i \neq j).$

The multiplication of sedenionic basic elements is given in Table 1. Any sedenion is written as a linear combination of E_{16} as $\mathcal{X} = \sum_{i=0}^{15} a_i e_i$. The expression $a_0 e_0 = Re(\mathcal{X}) = S_{\mathcal{X}}$ is called the real part of sedenion and $\sum_{i=1}^{15} a_i e_i = Im(\mathcal{X}) = \vec{V}_{\mathcal{X}}$ is called the vectorial part of sedenion. The sedenion can be written as

$$\mathcal{X} = a_0 e_0 + \sum_{i=1}^{15} a_i e_i = S_{\mathcal{X}} + \vec{V}_{\mathcal{X}}.$$

A set of sedenions can be written in the form [1, 4]

$$\mathbb{S} = \left\{ \mathcal{X} = \sum_{i=0}^{15} a_i e_i \mid a_i \in \mathbb{R}, 0 \leq i \leq 15 \right\}.$$

The sum of two sedenions is defined by

$$\mathcal{X} + \mathcal{Y} = \sum_{i=0}^{15} (a_i + b_i) e_i = \left(S_{\mathcal{X}} + S_{\mathcal{Y}}\right) + \left(\vec{V}_{\mathcal{X}} + \vec{V}_{\mathcal{Y}}\right).$$

The sedenionic multiplication can be written as

$$\mathcal{X}\mathcal{Y} = \left(\sum_{i=0}^{15} a_i e_i\right)\left(\sum_{i=0}^{15} b_i e_i\right) = \sum_{i,j,k=1}^{15} f_{ij} \gamma_{ij}^{k} e_k \ ,$$

where $e_i, e_j, e_k \in E_{16}$, $f_{ij} = \mathcal{x}_i \mathcal{y}_j$ and $\gamma_{ij}^{k} \in \{-1,0,+1\}$, [1, 4]. The coefficient γ_{ij}^{k} is called field parameter. $\overline{\mathcal{X}}$ is called the conjugate of sedenion $\mathcal{X}$. The conjugate of sedenion is defined by

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

$$\bar{\mathcal{X}} = a_0 e_0 - \sum_{\mathrm{i}=1}^{15} a_i e_i = S_{\mathcal{X}} - \vec{V}_{\mathcal{X}}.$$

The list of all triplet indices of ordered triplets $(\mathrm{i}, \mathrm{j}, \mathrm{k})$ that provide the loops here is given in Table 2.

Table 1. Multiplication conditions of unit sedenions basic elements

$e_i e_j$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	$-e_1$
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

Table 2. Sedenionic triplets

$(\mathbf{i}, \mathbf{j}, \mathbf{k})$
$(1,2,3), (1,4,5), (1,7,6), (1,8,9), (1,11,10),$
$(1,13,12), (1,14,15), (2,4,6), (2,5,7), (2,8,10),$
$(2,9,11), (2,14,12), (2,15,13), (3,4,7), (3,6,5),$
$(3,8,11), (3,10,9), (3,13,14), (3,15,12), (4,8,12),$
$(4,9,13), (4,10,14), (4,11,15), (5,8,13), (5,10,15),$
$(5,12,9), (5,14,11), (6,8,14), (6,11,13), (6,12,10),$
$(6,15,9), (7,8,15), (7,9,14), (7,12,11), (7,13,10)$

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

Corollary 1. $(\mathbb{S},+)$ is an Abelian group.

Corollary 2. The set of sedenions is 16-dimensional vector space over real numbers.

For all $\mathcal{X} = \sum_{i=0}^{15} a_i e_i$, $\mathcal{Y} = \sum_{i=0}^{15} b_i e_i \in \mathbb{S}$, the inner product of two sedenions is defined by [1, 4]:

$$<,> : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$$
$$(\mathcal{X},\mathcal{Y}) \to <\mathcal{X},\mathcal{Y}> = \frac{1}{2}(\mathcal{X}\bar{\mathcal{Y}} + \bar{\mathcal{X}}\mathcal{Y}) = \sum_{i=0}^{15} a_i b_i.$$

The norm of the sedenion $\mathcal{X}$ is denoted by

$$\|\mathcal{X}\| = \sqrt{\mathcal{X}\bar{\mathcal{X}}} = \sqrt{\sum_{i=0}^{15} (a_i)^2}\ .$$

If $\|\mathcal{X}\| = 1$, then $\mathcal{X}$ is called the unit sedenion. The inner product and norm operations provide the following properties [4]:

1. $<\mathcal{X},\mathcal{Y}> = <\mathcal{Y},\mathcal{X}>$
2. $<\mathcal{X},\mathcal{X}> = \|\mathcal{X}\|^2 \geq 0$
3. $<\mathcal{X}\mathcal{Y},\mathcal{Z}> = <\mathcal{Y},\bar{\mathcal{X}}\mathcal{Z}> = <\mathcal{X},\mathcal{Z}\bar{\mathcal{Y}}>$
4. $\|\mathcal{X}+\mathcal{Y}\| \leq \|\mathcal{X}\| + \|\mathcal{Y}\|$
5. $\|\mathcal{X}\| = \|-\mathcal{X}\| = \|\bar{\mathcal{X}}\| = \|-\bar{\mathcal{X}}\|$
6. $\|\mathcal{X}\|^2 + \|\mathcal{Y}\|^2 = \frac{1}{2}(\|\mathcal{X}+\mathcal{Y}\|^2 + \|\mathcal{X}-\mathcal{Y}\|^2)$
7. $\|\mathcal{X}\mathcal{Y}\| = \|\mathcal{Y}\mathcal{X}\| = \|\bar{\mathcal{X}}\mathcal{Y}\| = \|\mathcal{X}\bar{\mathcal{Y}}\|$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-zero sedenions. If $\mathcal{X}\mathcal{Y} = 0$, then $\mathcal{X}$ and $\mathcal{Y}$ are called zero divisors. If $\mathcal{X} = (e_3 + e_{10})$ and $\mathcal{Y} = (e_6 - e_{15})$ are two non-zero sedenions, then we get

$$\mathcal{X}\mathcal{Y} = (e_3 + e_{10})(e_6 - e_{15}) = e_5 - e_{12} + e_{12} - e_5 = 0.$$

If $\mathcal{X}$ is non-zero sedenion, then the inverse of sedenion is defined by

$$\mathcal{X}^{-1} = \frac{\bar{\mathcal{X}}}{\|\mathcal{X}\|^2}.$$

Fuzzy Real Numbers

In this section information about fuzzy real numbers is given.

Definition 1 [23, 24]**.** Let $\mathbb{R}$ be a set of real numbers. A fuzzy real set is a function $\bar{A}: \mathbb{R} \longrightarrow [0,1]$.

Definition 2 [23, 24]**.** A fuzzy real set $\bar{A}$ is a fuzzy real number if and only if it satisfies the following:

i. $\bar{A}$ is normal; there exists $x \in \mathbb{R}$ such that $\bar{A}(x) = 1$.

ii. $\bar{A}$ is fuzzy convex; $\bar{A}\,(tx + (1-t)y) \geq \min\{\bar{A}\,(x), \bar{A}\,(y)\}$, where $t \in [0, 1]$ and $x, y \in \mathbb{R}$.

iii. $\bar{A}$ is upper semicontinuous on $\mathbb{R}$; given an arbitrary $x_0 \in$ R and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then $|\bar{A}(x) - \bar{A}(x_0)| < \varepsilon$.

iv. $\bar{A}$ is compactly supported; cl$\{x \in$ R $|\ \bar{A}\,(x) > 0\}$ is compact, where cl(B) denotes closure of set B. A set of fuzzy real numbers is denoted by $\mathbb{R}_{\mathcal{F}}$.

