Maejo Int. J. Sci. Technol. **2023**, *17*(03), 187-200

Maejo International Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

Hyper dual number matrices

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Received: 23 June 2023 / Accepted: 12 September 2023 / Published: 15 September 2023

Abstract: We study hyper dual numbers and hyper dual number matrices. Firstly, we give some basic properties of dual numbers and hyper dual numbers. Next, we investigate hyper dual number matrices using the properties of dual number matrices. Then we define the dual adjoint matrix of hyper dual matrices and describe some of their properties.

Keywords: dual numbers, hyper dual numbers, matrices of hyper dual numbers, eigenvalue, dual adjoint matrix

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INTRODUCTION

Dual numbers were introduced in 1873 by Clifford [1]. In 1903 Study [2] defined the relationship between two lines in Euclidean space as a dual angle and proved that in the E.Study transformation, which he named himself, each point on the unit dual sphere corresponds exactly to the directional lines in $\mathbb{R}^3$. In this manner it allows the theory of directional lines in $\mathbb{R}^3$ to be examined with the help of dual numbers. Just as the geometry of the Euclidean plane can be described by complex numbers, the geometry of the Galilean plane

$$\mathbb{D} = \{z = x + \varepsilon y : \ \varepsilon^2 = 0, \varepsilon \neq 0, x, y \in \mathbb{R}\}$$

can be described by dual numbers, which are extensively used in quantum mechanics and classical mechanics of screws [3-6].

Matrix representation of dual numbers can be given as

$$x + \varepsilon y \leftrightarrow \begin{bmatrix} x & y \\ 0 & x \end{bmatrix},$$

where

$$\varepsilon \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is dual unit.

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This mapping is an isomorphism, as the operations here correspond to matrix addition and multiplication. Thus, we have the matrix forms of Euler formula for dual numbers as given:

$$1+\varepsilon\theta=\exp\begin{bmatrix}0 & \theta\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & \theta\\ 0 & 1\end{bmatrix}.$$

This matrix is the rotation matrix in the Galilean plane. For any $z=x+\varepsilon y\in\mathbb{D}$, $\mathrm{Re}(z)=x$ is the real part of z and $\mathrm{Im}(z)=y$ is the imaginary part of z. The conjugate of a dual number is denoted by z and defined by $\overline{z}=x-\varepsilon y$.

The norm of a dual number is defined as

$$|z|=|x|.$$

If $x=\mp1$, then z is called unit dual number. Note that $\frac{z}{|z|}$ is a unit dual number for $z\in\mathbb{D}$ with $|z|\neq 0$. The square root of a dual number $z=x+\varepsilon y$ is defined as

$$\sqrt{x+\varepsilon y}=\sqrt{x}+\varepsilon\frac{y}{2\sqrt{x}},x>0.$$

The inverse of the dual number z is

$$z^{-1}=\frac{\overline{z}}{|z|^2},\text{if } x\neq 0.$$

Dual numbers play an important role in the field of robot technology [7]. The concept of dual numbers is used to show the geometry of lines. Robotic applications can be made with the help of this one-to-one mapping [8]. Using this advantage of dual numbers in algebra of quaternions, robotic technology and motion theory have been taken to another dimension [9]. The innovations brought to technology with the use of number systems in this way are quite remarkable. A list of references to dual number applications in linear algebra and kinematic and numerical algorithms can be found [e.g. 10]. One of the number systems that has attracted attention in recent years is the hyper dual number system. This number system was first given by Fike et al. and Fike and Alonso [11-13]. Cil [14] worked on hyper dual numbers and their properties and presented it as a master thesis. Cohen and Shoham [15, 16] applied the hyper dual number system to rigid body kinematics. Nalbant and Yuce [17] described some new properties of real quaternion matrices. Next, they gave a Matlab algorithm that easily finds matrix representations of the real quaternion matrices.

Hyper dual numbers are a higher dimensional extension of dual numbers in the same way as quaternions are a higher dimensional extension of ordinary complex numbers [12]. Fike et al. [11] focused on the numerical approach and developed a Fortran algorithm to calculate numerically the velocity and acceleration of the coupler curve of a spherical 4R mechanism. Later, Fike and Alonso [12] demonstrated the accuracy of hyper dual number calculations by giving a comparison of the various first and second derivative calculation methods for a simple, analytic function.