FUZZY SEDENION NUMBERS AND FUZZY SEDENION VALUED SERIES

Fuzzy Sedenion Numbers

In this section fuzzy sedenion numbers are presented.

Definition 3 [20]. A fuzzy sedenion number is defined by $\acute{s}: S \rightarrow [0,1]$ such that

$$\acute{s}\left(\sum_{i=0}^{15} a_i\, e_i\right) = \bigwedge_{i=0}^{15} \overline{A}_{\mathrm{i}}(a_i),$$

where $\wedge$ is the minimum operator, $x_i \in \mathbb{R}$ and $A_i \in \mathbb{R}_{\mathcal{F}}$ for $i = 0, \ldots, 15$.

The fuzzy real part is represented by $Re_{\mathcal{F}}(\acute{s}) = \overline{A_0}$ and the fuzzy imaginary part is represented by $Im_{\mathcal{F}}(\acute{s}) = \overline{A}_{\mathrm{i}}$ for i = 1,...,15. The set of the fuzzy sedenion numbers is denoted by $\mathbb{S}_{\mathcal{F}}$ and identified by $\mathbb{R}^{16}{}_{\mathcal{F}}$.

Definition 4 [20]. Let $\acute{s} = (\overline{A_0}, \ldots, \overline{A_{15}})$ and $\acute{t} = (\overline{B_0}, \ldots, \overline{B_{15}}) \in \mathbb{S}_{\mathcal{F}}$. Then the sum of two fuzzy sedenion numbers is defined as

$$\acute{s} + \acute{t} = (\overline{A_0} + \overline{B_0}, \ldots, \overline{A_{15}} + \overline{B_{15}}).$$

It is possible to see that the addition operation is calculated similarly for the complex, quaternion, octonion, sedenion and all other hypercomplex numbers, although calculating the product of two fuzzy sedenion numbers is more complicated.

Definition 5. Let $\acute{s} = (\overline{A_0}, \ldots, \overline{A_{15}})$ and $\acute{t} = (\overline{B_0}, \ldots, \overline{B_{15}})$ be two fuzzy sedenion numbers. Then their multiplication is defined as

$$\acute{s}.\acute{t} = (\overline{C_0}, \ldots, \overline{C_{15}}),$$

where

$$\overline{C_0} = \overline{\mathrm{A}_0}\,\overline{\mathrm{B}_0} - \overline{\mathrm{A}_1}\,\overline{\mathrm{B}_1} - \overline{\mathrm{A}_2}\,\overline{\mathrm{B}_2} - \overline{\mathrm{A}_3}\,\overline{\mathrm{B}_3} - \overline{\mathrm{A}_4}\,\overline{\mathrm{B}_4} - \overline{\mathrm{A}_5}\,\mathrm{B}_5 - \overline{\mathrm{A}_6}\,\overline{\mathrm{B}_6} - \overline{\mathrm{A}_7}\,\overline{\mathrm{B}_7} - \overline{\mathrm{A}_8}\,\overline{\mathrm{B}_8}$$
$$-\overline{\mathrm{A}_9}\,\overline{\mathrm{B}_9} - \overline{\mathrm{A}_{10}}\,\overline{\mathrm{B}_{10}} - \overline{\mathrm{A}_{11}}\,\overline{\mathrm{B}_{11}} - \overline{\mathrm{A}_{12}}\,\overline{\mathrm{B}_{12}} - \overline{\mathrm{A}_{13}}\,\overline{\mathrm{B}_{13}} - \overline{\mathrm{A}_{14}}\,\overline{\mathrm{B}_{14}} - \overline{\mathrm{A}_{15}}\,\overline{\mathrm{B}_{15}},$$

$$\overline{C_1} = \overline{\mathrm{A}_0}\,\overline{\mathrm{B}_1} + \overline{\mathrm{A}_1}\,\overline{\mathrm{B}_0} + \overline{\mathrm{A}_2}\,\overline{\mathrm{B}_3} - \overline{\mathrm{A}_3}\,\overline{\mathrm{B}_2} + \overline{\mathrm{A}_4}\,\overline{\mathrm{B}_5} - \overline{\mathrm{A}_5}\,\mathrm{B}_4 - \overline{\mathrm{A}_6}\,\overline{\mathrm{B}_7} + \overline{\mathrm{A}_7}\,\overline{\mathrm{B}_6} + \overline{\mathrm{A}_8}\,\overline{\mathrm{B}_9}$$
$$-\overline{\mathrm{A}_9}\,\overline{\mathrm{B}_8} - \overline{\mathrm{A}_{10}}\,\overline{\mathrm{B}_{11}} + \overline{\mathrm{A}_{11}}\,\overline{\mathrm{B}_{10}} - \overline{\mathrm{A}_{12}}\,\overline{\mathrm{B}_{13}} - \overline{\mathrm{A}_{13}}\,\overline{\mathrm{B}_{12}} + \overline{\mathrm{A}_{14}}\,\overline{\mathrm{B}_{15}} - \overline{\mathrm{A}_{15}}\,\overline{\mathrm{B}_{14}},$$

$$\overline{C_2} = \overline{\mathrm{A}_0}\,\overline{\mathrm{B}_2} - \overline{\mathrm{A}_1}\,\overline{\mathrm{B}_3} + \overline{\mathrm{A}_2}\,\overline{\mathrm{B}_0} + \overline{\mathrm{A}_3}\,\overline{\mathrm{B}_1} + \overline{\mathrm{A}_4}\,\overline{\mathrm{B}_6} + \overline{\mathrm{A}_5}\,\mathrm{B}_7 - \overline{\mathrm{A}_6}\,\overline{\mathrm{B}_4} - \overline{\mathrm{A}_7}\,\overline{\mathrm{B}_5} + \overline{\mathrm{A}_8}\,\overline{\mathrm{B}_{10}}$$
$$+\overline{\mathrm{A}_9}\,\overline{\mathrm{B}_{11}} - \overline{\mathrm{A}_{10}}\,\overline{\mathrm{B}_8} - \overline{\mathrm{A}_{11}}\,\overline{\mathrm{B}_9} - \overline{\mathrm{A}_{12}}\,\overline{\mathrm{B}_{14}} - \overline{\mathrm{A}_{13}}\,\overline{\mathrm{B}_{15}} + \overline{\mathrm{A}_{14}}\,\overline{\mathrm{B}_{12}} + \overline{\mathrm{A}_{15}}\,\overline{\mathrm{B}_{13}},$$

$$\overline{C_3} = \overline{\mathrm{A}_0}\,\overline{\mathrm{B}_3} + \overline{\mathrm{A}_1}\,\overline{\mathrm{B}_2} - \overline{\mathrm{A}_2}\,\overline{\mathrm{B}_1} + \overline{\mathrm{A}_3}\,\overline{\mathrm{B}_0} + \overline{\mathrm{A}_4}\,\overline{\mathrm{B}_7} - \overline{\mathrm{A}_5}\,\overline{\mathrm{B}_6} + \overline{\mathrm{A}_6}\,\overline{\mathrm{B}_5} - \overline{\mathrm{A}_7}\,\overline{\mathrm{B}_4} + \overline{\mathrm{A}_8}\,\overline{\mathrm{B}_{11}}$$
$$-\overline{\mathrm{A}_9}\,\overline{\mathrm{B}_{10}} + \overline{\mathrm{A}_{10}}\,\overline{\mathrm{B}_9} - \overline{\mathrm{A}_{11}}\,\overline{\mathrm{B}_8} - \overline{\mathrm{A}_{12}}\,\overline{\mathrm{B}_{15}} + \overline{\mathrm{A}_{13}}\,\overline{\mathrm{B}_{14}} - \overline{\mathrm{A}_{14}}\,\overline{\mathrm{B}_{13}} + \overline{\mathrm{A}_{15}}\,\overline{\mathrm{B}_{12}},$$

$$\overline{C_4} = \overline{\mathrm{A}_0}\,\overline{\mathrm{B}_4} - \overline{\mathrm{A}_1}\,\overline{\mathrm{B}_5} - \overline{\mathrm{A}_2}\,\overline{\mathrm{B}_6} - \overline{\mathrm{A}_3}\,\overline{\mathrm{B}_7} + \overline{\mathrm{A}_4}\,\overline{\mathrm{B}_0} + \overline{\mathrm{A}_5}\,\overline{\mathrm{B}_1} + \overline{\mathrm{A}_6}\,\overline{\mathrm{B}_2} + \overline{\mathrm{A}_7}\,\overline{\mathrm{B}_3} + \overline{\mathrm{A}_8}\,\overline{\mathrm{B}_{12}}$$
$$+\overline{\mathrm{A}_9}\,\overline{\mathrm{B}_{13}} + \overline{\mathrm{A}_{10}}\,\overline{\mathrm{B}_{14}} + \overline{\mathrm{A}_{11}}\,\overline{\mathrm{B}_{15}} - \overline{\mathrm{A}_{12}}\,\overline{\mathrm{B}_8} - \overline{\mathrm{A}_{13}}\,\overline{\mathrm{B}_9} - \overline{\mathrm{A}_{14}}\,\overline{\mathrm{B}_{10}} - \overline{\mathrm{A}_{15}}\,\overline{\mathrm{B}_{11}},$$

$$\overline{C_5} = \overline{\mathrm{A}_0}\,\overline{\mathrm{B}_5} - \overline{\mathrm{A}_1}\,\overline{\mathrm{B}_4} - \overline{\mathrm{A}_2}\,\overline{\mathrm{B}_7} + \overline{\mathrm{A}_3}\,\overline{\mathrm{B}_6} - \overline{\mathrm{A}_4}\,\overline{\mathrm{B}_1} + \overline{\mathrm{A}_5}\,\overline{\mathrm{B}_0} - \overline{\mathrm{A}_6}\,\overline{\mathrm{B}_3} + \overline{\mathrm{A}_7}\,\overline{\mathrm{B}_2} + \overline{\mathrm{A}_8}\,\overline{\mathrm{B}_{13}}$$
$$-\overline{\mathrm{A}_9}\,\overline{\mathrm{B}_{12}} + \overline{\mathrm{A}_{10}}\,\overline{\mathrm{B}_{15}} - \overline{\mathrm{A}_{11}}\,\overline{\mathrm{B}_{14}} + \overline{\mathrm{A}_{12}}\,\overline{\mathrm{B}_9} - \overline{\mathrm{A}_{13}}\,\overline{\mathrm{B}_8} + \overline{\mathrm{A}_{14}}\,\overline{\mathrm{B}_{11}} - \overline{\mathrm{A}_{15}}\,\overline{\mathrm{B}_{10}},$$

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

$$\overline{C_6} = \overline{A_0}\,\overline{B_6} - \overline{A_1}\,\overline{B_3} + \overline{A_2}\,\overline{B_4} - \overline{A_3}\,\overline{B_5} - \overline{A_4}\,\overline{B_2} + \overline{A_5}\,\overline{B_3} + \overline{A_6}\,\overline{B_0} - \overline{A_7}\,\overline{B_1} - \overline{A_8}\,\overline{B_{14}}$$
$$-\overline{A_9}\,\overline{B_{15}} - \overline{A_{10}}\,\overline{B_{12}} + \overline{A_{11}}\,\overline{B_{13}} + \overline{A_{12}}\,\overline{B_{10}} - \overline{A_{13}}\,\overline{B_{11}} - \overline{A_{14}}\,\overline{B_8} - \overline{A_{15}}\,\overline{B_9},$$

$$\overline{C_7} = \overline{A_0}\,\overline{B_7} - \overline{A_1}\,\overline{B_6} + \overline{A_2}\,\overline{B_5} + \overline{A_3}\,\overline{B_4} - \overline{A_4}\,\overline{B_3} - \overline{A_5}\,\overline{B_2} + \overline{A_6}\,\overline{B_1} + \overline{A_7}\,\overline{B_0} + \overline{A_8}\,\overline{B_{15}}$$
$$+\overline{A_9}\,\overline{B_{14}} - \overline{A_{10}}\,\overline{B_{13}} + \overline{A_{11}}\,\overline{B_{12}} + \overline{A_{12}}\,\overline{B_{11}} + \overline{A_{13}}\,\overline{B_{10}} - \overline{A_{14}}\,\overline{B_9} - \overline{A_{15}}\,\overline{B_8},$$

$$\overline{C_8} = \overline{A_0}\,\overline{B_8} - \overline{A_1}\,\overline{B_9} - \overline{A_2}\,\overline{B_{10}} - \overline{A_3}\,\overline{B_9} - \overline{A_4}\,\overline{B_{12}} - \overline{A_5}\,\overline{B_{13}} - \overline{A_6}\,\overline{B_{14}} - \overline{A_7}\,\overline{B_{15}} + \overline{A_8}\,\overline{B_0}$$
$$+\overline{A_9}\,\overline{B_1} + \overline{A_{10}}\,\overline{B_2} + \overline{A_{11}}\,\overline{B_3} + \overline{A_{12}}\,\overline{B_4} + \overline{A_{13}}\,\overline{B_5} + \overline{A_{14}}\,\overline{B_6} + \overline{A_{15}}\,\overline{B_7},$$

$$\overline{C_9} = \overline{A_0}\,\overline{B_9} + \overline{A_1}\,\overline{B_8} - \overline{A_2}\,\overline{B_{11}} + \overline{A_3}\,\overline{B_{10}} - \overline{A_4}\,\overline{B_{13}} + \overline{A_5}\,\overline{B_{12}} + \overline{A_6}\,\overline{B_{15}} - \overline{A_7}\,\overline{B_{14}} - \overline{A_8}\,\overline{B_1}$$
$$+\overline{A_9}\,\overline{B_0} - \overline{A_{10}}\,\overline{B_{13}} + \overline{A_{11}}\,\overline{B_2} - \overline{A_{12}}\,\overline{B_5} + \overline{A_{13}}\,\overline{B_4} + \overline{A_{14}}\,\overline{B_7} - \overline{A_{15}}\,\overline{B_6},$$