Many authors have studied quaternion matrices, split quaternion matrices and hybrid number matrices [e.g. 18-20]. Considering the studies of matrices, together with the developments in robot technology and rigid body kinematics, it is clear that the definition and applications of hyper dual number matrices will make important contributions to robot technology. Hyper dual number matrices can be written as block matrices whose elements are dual numbers. The inverse, conjugate and similar properties of hyper dual number matrices that can be written in this way are given in this study. In this respect, it is thought that these features will play a key role in studies and applications in kinematic and robotic technology.

In this paper we investigate hyper dual number matrices. Firstly, we give some properties of hyper dual numbers. Then we introduce hyper dual number matrices and give some of their

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properties. Finally, we define the dual adjoint matrix of hyper dual number matrices and give some properties of these matrices. Also, we give an application of matrices in the last part.

HYPER DUAL NUMBERS

A hyper dual number **x** has the form

$$\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2,$$

where x_1, x_2, x_3, x_4 are real numbers while $\varepsilon_1, \varepsilon_2$ are dual units. Hyper dual number algebra is a unitary and commutative ring with four basic elements $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$ that provide the equations

$$\varepsilon_1^2 = \varepsilon_2^2 = 0,$$
$$\varepsilon_1 \neq 0, \varepsilon_2 \neq 0, \varepsilon_1 \neq \varepsilon_2,$$
$$\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1.$$

On the other hand, hyper dual number $\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2$ can be given in the following form

$$\mathbf{x} = (x_1 + \varepsilon x_2) + \varepsilon^*(x_3 + \varepsilon x_4).$$

Also, hyper dual number is written as the sum of two dual numbers:

$$\mathbf{x} = d_1 + \varepsilon^* d_2,$$
$$\varepsilon = \varepsilon_1, \varepsilon^* = \varepsilon_2,$$
$$\varepsilon^2 = (\varepsilon^*)^2 = 0,$$
$$\varepsilon \neq 0, \varepsilon^* \neq 0, \varepsilon \neq \varepsilon^*,$$
$$\varepsilon\varepsilon^* = \varepsilon^*\varepsilon \neq 0,$$

where ε^* is called the hyper dual unit and and $d_1, d_2 \in \mathbb{D}$. Let us denote the algebra of hyper dual number by $\mathbb{HDN}$.

A set of hyper dual numbers can be represented as

$$\mathbb{HDN} = \{\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2 : x_1, x_2, x_3, x_4 \in \mathbb{R}\}.$$

The sum and product of hyper dual numbers $\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2$ and $\mathbf{y} = y_1 + y_2\varepsilon_1 + y_3\varepsilon_2 + y_4\varepsilon_1\varepsilon_2$ are

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1) + (x_2 + y_2)\varepsilon_1 + (x_3 + y_3)\varepsilon_2 + (x_4 + y_4)\varepsilon_1\varepsilon_2,$$
$$\mathbf{xy} = x_1y_1 + (x_1y_2 + x_2y_1)\varepsilon_1 + (x_1y_3 + x_3y_1)\varepsilon_2 + (x_1y_4 + x_2y_3 + x_4y_1 + x_3y_2)\varepsilon_1\varepsilon_2.$$

For any hyper dual number $\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2$, $\mathrm{Re}(\mathbf{x}) = x_1$ is the real part of **x** and $\mathrm{HDu}(\mathbf{x}) = x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2$ is the hyper dual part of **x**. The conjugate of a hyper dual number **x** is denoted by $\bar{\mathbf{x}}$ and defined by $\bar{\mathbf{x}} = x_1 - x_2\varepsilon_1 - x_3\varepsilon_2 - x_4\varepsilon_1\varepsilon_2$ or $\bar{\mathbf{x}} = \overline{d_1} - \varepsilon^* d_2$.

The inner product of $\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2$ and $\mathbf{y} = y_1 + y_2\varepsilon_1 + y_3\varepsilon_2 + y_4\varepsilon_1\varepsilon_2$ is defined by

$$\langle\, , \rangle : \mathbb{HDN} \times \mathbb{HDN} \longrightarrow \mathbb{HDN}$$
$$(\mathbf{x}, \mathbf{y}) \longrightarrow \langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1.$$

The norm of a hyper dual number $\boldsymbol{x}$ is defined by

$$\|\ \| : \mathbb{HDN} \longrightarrow \mathbb{R}$$
$$\mathbf{x} \longrightarrow \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = |x_1|.$$

Since an $\mathbb{HDN}$ is a dual number with dual number entries, we use the definition of the square root of a dual number to compute the square root of the $\mathbb{HDN}$. So the square root of any hyper dual number $\mathbf{x} = d_1 + \varepsilon^* d_2 = x_1 + \varepsilon x_2 + \varepsilon^*(x_3 + \varepsilon x_4)$ is found as follows:

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$$\sqrt{\mathbf{x}}=\sqrt{d_1+\varepsilon^* d_2}=\sqrt{d_1}+\varepsilon^*\frac{d_2}{2\sqrt{d_1}}, x_1>0$$
$$=\sqrt{x_1}\left(1+\frac{x_2}{2x_1}\varepsilon+\frac{x_3}{2x_1}\varepsilon^*+\left(\frac{x_4}{2x_1}-\frac{x_2x_3}{4x_1^2}\right)\varepsilon\varepsilon^*\right), x_1>0.$$

The inverse of the hyper dual number $\mathbf{x}$ is [14]

$$\mathbf{x}^{-1}=\frac{1}{x_1}-\frac{x_2}{x_1^2}\varepsilon-\frac{x_3}{x_1^2}\varepsilon^*-\left(\frac{x_4}{x_1^2}-\frac{2x_2x_3}{x_1^3}\right)\varepsilon\varepsilon^*, x_1\neq 0.$$

Corollary 1. If $x_1=0$, there is no inverse of the hyper dual number $\mathbf{x}=x_1+x_2\varepsilon_1+x_3\varepsilon_2+x_4\varepsilon_1\varepsilon_2$.

Theorem 1. The following properties are satisfied for any $\mathbf{x},\mathbf{y}\in\mathbb{HDN}$.

1. $\mathbf{x}\overline{\mathbf{x}}=\overline{\mathbf{x}}\mathbf{x}$.
2. $\mathbf{xy}=\mathbf{yx}$.
3. $\varepsilon^* d=d\varepsilon^*$, for any $d\in\mathbb{D}$.
4. $\|\mathbf{x}\|^2+\|\mathbf{y}\|^2=\frac{1}{2}(\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2)$.
5. If $x_2y_3=-x_3y_2$, then $\overline{\mathbf{xy}}=\overline{\mathbf{x}}\,\overline{\mathbf{y}}=\overline{\mathbf{y}}\,\overline{\mathbf{x}}$.
6. $\|\mathbf{xy}\|=\|\mathbf{x}\|\|\mathbf{y}\|$.
7. $\overline{\mathbf{x}}=\mathbf{x}$ if and only if $\mathbf{x}\in\mathbb{R}$.
8. If $x_2x_3=0$ and $x_1\neq 0$, then $\mathbf{x}^{-1}=\frac{\overline{\mathbf{x}}}{\|\mathbf{x}\|^2}$.
9.There exists a unique representation of the form $\mathbf{x}=d_1+\varepsilon^* d_2$ for any $\mathbf{x}\in\mathbb{HDN}$, where $d_1,d_2\in\mathbb{D}$.

Proof. We prove some of these properties, and others can be proved similarly.

5. Let $x_2y_3=-x_3y_2$ for $\mathbf{x}=x_1+x_2\varepsilon_1+x_3\varepsilon_2+x_4\varepsilon_1\varepsilon_2\in\mathbb{HDN}$ and $\mathbf{y}=y_1+y_2\varepsilon_1+y_3\varepsilon_2+y_4\varepsilon_1\varepsilon_2\in\mathbb{HDN}$. Then

$$\begin{aligned}\overline{\mathbf{xy}}&=x_1y_1-(x_1y_2+x_2y_1)\varepsilon_1-(x_1y_3+x_3y_1)\varepsilon_2-(x_1y_4+x_2y_3+x_4y_1+x_3y_2)\varepsilon_1\varepsilon_2\\&=x_1y_1-(x_1y_2+x_2y_1)\varepsilon_1-(x_1y_3+x_3y_1)\varepsilon_2-(x_1y_4-x_3y_2+x_4y_1+x_3y_2)\varepsilon_1\varepsilon_2\\&=x_1y_1-(x_1y_2+x_2y_1)\varepsilon_1-(x_1y_3+x_3y_1)\varepsilon_2-(x_1y_4+x_4y_1)\varepsilon_1\varepsilon_2\end{aligned}$$