$$\overline{C_{10}} = \overline{A_0}\,\overline{B_{10}} + \overline{A_1}\,\overline{B_{11}} + \overline{A_2}\,\overline{B_8} - \overline{A_3}\,\overline{B_9} - \overline{A_4}\,\overline{B_{14}} - \overline{A_5}\,\overline{B_{15}} + \overline{A_6}\,\overline{B_{12}} + \overline{A_7}\,\overline{B_{13}} - \overline{A_8}\,\overline{B_2}$$
$$+\overline{A_9}\,\overline{B_3} + \overline{A_{10}}\,\overline{B_0} - \overline{A_{11}}\,\overline{B_1} - \overline{A_{12}}\,\overline{B_6} - \overline{A_{13}}\,\overline{B_7} + \overline{A_{14}}\,\overline{B_4} + \overline{A_{15}}\,\overline{B_5},$$

$$\overline{C_{11}} = \overline{A_0}\,\overline{B_{11}} - \overline{A_1}\,\overline{B_{10}} + \overline{A_2}\,\overline{B_9} + \overline{A_3}\,\overline{B_8} - \overline{A_4}\,\overline{B_{15}} + \overline{A_5}\,\overline{B_{14}} - \overline{A_6}\,\overline{B_{13}} + \overline{A_7}\,\overline{B_{12}} - \overline{A_8}\,\overline{B_3}$$
$$+\overline{A_9}\,\overline{B_2} + \overline{A_{10}}\,\overline{B_1} + \overline{A_{11}}\,\overline{B_0} - \overline{A_{12}}\,\overline{B_7} + \overline{A_{13}}\,\overline{B_6} - \overline{A_{14}}\,\overline{B_5} + \overline{A_{15}}\,\overline{B_4},$$

$$\overline{C_{12}} = \overline{A_0}\,\overline{B_{12}} + \overline{A_1}\,\overline{B_{13}} + \overline{A_2}\,\overline{B_{14}} + \overline{A_3}\,\overline{B_{15}} + \overline{A_4}\,\overline{B_8} - \overline{A_5}\,\overline{B_9} - \overline{A_6}\,\overline{B_{10}} - \overline{A_7}\,\overline{B_{11}} - \overline{A_8}\,\overline{B_4}$$
$$+\overline{A_9}\,\overline{B_5} + \overline{A_{10}}\,\overline{B_6} + \overline{A_{11}}\,\overline{B_7} + \overline{A_{12}}\,\overline{B_0} - \overline{A_{13}}\,\overline{B_1} - \overline{A_{14}}\,\overline{B_2} - \overline{A_{15}}\,\overline{B_3},$$

$$\overline{C_{13}} = \overline{A_0}\,\overline{B_{13}} - \overline{A_1}\,\overline{B_{12}} + \overline{A_2}\,\overline{B_{15}} - \overline{A_3}\,\overline{B_{14}} + \overline{A_4}\,\overline{B_9} + \overline{A_5}\,\overline{B_8} - \overline{A_6}\,\overline{B_{11}} - \overline{A_7}\,\overline{B_{10}} - \overline{A_8}\,\overline{B_5}$$
$$-\overline{A_9}\,\overline{B_4} + \overline{A_{10}}\,\overline{B_7} - \overline{A_{11}}\,\overline{B_6} + \overline{A_{12}}\,\overline{B_1} + \overline{A_{13}}\,\overline{B_0} + \overline{A_{14}}\,\overline{B_3} - \overline{A_{15}}\,\overline{B_2},$$

$$\overline{C_{14}} = \overline{A_0}\,\overline{B_{14}} - \overline{A_1}\,\overline{B_{15}} - \overline{A_2}\,\overline{B_{12}} + \overline{A_3}\,\overline{B_{13}} + \overline{A_4}\,\overline{B_{10}} - \overline{A_5}\,\overline{B_{11}} + \overline{A_6}\,\overline{B_8} + \overline{A_7}\,\overline{B_9} - \overline{A_8}\,\overline{B_6}$$
$$-\overline{A_9}\,\overline{B_7} - \overline{A_{10}}\,\overline{B_4} + \overline{A_{11}}\,\overline{B_5} + \overline{A_{12}}\,\overline{B_2} - \overline{A_{13}}\,\overline{B_3} + \overline{A_{14}}\,\overline{B_0} + \overline{A_{15}}\,\overline{B_1},$$

$$\overline{C_{15}} = \overline{A_0}\,\overline{B_{15}} + \overline{A_1}\,\overline{B_{14}} - \overline{A_2}\,\overline{B_{13}} - \overline{A_3}\,\overline{B_{12}} + \overline{A_4}\,\overline{B_{11}} + \overline{A_5}\,\overline{B_{10}} - \overline{A_6}\,\overline{B_9} + \overline{A_7}\,\overline{B_8} - \overline{A_8}\,\overline{B_7}$$
$$+\overline{A_9}\,\overline{B_6} - \overline{A_{10}}\,\overline{B_5} - \overline{A_{11}}\,\overline{B_4} + \overline{A_{12}}\,\overline{B_3} + \overline{A_{13}}\,\overline{B_2} - \overline{A_{14}}\,\overline{B_1} + \overline{A_{15}}\,\overline{B_0}.$$

Interval Sedenion Numbers

The concept of interval real numbers was previously introduced and their properties were previously examined [25, 26]. In this section they will be defined using sedenion numbers and some properties will be examined.

Definition 6. An interval sedenion number is a hexadecimal $S = (A_0, \dots, A_{15})$, where $A_i \in I(\mathbb{R}) = \{[a,b]: a, b \in \mathbb{R}\}$, with $i = 0, \dots, 7$. A set of the interval sedenion numbers is denoted by $I(\mathbb{S}) = \{[a,b]: a, b \in \mathbb{S}\}$.

Definition 7. Let $S = (A_0, \dots, A_{15})$ and $T = (B_0, \dots, B_{15})$ be two interval sedenion numbers. If $A_0 = B_0, \dots, A_{15} = B_{15}$, then S is to be T, denoted by $S = T$.

Definition 8. Let $S = (A_0, \dots, A_{15})$ and $T = (B_0, \dots, B_{15})$ be two interval sedenion numbers. Then the sum of two interval sedenion numbers is defined as

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

$S+T=(A_0+B_0 \dots, A_{15}+B_{15})$.