$$\begin{aligned}\overline{\mathbf{x}}\,\overline{\mathbf{y}}&=(x_1-x_2\varepsilon_1-x_3\varepsilon_2-x_4\varepsilon_1\varepsilon_2)(y_1-y_2\varepsilon_1-y_3\varepsilon_2-y_4\varepsilon_1\varepsilon_2)\\&=x_1y_1-x_1y_2\varepsilon_1-x_1y_3\varepsilon_2-x_1y_4\varepsilon_1\varepsilon_2-x_2y_1\varepsilon_1+x_2y_3\varepsilon_1\varepsilon_2-x_3y_1\varepsilon_2+x_3y_2\varepsilon_1\varepsilon_2-x_4y_1\varepsilon_1\varepsilon_2\\&=x_1y_1-(x_1y_2+x_2y_1)\varepsilon_1-(x_1y_3+x_3y_1)\varepsilon_2-(x_1y_4+x_3y_2-x_3y_2+x_4y_1)\varepsilon_1\varepsilon_2\\&=x_1y_1-(x_1y_2+x_2y_1)\varepsilon_1-(x_1y_3+x_3y_1)\varepsilon_2-(x_1y_4+x_4y_1)\varepsilon_1\varepsilon_2\end{aligned}$$

Since hyper dual numbers are commutative, $\overline{\mathbf{x}}\,\overline{\mathbf{y}}=\overline{\mathbf{y}}\,\overline{\mathbf{x}}$. Thus, $\overline{\mathbf{xy}}=\overline{\mathbf{x}}\,\overline{\mathbf{y}}=\overline{\mathbf{y}}\,\overline{\mathbf{x}}$.

8. Let $x_2x_3=0$ and $x_1\neq 0$ for $\mathbf{x}=x_1+x_2\varepsilon_1+x_3\varepsilon_2+x_4\varepsilon_1\varepsilon_2\in\mathbb{HDN}$. Thus, we obtain

$$\begin{aligned}\mathbf{x}^{-1}&=\frac{1}{x_1}-\frac{x_2}{x_1^2}\varepsilon_1-\frac{x_3}{x_1^2}\varepsilon_2-\left(\frac{x_4}{x_1^2}-\frac{2x_2x_3}{x_1^3}\right)\varepsilon_1\varepsilon_2=\frac{x_1}{x_1^2}-\frac{x_2}{x_1^2}\varepsilon_1-\frac{x_3}{x_1^2}\varepsilon_2-\frac{x_4}{x_1^2}\varepsilon_1\varepsilon_2\\&=\frac{x_1-x_2\varepsilon_1-x_3\varepsilon_2-x_4\varepsilon_1\varepsilon_2}{x_1^2}=\frac{\overline{\mathbf{x}}}{\|\mathbf{x}\|^2}.\end{aligned}$$

Theorem 2. Every hyper dual number can be represented by a 2×2 dual matrix [14].

Proof. Let $\mathbf{x}\in\mathbb{HDN}$. Then there exist dual numbers d_1 and d_2 such that $\mathbf{x}=d_1+\varepsilon^* d_2$ by theorem 1(9). The linear map $f_{\mathbf{x}}:\mathbb{HDN}\longrightarrow\mathbb{HDN}$ is defined by $f_{\mathbf{x}}(\mathbf{y})=\mathbf{xy}$ for all $\mathbf{y}\in\mathbb{HDN}$. This map is bijective and

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$$f_{\mathbf{x}}(1) = 1(d_1 + \varepsilon^* d_2) = d_1 + \varepsilon^* d_2,$$
$$f_{\mathbf{x}}(\varepsilon^*) = \varepsilon^*(d_1 + \varepsilon^* d_2) = \varepsilon^* d_1.$$

With this transformation, hyper dual numbers are defined as subset of the matrix ring $\mathbb{M}_{2\times 2}(\mathbb{D})$, the set of 2×2 dual matrices:

$$\mathbb{HDN}' = \left\{ \begin{bmatrix} d_1 & 0 \\ d_2 & d_1 \end{bmatrix} : d_1, d_2 \in \mathbb{D} \right\}.$$

$\mathbb{HDN}$ and $\mathbb{HDN}'$ are basically the same. Note that

$$\mathcal{M} : \mathbf{x} = d_1 + \varepsilon^* d_2 \in \mathbb{HDN} \longrightarrow \mathbf{x}' = \begin{bmatrix} d_1 & 0 \\ d_2 & d_1 \end{bmatrix} \in \mathbb{HDN}'$$

is bijective and preserves the operations. Furthermore, $\|\mathbf{x}\|^2 = \mathrm{Re}(\det \mathbf{x}')$.

Theorem 3. Every hyper dual number can be represented by a 4×4 real matrix [14].

Proof. The linear map $\varphi_{\mathbf{x}} : \mathbb{HDN} \longrightarrow \mathbb{HDN}$ is defined by $\varphi_{\mathbf{x}}(\mathbf{y}) = \mathbf{xy}$ for any $\mathbf{y} \in \mathbb{HDN}$, where $\mathbf{x} = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2$. This map is bijective and

$$\varphi_{\mathbf{x}}(1) = (x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2)1 = x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2,$$
$$\varphi_{\mathbf{x}}(\varepsilon_1) = (x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2)\varepsilon_1 = x_1\varepsilon_1 + x_3\varepsilon_1\varepsilon_2,$$
$$\varphi_{\mathbf{x}}(\varepsilon_2) = (x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2)\varepsilon_2 = x_1\varepsilon_2 + x_2\varepsilon_1\varepsilon_2,$$
$$\varphi_{\mathbf{x}}(\varepsilon_1\varepsilon_2) = (x_1 + x_2\varepsilon_1 + x_3\varepsilon_2 + x_4\varepsilon_1\varepsilon_2)\varepsilon_1\varepsilon_2 = x_1\varepsilon_1\varepsilon_2.$$

With this transformation, hyper dual numbers are defined as subset of the matrix ring $\mathbb{M}_{4\times 4}(\mathbb{R})$, the set of 4×4 dual matrices:

$$\mathbb{HDN}'' = \left\{ \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ x_3 & 0 & x_1 & 0 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}.$$

$\mathbb{HDN}$ and $\mathbb{HDN}''$ are essentially the same.

HYPER DUAL NUMBER MATRICES

The set of $m \times n$ matrices with hyper dual number entries, which is denoted by $\mathbb{M}_{m\times n}(\mathbb{HDN})$ with ordinary matrix addition and multiplication, is a ring with unity. The set of hyper dual number matrices can be represented by

$$\mathbb{M}_{m\times n}(\mathbb{HDN}) = \{\mathbf{A} = (\mathbf{a}_{st}) : \mathbf{a}_{st} \in \mathbb{HDN}\},$$
$$\mathbb{M}_{m\times n}(\mathbb{HDN}) = \{\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 : A_1, A_2, A_3, A_4 \in \mathbb{M}_{m\times n}(\mathbb{R})\}$$

or

$$\mathbb{M}_{m\times n}(\mathbb{HDN}) = \{\mathbf{A} = D_1 + \varepsilon^* D_2 : (\varepsilon^*)^2 = 0, D_1, D_2 \in \mathbb{M}_{m\times n}(\mathbb{D})\}.$$

Right and left scalar multiplication are defined by

$$\mathbf{Ax} = (\mathbf{a}_{st}\mathbf{x}) \text{ and } \mathbf{xA} = (\mathbf{x}\mathbf{a}_{st})$$

respectively, for $\mathbf{A} = (a_{st}) \in \mathbb{M}_{m\times n}(\mathbb{HDN})$ and $\mathbf{x} \in \mathbb{HDN}$.

The following are satisfied:

$$\mathbf{x}(\mathbf{AB}) = (\mathbf{xA})\mathbf{B},$$
$$(\mathbf{Ax})\mathbf{B} = \mathbf{A}(\mathbf{xB}),$$
$$(\mathbf{xy})\mathbf{A} = \mathbf{x}(\mathbf{yA}),$$

for $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m\times n}(\mathbb{HDN})$ and $\mathbf{x}, \mathbf{y} \in \mathbb{HDN}$. $\mathbb{M}_{m\times n}(\mathbb{HDN})$ is a module over the ring $\mathbb{HDN}$.