Definition 9. Let $S=(A_0,\dots,A_{15})$ and $T=(B_0,\dots,B_{15})$ be two interval sedenion numbers. Then the multiplication of two interval sedenion numbers is defined as

$S.T=(C_0,\dots,C_{15})$, where

$$
\begin{aligned}
C_0 &= A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 - A_4B_4 - A_5B_5 - A_6B_6 - A_7B_7 - A_8B_8 \\
&\quad -A_9B_9 - A_{10}B_{10} - A_{11}B_{11} - A_{12}B_{12} - A_{13}B_{13} - A_{14}B_{14} - A_{15}B_{15}, \\
C_1 &= A_0B_1 + A_1B_0 + A_2B_3 - A_3B_2 + A_4B_5 - A_5B_4 - A_6B_7 + A_7B_6 + A_8B_9 \\
&\quad -A_9B_1 - A_{10}B_{11} + A_{11}B_{10} - A_{12}B_{13} - A_{13}B_9 + A_{14}B_{15} - A_{15}B_{14}, \\
C_2 &= A_0B_2 - A_1B_3 + A_2B_0 + A_3B_1 + A_4B_6 + A_5B_7 - A_6B_4 - A_7B_5 + A_8B_{10} \\
&\quad +A_9B_{11} - A_{10}B_8 - A_{11}B_9 - A_{12}B_{14} - A_{13}B_{15} + A_{14}B_{12} + A_{15}B_{13}, \\
C_3 &= A_0B_3 + A_1B_2 - A_2B_1 + A_3B_0 + A_4B_7 - A_5B_6 + A_6B_5 - A_7B_4 + A_8B_{11} \\
&\quad -A_9B_{10} + A_{10}B_9 - A_{11}B_8 - A_{12}B_{15} + A_{13}B_{14} - A_{14}B_{13} + A_{15}B_{12}, \\
C_4 &= A_0B_4 - A_1B_5 - A_2B_6 - A_3B_7 + A_4B_0 + A_5B_1 + A_6B_2 + A_7B_3 + A_8B_{12} \\
&\quad +A_9B_{13} + A_{10}B_{14} + A_{11}B_{15} - A_{12}B_8 - A_{13}B_9 - A_{14}B_{10} - A_{15}B_{11}, \\
C_5 &= A_0B_5 + A_1B_4 - A_2B_7 + A_3B_6 - A_4B_1 + A_5B_0 - A_6B_3 + A_7B_2 + A_8B_{13} \\
&\quad -A_9B_{12} + A_{10}B_{15} - A_{11}B_{14} + A_{12}B_9 - A_{13}B_8 + A_{14}B_{11} - A_{15}B_{10}, \\
C_6 &= A_0B_6 + A_1B_3 + A_2B_4 - A_3B_5 - A_4B_2 + A_5B_3 + A_6B_0 - A_7B_1 - A_8B_{14} \\
&\quad -A_9B_{15} - A_{10}B_{12} + A_{11}B_{13} + A_{12}B_{10} - A_{13}B_{11} - A_{14}B_8 + A_{15}B_9, \\
C_7 &= A_0B_7 - A_1B_6 + A_2B_5 + A_3B_4 - A_4B_3 - A_5B_2 + A_6B_1 + A_7B_0 + A_8B_{15} \\
&\quad +A_9B_{14} - A_{10}B_{13} + A_{11}B_{12} + A_{12}B_{11} + A_{13}B_{10} - A_{14}B_9 - A_{15}B_8, \\
C_8 &= A_0B_8 - A_1B_9 - A_2B_{10} - A_3B_9 - A_4B_{12} - A_5B_{13} - A_6B_{14} - A_7B_{15} + A_8B_0 \\
&\quad +A_9B_1 + A_{10}B_2 + A_{11}B_3 + A_{12}B_4 + A_{13}B_5 + A_{14}B_6 + A_{15}B_7, \\
C_9 &= A_0B_9 + A_1B_8 - A_2B_{11} + A_3B_{10} - A_4B_{13} + A_5B_{12} + A_6B_{15} - A_7B_{14} - A_8B_1 \\
&\quad +A_9B_0 - A_{10}B_3 + A_{11}B_2 - A_{12}B_5 + A_{13}B_4 + A_{14}B_7 - A_{15}B_6, \\
C_{10} &= A_0B_{10} + A_1B_{11} + A_2B_8 - A_3B_9 - A_4B_{14} - A_5B_{15} + A_6B_{12} + A_7B_{13} - A_8B_2 \\
&\quad +A_9B_3 + A_{10}B_0 - A_{11}B_1 - A_{12}B_6 - A_{13}B_7 + A_{14}B_4 + A_{15}B_5, \\
C_{11} &= A_0B_{11} - A_1B_{10} + A_2B_9 + A_3B_8 - A_4B_{15} + A_5B_{14} - A_6B_{13} + A_7B_{12} - A_8B_3 \\
&\quad -A_9B_2 + A_{10}B_1 + A_{11}B_0 - A_{12}B_7 + A_{13}B_6 - A_{14}B_5 + A_{15}B_4, \\
C_{12} &= A_0B_{12} + A_1B_{13} + A_2B_{14} + A_3B_{15} + A_4B_8 - A_5B_9 - A_6B_{10} - A_7B_{11} - A_8B_4 \\
&\quad +A_9B_5 + A_{10}B_6 + A_{11}B_7 + A_{12}B_0 - A_{13}B_1 - A_{14}B_2 - A_{15}B_3, \\
C_{13} &= A_0B_{13} - A_1B_{12} + A_2B_{15} - A_3B_{14} + A_4B_9 + A_5B_8 + A_6B_{11} - A_7B_{10} - A_8B_5 \\
&\quad -A_9B_4 + A_{10}B_7 - A_{11}B_6 + A_{12}B_1 + A_{13}B_0 + A_{14}B_3 - A_{15}B_2, \\
C_{14} &= A_0B_{14} - A_1B_{15} - A_2B_{12} + A_3B_{13} + A_4B_{10} - A_5B_{11} + A_6B_8 + A_7B_9 - A_8B_6 \\
&\quad -A_9B_7 - A_{10}B_4 + A_{11}B_5 + A_{12}B_2 - A_{13}B_3 + A_{14}B_0 + A_{15}B_1, \\
C_{15} &= A_0B_{15} + A_1B_{14} - A_2B_{13} - A_3B_{12} + A_4B_{11} + A_5B_{10} - A_6B_9 + A_7B_8 - A_8B_7 \\
&\quad +A_9B_6 - A_{10}B_5 - A_{11}B_4 + A_{12}B_3 + A_{13}B_2 - A_{14}B_1 + A_{15}B_0.
\end{aligned}
$$

Let $S=(A_0,\dots,A_{15})$, $T=(B_0,\dots,B_{15})$ and $R=(R_0,\dots,R_{15})\in I(\mathbb{S})$ be given. Then the following properties are provided for interval sedenion numbers:

i. $S+T\in I(\mathbb{S})$ and $ST\in I(\mathbb{S})$

ii. $S+T=(A_0+B_0,\dots,A_{15}+B_{15})=(B_0+A_0,\dots,B_{15}+A_{15})=T+S$

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

iii. $(S+T)+R=S+(T+R)$
iv. $S+0=(A_0,\dots,A_{15})+(0,\dots,0)=S$ for $0=(0,\dots,0)\in I(\mathbb{S})$
v. $S.1=(A_0,\dots,A_{15}).(1,0\dots,0)=S$ for $1=(1,0,\dots,0)\in I(\mathbb{S})$.

We examine the metric concept previously defined [19, 21] for interval sequence numbers.

Definition 10. Let $S=(A_0,\dots,A_{15})$ and $T=(B_0,\dots,B_{15})$ be two interval sedenion numbers. Then a function
$d: I(\mathbb{S})\times I(\mathbb{S})\rightarrow \mathbb{R}_+ ,\quad d(S,T)=\sum_{i=0}^{15} d(A_i,B_i)$
is called a metric.

With the help of the metric definition, it has been shown that the number of intervals $I(\mathbb{R})$, $I(\mathbb{C})$), $I(\mathbb{H})$ and $I(\mathbb{O})$ are dense and metric spaces. Similarly, interval sedenion number $I(\mathbb{S})$ is a metric space with the d metric defined on the set $I(\mathbb{S})$ and is also dense. It is easy to define a metric on $\mathbb{S}_{\mathcal{F}}$ in a similar way. Also, the set $\mathbb{S}_{\mathcal{F}}$ is a metric space.

Supremum and Infimum for Fuzzy Sedenion Numbers

In this section we define and establish results on the least upper bound and the greatest lower bound of fuzzy sedenion numbers.

Definition 11. Let $\acute{s}=(\overline{A_0},\dots,\overline{A_{15}})$ be the fuzzy sedenion number. The number $\acute{s}$ is called an infinite fuzzy sedenion number if and only if $\acute{s}=(\overline{\infty},\dots,\overline{\infty})=\acute{\infty}$.

Definition 12. Let $\acute{s}=(\overline{A_0},\dots,\overline{A_{15}})$ and $\acute{t}=(\overline{B_0},\dots,\overline{B_{15}})$ be two fuzzy sedenion numbers. Then $\acute{s}\leq\acute{t}$ if and only if $\overline{A_0}\leq\overline{B_0}\wedge\overline{A_1}\leq\overline{B_1}\wedge\dots\overline{A_{15}}\leq\overline{B_{15}}$.

Definition 13. Let $M\subseteq\mathbb{S}_{\mathcal{F}}$. If there exists $'\in\mathbb{S}_{\mathcal{F}}$, where $K'\neq\acute{\infty}$ such that $\acute{t}\leq\acute{K}$ for every $\acute{t}\in M$, then M is said to have an upper bound K' . Analogously, if there exists $\acute{s}\in\mathbb{S}_{\mathcal{F}}$, where $\acute{s}\neq\acute{\infty}$ such that $\acute{s}\leq\acute{t}$ for every $\acute{t}\in M$, then M is said to have a lower bound $\acute{s}$. A set with lower and upper bounds is said to be bounded.

Definition 14. Let $\acute{s}$ be a fuzzy sedenion number. Then $\acute{s}$ is called the least upper bound for $M\subseteq\mathbb{S}_{\mathcal{F}}$ if the following properties are valid for $\acute{s}$:
i. $\acute{t}\leq\acute{s},\forall\acute{t}\in M,$
ii. For any real number $\varepsilon>0$, there is a number $\acute{t}\in M$ such that $s<\acute{\acute{t}}+\varepsilon$.

Definition 15. Let $\acute{r}$ be a fuzzy sedenion number. Then $\acute{r}$ is called the greatest upper bound for $M\subseteq\mathbb{S}_{\mathcal{F}}$ if the following properties are valid for $\acute{r}$:
i. $\acute{r}\leq\acute{t},\forall\acute{t}\in M$
ii. For any real number $\varepsilon>0$, there is a number $\acute{t}\in M$ such that $\acute{t}-\varepsilon<\acute{r}$. Thus, we get $SupM=\acute{s}$ and $InfM=\acute{r}$.

Let $M\subseteq\mathbb{S}_{\mathcal{F}}$. Then we can give the following statements:
i. $Re_{(M)}=\{Re(\acute{t})\epsilon\mathbb{R}_{\mathcal{F}}:\acute{t}\in M\}$
ii. $Im1_{(M)}=\{Im1(\acute{t})\epsilon\mathbb{R}_{\mathcal{F}}:\acute{t}\in M\}$
$Im2_{(M)}=\{Im2(\acute{t})\epsilon\mathbb{R}_{\mathcal{F}}:\acute{t}\in M\}$
$\vdots$
$Im15_{(M)}=\{Im15(\acute{t})\epsilon\mathbb{R}_{\mathcal{F}}:\acute{t}\in M\}$.

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

As a result, $M = Re_{(M)} \times Im1_{(M)} \times Im2_{(M)} \times \cdots \times Im15_{(M)}$ can be written. In addition, the following statements are given: $SupM =$
$\left(Sup\left(Re_{(M)}\right), Sup\left(Im1_{(M)}\right), Sup\left(Im2_{(M)}\right), \cdots, Sup\left(Im15_{(M)}\right)\right)$
$InfM = \left(Inf\left(Re_{(M)}\right), Inf\left(Im1_{(M)}\right), Inf\left(Im2_{(M)}\right), \cdots, Inf\left(Im15_{(M)}\right)\right).$

Limit of Sequence of Fuzzy Sedenion Numbers

In this section the limits of fuzzy sedenion numbers are examined with the help of the limit concepts of different types of fuzzy numbers [17-21, 27].

Definition 16. Let d be a metric on $\mathbb{R}_{\mathcal{F}}$, $\{\acute{s}_n\} \subset S_F$ and $\acute{s} \in \mathbb{S}_{\mathcal{F}}$. Then $\{\acute{s}_n\}$ is said to converge to $\acute{s}$ if, for an arbitrary $\varepsilon > 0$, there exists an integer $N > 0$ such that $d(\acute{s}_n, \acute{s}) < \varepsilon$, as $n \geq N$. We denote it by $\lim\limits_{n\to\infty} \acute{s}_n = \acute{s}$.

Theorem 1. The limit $\lim\limits_{n\to\infty} \acute{s}_n = \acute{s}$ if and only if $\lim\limits_{n\to\infty} Re(\acute{s}_n) = Re(\acute{s})$, $\lim\limits_{n\to\infty} Im1(\acute{s}_n) = Im1(\acute{s})$, $\lim\limits_{n\to\infty} Im2(\acute{s}_n) = Im2(\acute{s})$, … , $\lim\limits_{n\to\infty} Im15(\acute{s}_n) = Im15(\acute{s})$.

Proof. It follows immediately from Definition 16.

Theorem 2. Let $\{\acute{s}_n\}$, $\{\acute{r}_n\} \subset \mathbb{S}_{\mathcal{F}}$ and $\acute{s}, \acute{r} \in \mathbb{S}_{\mathcal{F}}$ and $c \in \mathbb{R}$. Then

i. $\lim\limits_{n\to\infty} (\acute{s}_n + \acute{r}_n) = \lim\limits_{n\to\infty} \acute{s}_n + \lim\limits_{n\to\infty} \acute{r}_n$
ii. $\lim\limits_{n\to\infty} c.\acute{s}_n = c.\lim\limits_{n\to\infty} \acute{s}_n$.

Proof.

i. Let $\acute{s}_n = \left(\overline{A_{0_n}}, \ldots, \overline{A_{15_n}}\right)$ and $\acute{t}_n = \left(\overline{B_{0_n}}, \ldots, \overline{B_{15_n}}\right)$. Then we get

$$\begin{aligned}
\lim_{n\to\infty} (\acute{s}_n + \acute{r}_n) &= \lim_{n\to\infty} \left[\left(\overline{A_{0_n}}, \ldots, \overline{A_{15_n}}\right) + \left(\overline{B_{0_n}}, \ldots, \overline{B_{15_n}}\right)\right] \\
&= \lim_{n\to\infty} \left(\overline{A_{0_n}} + \overline{B_{0_n}}, \ldots, \overline{A_{15_n}} + \overline{B_{15_n}}\right) \\
&= \left(\lim_{n\to\infty} \left(\overline{A_{0_n}} + \overline{B_{0_n}}\right), \ldots, \lim_{n\to\infty} \left(\overline{A_{15_n}} + \overline{B_{15_n}}\right)\right) \\
&= \lim_{n\to\infty} \left(\overline{A_{0_n}}, \ldots, \overline{A_{15_n}}\right) + \lim_{n\to\infty} \left(\overline{B_{0_n}}, \ldots, \overline{B_{15_n}}\right) \\
&= \lim_{n\to\infty} \acute{s}_n + \lim_{n\to\infty} \acute{r}_n .
\end{aligned}$$

ii. Let $\acute{s}_n = \left(\overline{A_{0_n}}, \ldots, \overline{A_{15_n}}\right)$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} c.\acute{s}_n &= \lim_{n\to\infty} \left(c.\overline{A_{0_n}}, \ldots, c.\overline{A_{15_n}}\right) \\
&= c.\lim_{n\to\infty} \left(\overline{A_{0_n}}, \ldots, \overline{A_{15_n}}\right) \\
&= c.\lim_{n\to\infty} \acute{s}_n .
\end{aligned}$$

Theorem 3 (Limit uniqueness theorem). If $\lim\limits_{n\to\infty} \acute{s}_n = \acute{s}$ and $\lim\limits_{n\to\infty} \acute{s}_n = \acute{r}$, then $\acute{s} = \acute{r}$.