For $\mathbf{A} = (\mathbf{a}_{st}) \in \mathbb{M}_{m\times n}(\mathbb{HDN})$, $\overline{\mathbf{A}} = (\overline{\mathbf{a}_{st}}) \in \mathbb{M}_{m\times n}(\mathbb{HDN})$ is the conjugate of $\mathbf{A}$; $\mathbf{A}^{\mathrm{T}} = (\mathbf{a}_{ts}) \in \mathbb{M}_{n\times m}(\mathbb{HDN})$ is the transpose of $\mathbf{A}$; and $\mathbf{A}^{*} = (\overline{\mathbf{A}})^{\mathrm{T}} \in \mathbb{M}_{n\times m}(\mathbb{HDN})$ is the conjugate transpose of $\mathbf{A}$. Also, since it can be written as $\mathbf{A} = D_1 + \varepsilon^* D_2$, the conjugate of $\mathbf{A}$ can be given as $\overline{D_1} - \varepsilon^* D_2$. For a square hyper dual number matrix $\mathbf{A} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$, if $\mathbf{AA}^* = \mathbf{A}^*\mathbf{A}$, then $\mathbf{A}$ is called normal matrix; if $\mathbf{A} = \mathbf{A}^*$, then $\mathbf{A}$ is called hermitian matrix; if $\mathbf{AA}^* = I_n$, then $\mathbf{A}$ is called unitary matrix. For $\mathbf{B} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$, if $\mathbf{AB} = \mathbf{BA} = I_n$, then $\mathbf{A}$ is called invertible matrix and $\mathbf{B}$ is called the inverse of $\mathbf{A}$.

Considering all this, if D_1 is normal matrix and $D_2^{\mathrm{T}} D_1 - D_1 D_2^{\mathrm{T}} = D_1^* D_2 + D_2 D_1^*$, then $\mathbf{A}$ is normal matrix. If D_1 is unitary matrix and $D_2 D_1^* = D_1 D_2^{\mathrm{T}}$, then $\mathbf{A}$ is unitary matrix. If D_1 is hermitian matrix and $-D_2 = D_2^{\mathrm{T}}$, then $\mathbf{A}$ is hermitian matrix.

Theorem 4. Let $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. If $\mathbf{AB} = I_n$, then $\mathbf{BA} = I_n$.

Proof. Let $\mathbf{AB} = I_n$ for $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathbf{B} = B_1 + B_2\varepsilon_1 + B_3\varepsilon_2 + B_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. From this we get

$$\mathbf{AB} = A_1B_1 + (A_1B_2 + A_2B_1)\varepsilon_1 + (A_1B_3 + A_3B_1)\varepsilon_2 + (A_1B_4 + A_2B_3 + A_4B_1 + A_3B_2)\varepsilon_1\varepsilon_2,$$

so

$$\begin{aligned} &A_1B_1 = I_n, \\ &A_1B_2 + A_2B_1 = I_n, \\ &A_1B_3 + A_3B_1 = I_n, \\ &A_1B_4 + A_2B_3 + A_4B_1 + A_3B_2 = I_n. \end{aligned}$$

Since $A_i, B_j, i, j = 1, \ldots, 4$ are $n \times n$ real matrices, we get

$$\begin{aligned} B_1 &= A_1^{-1}, \\ B_2 &= -A_1^{-1}A_2B_1 = -A_1^{-1}A_2A_1^{-1}, \\ B_3 &= -A_1^{-1}A_3B_1 = -A_1^{-1}A_3A_1^{-1}, \\ B_4 &= -A_1^{-1}(A_2B_3 + A_4B_1 + A_3B_2) = -A_1^{-1}(-A_2A_1^{-1}A_3A_1^{-1} + A_4A_1^{-1} - A_3A_1^{-1}A_2A_1^{-1}) \end{aligned}$$

from real matrix properties. Thus, we obtain

$$\begin{aligned} \mathbf{BA} &= B_1A_1 + (B_2A_1 + B_1A_2)\varepsilon_1 + (B_3A_1 + B_1A_3)\varepsilon_2 \\ &\quad + (B_4A_1 + B_3A_2 + B_1A_4 + B_2A_3)\varepsilon_1\varepsilon_2 \\ &= A_1^{-1}A_1 + (-A_1^{-1}A_2A_1^{-1}A_1 + A_1^{-1}A_2)\varepsilon_1 + (-A_1^{-1}A_3A_1^{-1}A_1 + A_1^{-1}A_3)\varepsilon_2 \\ &\quad + [-A_1^{-1}(-A_2A_1^{-1}A_3A_1^{-1} + A_4A_1^{-1} - A_3A_1^{-1}A_2A_1^{-1})A_1 \\ &\quad - A_1^{-1}A_3A_1^{-1}A_2 + A_1^{-1}A_4 - A_1^{-1}A_2A_1^{-1}A_3]\varepsilon_1\varepsilon_2 \\ &= I_n. \end{aligned}$$

Theorem 5. If the real matrix A_1 is invertible ($\det A_1 \neq 0$), $\mathbf{A}$ is also invertible and the inverse is

$$\mathbf{A}^{-1} = A_1^{-1}\left[\overline{\mathbf{A}} + (A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2)\varepsilon_1\varepsilon_2\right]A_1^{-1}$$

for $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$.