Proof. Let $\acute{s}_n = \left(\overline{A_{0_n}}, \ldots, \overline{A_{15_n}}\right)$, $\acute{s} = \left(\overline{A_0}, \ldots, \overline{A_{15}}\right)$, $\acute{r} = \left(\overline{B_0}, \ldots, \overline{B_{15}}\right)$ and $\lim\limits_{n\to\infty} \acute{s}_n = \acute{s}$ and $\lim\limits_{n\to\infty} \acute{s}_n = \acute{r}$. Then we get $\lim\limits_{n\to\infty} \acute{s}_n = \left(\overline{A_0}, \ldots, \overline{A_{15}}\right) = \left(\overline{B_0}, \ldots, \overline{B_{15}}\right)$ and $\acute{s} = \acute{r}$.

Theorem 4. Let $\lim\limits_{n\to\infty} \acute{s}_n = \acute{s}$ and $\lim\limits_{n\to\infty} \acute{r}_n = \acute{r}$, then $\lim\limits_{n\to\infty} d(\acute{s}_n, \acute{r}_n) = d(\acute{s}, \acute{r})$.

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

Proof. Let $\lim_{n\to\infty} \acute{s}_n = \acute{s}$ and $\lim_{n\to\infty} \acute{r}_n = \acute{r}$. Then we get $\lim_{n\to\infty} Re(\acute{s}_n) = Re(\acute{s})$, $\lim_{n\to\infty} Im1(\acute{s}_n) = Im1(\acute{s})$, $\lim_{n\to\infty} Im2(\acute{s}_n) = Im2(\acute{s})$, … , $\lim_{n\to\infty} Im15(\acute{s}_n) = Im15(\acute{s})$ and $\lim_{n\to\infty} Re(\acute{r}_n) = Re(\acute{r})$, $\lim_{n\to\infty} Im1(\acute{r}_n) = Im1(\acute{r})$, $\lim_{n\to\infty} Im2(\acute{r}_n) = Im2(\acute{r})$, … , $\lim_{n\to\infty} Im15(\acute{r}_n) = Im15(\acute{r})$.

If we use the fuzzy distance, we obtain

$\lim_{n\to\infty} d_F\big(Re(\acute{s}_n), Re(\acute{r}_n)\big) = d_F(Re(\acute{s}), Re(\acute{r}))$,
$\lim_{n\to\infty} d_F\big(Im1(\acute{s}_n), Im1(\acute{r}_n)\big) = d_F(Im1(\acute{s}), Im1(\acute{r}))$,
⋮
$\lim_{n\to\infty} d_F\big(Im15(\acute{s}_n), Im15(\acute{r}_n)\big) = d_F(Im15(\acute{s}), Im15(\acute{r}))$.

Therefore, for any $\varepsilon > 0$, there exists $N_1, N_2, \dots, N_{15} > 0$ such that

$d_F(Re(\acute{s}), Re(\acute{r})) - \varepsilon \le d_F\big(Re(\acute{s}_n), Re(\acute{r}_n)\big) \le d_F(Re(\acute{s}), Re(\acute{r})) + \varepsilon$ as $n \ge N_1$,
$d_F(Im1(\acute{s}), Im1(\acute{r})) - \varepsilon \le d_F\big(Im1(\acute{s}_n), Im1(\acute{r}_n)\big) \le d_F(Im1(\acute{s}), Im1(\acute{r})) + \varepsilon$ as $n \ge N_2$,
⋮
$d_F(Im15(\acute{s}), Im15(\acute{r})) - \varepsilon \le d_F\big(Im15(\acute{s}_n), Im15(\acute{r}_n)\big) \le d_F(Im15(\acute{s}), Im15(\acute{r})) + \varepsilon$ as $n \ge N_{15}$.

Thus we get

$(d_F(Re(\acute{s}), Re(\acute{r})) - \varepsilon) \vee (d_F(Im1(\acute{s}), Im1(\acute{r})) - \varepsilon) \vee \dots \vee (d_F(Im15(\acute{s}), Im15(\acute{r})) - \varepsilon) \le \Big(d_F\big(Re(\acute{s}_n), Re(\acute{r}_n)\big)\Big) \vee \Big(d_F\big(Im1(\acute{s}_n), Im1(\acute{r}_n)\big)\Big) \vee \dots \vee \Big(d_F\big(Im15(\acute{s}_n), Im15(\acute{r}_n)\big)\Big) \le (d_F(Re(\acute{s}), Re(\acute{r})) + \varepsilon) \vee (d_F(Im1(\acute{s}), Im1(\acute{r})) + \varepsilon) \vee \dots \vee (d_F(Im15(\acute{s}), Im15(\acute{r})) + \varepsilon)$.

This results in

$d(\acute{s}, \acute{r}) - \varepsilon \le d(\acute{s}_n, \acute{r}_n) \le d(\acute{s}, \acute{r}) + \varepsilon$ as $n \ge maks\{N_1, N_2, \dots, N_{15}\}$,
that is $\lim_{n\to\infty} d(\acute{s}_n, \acute{r}_n) = d(\acute{s}, \acute{r})$.

Theorem 5 (Sandwich theorem). Let $\{\acute{s}_n\}$, $\{\acute{p}_n\}$, $\{\acute{r}_n\} \subset \mathbb{S}_{\mathcal{F}}$ and $\acute{s} \in \mathbb{S}_{\mathcal{F}}$. If for every n, $\acute{s}_n \le \acute{p}_n \le \acute{r}_n$ and $\lim_{n\to\infty} \acute{s}_n = \lim_{n\to\infty} \acute{r}_n = \acute{s}$, then $\lim_{n\to\infty} \acute{p}_n = \acute{s}$.

Proof. Let $\acute{s}_n \le \acute{p}_n \le \acute{r}_n$ and $\lim_{n\to\infty} \acute{s}_n = \lim_{n\to\infty} \acute{r}_n = \acute{s}$ for every n. If the limit is taken for each side of the inequality, then we have $\lim_{n\to\infty} \acute{s}_n \le \lim_{n\to\infty} \acute{p}_n \le \lim_{n\to\infty} \acute{r}_n$ and $\acute{s} \le \lim_{n\to\infty} \acute{p}_n \le \acute{s}$. Thus, we obtain $\lim_{n\to\infty} \acute{p}_n = \acute{s}$.