Proof. Let $\mathbf{AB} = I_n$ for $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathbf{B} = B_1 + B_2\varepsilon_1 + B_3\varepsilon_2 + B_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. In this case if we calculate $\mathbf{B}$ when $\mathbf{AB} = I_n$, we get the inverse of $\mathbf{A}$. Since

$$\begin{aligned} \mathbf{AB} &= A_1B_1 + (A_1B_2 + A_2B_1)\varepsilon_1 + (A_1B_3 + A_3B_1)\varepsilon_2 + (A_1B_4 + A_2B_3 + A_4B_1 + A_3B_2)\varepsilon_1\varepsilon_2 \\ &= I_n + 0_n\varepsilon_1 + 0_n\varepsilon_2 + 0_n\varepsilon_1\varepsilon_2 \end{aligned}$$

and

$$A_1B_1 = I_n \Rightarrow B_1 = A_1^{-1},$$

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$A_1B_2 + A_2B_1 = 0_n \Rightarrow B_2 = -A_1^{-1}A_2A_1^{-1}$,
$A_1B_3 + A_3B_1 = 0_n \Rightarrow B_3 = -A_1^{-1}A_3A_1^{-1}$,
$A_1B_4 + A_2B_3 + A_4B_1 + A_3B_2 = 0_n \Rightarrow B_4 = -A_1^{-1}(-A_2A_1^{-1}A_3A_1^{-1} + A_4A_1^{-1} - A_3A_1^{-1}A_2A_1^{-1})$,

we get

$$\mathbf{A}^{-1} = A_1^{-1} - A_1^{-1}A_2A_1^{-1}\varepsilon_1 - A_1^{-1}A_3A_1^{-1}\varepsilon_2 - A_1^{-1}(-A_2A_1^{-1}A_3A_1^{-1} + A_4A_1^{-1} - A_3A_1^{-1}A_2A_1^{-1})\varepsilon_1\varepsilon_2$$

or

$$\mathbf{A}^{-1} = A_1^{-1}\left[\overline{\mathbf{A}} + (A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2)\varepsilon_1\varepsilon_2\right]A_1^{-1}.$$

Example 1. Let $\mathbf{A} = \begin{bmatrix} 2 - \varepsilon_1 + 3\varepsilon_2 & -5\varepsilon_1 + 2\varepsilon_2 - 4\varepsilon_1\varepsilon_2 \\ 3 + \varepsilon_1 - \varepsilon_2 + \varepsilon_1\varepsilon_2 & 1 + 5\varepsilon_2 - 3\varepsilon_1\varepsilon_2 \end{bmatrix}$ be a hyper dual number matrix. Then the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 2 - 14\varepsilon_1 + 3\varepsilon_2 + 28\varepsilon_1\varepsilon_2 & 10\varepsilon_1 - 4\varepsilon_2 + \varepsilon_1\varepsilon_2 \\ -6 + 40\varepsilon_1 + 23\varepsilon_2 - 321\varepsilon_1\varepsilon_2 & 4 - 30\varepsilon_1 - 8\varepsilon_2 + 173\varepsilon_1\varepsilon_2 \end{bmatrix}.$$

We can really multiply $\mathbf{A}$ by the matrix $\mathbf{A}^{-1}$ to see that the result is the identity matrix.

Theorem 6. If the dual matrix D_1 is invertible ($\det D_1 \neq 0$), $\mathbf{A}$ is also invertible and the inverse is

$$\mathbf{A}^{-1} = D_1^{-1}(D_1 - \varepsilon^* D_2)D_1^{-1}$$

for $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$.