Theorem 6 (Boundedness theorem). Let $\{\acute{s}_n\} \subset \mathbb{S}_{\mathcal{F}}$ and $\acute{s} \neq \infty$. If $\{\acute{s}_n\}$ converges, then there exists $\acute{L}, \acute{K}$ $(\neq \infty)$ such that $\acute{K} \le \acute{s}_n \le \acute{L}$ for every n.

Proof. It is obvious from Definition 16.

Concept of Fuzzy Sedenion Valued Series

The concept of convergence is given for sedenions in this section.

Definition 17. Let the sequence of fuzzy sedenion numbers $\{\acute{s}_n\} \subset \mathbb{S}_{\mathcal{F}}$, $n = 1,2, \dots$. Then $\sum_{n=1}^{\infty} \acute{s}_n$ is called a fuzzy sedenion valued series. We can denote fuzzy sedenion valued series as follows:

Maejo Int. J. Sci. Technol. **2023**, *17*(03), 239-251

$\sum_{n=1}^{\infty} \acute{s}_n = (\sum_{n=1}^{\infty} Re\acute{s}_n, \sum_{n=1}^{\infty} Im1\acute{s}_n, \sum_{n=1}^{\infty} Im2\acute{s}_n, \dots, \sum_{n=1}^{\infty} Im15\acute{s}_n)$,

$\sum_{n=1}^{\infty} \acute{s}_n(X) = (\sum_{n=1}^{\infty} Re\acute{s}_n)(a_0) \wedge (\sum_{n=1}^{\infty} Im1\acute{s}_n)(a_1) \wedge \dots \wedge (\sum_{n=1}^{\infty} Im15\acute{s}_n)(a_{15})$,

where $\sum_{i=0}^{15} a_i e_i \in \mathbb{S}$ and $\acute{s}_n$ are called a general form of $\sum_{n=1}^{\infty} \acute{s}_n$.

Definition 18. Let $\acute{s} \in \mathbb{S}_{\mathcal{F}}$, $n = 1,2,\dots$.

Supp$\acute{s} = \{X = \sum_{i=0}^{15} a_i e_i \in \mathbb{S}\colon (Re\acute{s}_n)(a_0) > 0, (Im1\acute{s}_n)(a_1) > 0, \dots, (Im15\acute{s}_n)(a_{15}) > 0\}$.

Then Supp$\acute{s}$ is called the support of fuzzy sedenion number $\mathbb{S}$.

Definition 19. Let the sequence of fuzzy sedenion numbers $\{\acute{s}_n\} \subset \mathbb{S}_{\mathcal{F}}$, $n = 1,2,\dots$. If for arbitrary $X_n = \sum_{i=0}^{15} a_{i_n} e_i\, a_{i_n} e_i \in Supp\acute{s}_n$, $(n = 1,2,\dots)$, the relevant number-term series $\sum_{n=1}^{\infty} a_{0_n}$, $\sum_{n=1}^{\infty} a_{1_n}$, …, $\sum_{n=1}^{\infty} a_{15_n}$ are convergent, then we call fuzzy sedenion valued series $\sum_{n=1}^{\infty} \acute{s}_n$ convergent. Otherwise, if there exists at least a sequence of sedenion numbers $\{X_n\} \subset Supp\acute{s}_n$ such that the number-term series $\sum_{n=1}^{\infty} a_{0_n}$ or $\sum_{n=1}^{\infty} a_{1_n}$ or … or $\sum_{n=1}^{\infty} a_{15_n}$ diverges, then we call $\sum_{n=1}^{\infty} \acute{s}_n$ divergent.

Theorem 7. Let $\sum_{n=1}^{\infty} \acute{s}_n$ be a fuzzy sedenion valued series. Then $\sum_{n=1}^{\infty} \acute{s}_n$ is convergent if and only if any sequences $\{a_{0_n}\} \subset Supp(Re\acute{s}_n)$, $\{a_{1_n}\} \subset Supp(Im1\acute{s}_n)$, ..., $\{a_{15_n}\} \subset Supp(Im15\acute{s}_n)$, where $n = 1,2,\dots$ and the number-term series $\sum_{n=1}^{\infty} a_{0_n}$, $\sum_{n=1}^{\infty} a_{1_n}$, … , $\sum_{n=1}^{\infty} a_{15_n}$ are all convergent.

Proof. It is obvious from Definitions 18 and 19.

Theorem 8 (Comparison criterion). Let $\sum_{n=1}^{\infty} \acute{s}_n$, $\sum_{n=1}^{\infty} \acute{v}_n$ be two fuzzy sedenion valued series and they satisfy $\acute{s}_n \leq \acute{t}_n$, $n = 1,2,\dots$. Then

i. If $\sum_{n=1}^{\infty} \acute{v}_n$ converges, then $\sum_{n=1}^{\infty} \acute{s}_n$ is also convergent;

ii. If $\sum_{n=1}^{\infty} \acute{s}_n$ diverges, then $\sum_{n=1}^{\infty} \acute{v}_n$ is also divergent.

Proof.

i. Let $X_n = \sum_{i=0}^{15} a_{i_n} e_i \in Supp\acute{s}_n$, $n = 1,2,\dots$ and $\sum_{i=1}^{\infty} \acute{v}_n$ converge. Then

$(Re\acute{s}_n)(a_0) > 0$, $(Im1\acute{s}_n)(a_1) > 0, \dots, (Im15\acute{s}_n)(a_{15}) > 0$.

By definition of the order of fuzzy sedenion numbers, we get

$(Re\acute{v}_n)(a_0) > (Re\acute{s}_n)(a_0)$, $(Im1\acute{v}_n)(a_1) > (Im1\acute{s}_n)(a_1)$, …, $(Im15\acute{v}_n)(a_{15}) > (Im15\acute{s}_n)(a_{15})$.

Therefore,

$\{a_{0_n}\} \subset Supp(Re\acute{v}_n)$, $\{a_{1_n}\} \subset Supp(Im1\acute{v}_n)$, ..., $\{a_{15_n}\} \subset Supp(Im15\acute{v}_n)$.

From the convergent definition of $\sum_{n=1}^{\infty} \acute{v}_n$, it is clear that $\sum_{n=1}^{\infty} a_{0_n}$, $\sum_{n=1}^{\infty} a_{1_n}$, … and $\sum_{n=1}^{\infty} a_{15_n}$ are all convergent. Thus, from Theorem 7, we infer that $\sum_{n=1}^{\infty} \acute{s}_n$ is convergent.

ii. Let $X_n = \sum_{i=0}^{15} a_{i_n} e_i \in Supp\acute{s}_n$, $n = 1,2,\dots$ and $\sum_{n=1}^{\infty} \acute{v}_n$ converge. Then from the statement i., $\sum_{n=1}^{\infty} \acute{s}_n$ is convergent. This is a contradiction. Thus, $\sum_{n=1}^{\infty} \acute{v}_n$ is divergent.

CONCLUSIONS

When the fuzzy numbers defined in the sources from the literature are focused on, it is realised that it is important to define the fuzzy sedenion numbers that include all the numbers in these sources. Based on the definition of the fuzzy sedenion numbers, the analytical properties of fuzzy sedenion numbers have been investigated. Moreover, the concepts of metric, supremum, infimum, and limit and series have also been given.

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