Proof. Let $\mathbf{AB} = I_n$ for $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathbf{B} = K_1 + \varepsilon^* K_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. In this case if we calculate $\mathbf{B}$ when $\mathbf{AB} = I_n$, we get the inverse of $\mathbf{A}$. Since

$$\mathbf{AB} = D_1K_1 + D_1K_2\varepsilon^* + D_2K_1\varepsilon^* = I_n + \varepsilon^* 0_n$$

and

$$D_1K_1 = I_n \Rightarrow K_1 = D_1^{-1},$$
$$D_1K_2 + D_2K_1 = 0_n \Rightarrow K_2 = -D_1^{-1}D_2D_1^{-1},$$

we get

$$\mathbf{A}^{-1} = D_1^{-1}(D_1 - \varepsilon^* D_2)D_1^{-1}.$$

Example 2. Let $\mathbf{A} = \begin{bmatrix} 1 + \varepsilon - \varepsilon^* & 3 + \varepsilon\varepsilon^* \\ 2\varepsilon^* + \varepsilon\varepsilon^* - 2 & 1 + \varepsilon^* \end{bmatrix}$ be a hyper dual number matrix. Hyper dual number matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = D_1 + \varepsilon^* D_2 = \begin{bmatrix} 1 + \varepsilon & 3 \\ -2 & 1 \end{bmatrix} + \varepsilon^* \begin{bmatrix} -1 & \varepsilon \\ 2 + \varepsilon & 1 \end{bmatrix}.$$

If $D_1 - \varepsilon^* D_2$ and D_1^{-1} are calculated, it is found respectively as follows:

$$D_1 - \varepsilon^* D_2 = \begin{bmatrix} 1 + \varepsilon + \varepsilon^* & 3 - \varepsilon\varepsilon^* \\ -2 - 2\varepsilon^* - \varepsilon\varepsilon^* & 1 - \varepsilon^* \end{bmatrix}$$

and

$$D_1^{-1} = \frac{1}{49}\begin{bmatrix} 7 - \varepsilon & 3(\varepsilon - 7) \\ 14 - 2\varepsilon & 7 + 6\varepsilon \end{bmatrix}.$$

Therefore, the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = D_1^{-1}(D_1 - \varepsilon^* D_2)D_1^{-1} = \frac{1}{49}\begin{bmatrix} 7 - \varepsilon + 13\varepsilon^* + \frac{19}{7}\varepsilon\varepsilon^* & 3\varepsilon - 18\varepsilon^* - \frac{13}{7}\varepsilon\varepsilon^* - 21 \\ 14 - 2\varepsilon - 2\varepsilon^* - \frac{59}{7}\varepsilon\varepsilon^* & 7 + 6\varepsilon - \varepsilon^* + \frac{37}{7}\varepsilon\varepsilon^* \end{bmatrix}.$$

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We can really multiply $\mathbf{A}$ by the matrix $\mathbf{A}^{-1}$ to see that the result is the identity matrix.

Definition 1. A hyper dual number matrix that exists inversely is called a hyper dual regular matrix. Otherwise, it is called hyper dual singular matrix.

Theorem 7. Let $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathbf{x} \in \mathbb{HDN}$. Then the following are satisfied:
1. $\left(\overline{\mathbf{A}}\right)^{\mathrm{T}} = \overline{(\mathbf{A}^{\mathrm{T}})}$;
2. $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$ and $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$;
3. If $A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2 = 0$, then $\left(\overline{\mathbf{A}}\right)^{-1} = \overline{(\mathbf{A}^{-1})}$;
4. $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} + \mathbf{A}^{\mathrm{T}}$;
5. $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$;
6. If $A_2B_3 = -A_3B_2$, then $\overline{\mathbf{AB}} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$;
7. If A_1 and B_1 are invertible, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$;
8. If A_1 is invertible, then $(\mathbf{A}^*)^{-1} = (\mathbf{A}^{-1})^*$.

Proof. We prove a few of these properties and others can be proved similarly.
1. Let $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. Then

$$\begin{aligned}\left(\overline{\mathbf{A}}\right)^{\mathrm{T}} &= (A_1 - A_2\varepsilon_1 - A_3\varepsilon_2 - A_4\varepsilon_1\varepsilon_2)^{\mathrm{T}}\\ &= A_1^{\mathrm{T}} - A_2^{\mathrm{T}}\varepsilon_1 - A_3^{\mathrm{T}}\varepsilon_2 - A_4^{\mathrm{T}}\varepsilon_1\varepsilon_2\\ &= \overline{(\mathbf{A}^{\mathrm{T}})}.\end{aligned}$$

3. Let $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2 = 0$. Then

$$\begin{aligned}\left(\overline{\mathbf{A}}\right)^{-1} &= (A_1 - A_2\varepsilon_1 - A_3\varepsilon_2 - A_4\varepsilon_1\varepsilon_2)^{-1}\\ &= A_1^{-1}[\mathbf{A} + (A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2)\varepsilon_1\varepsilon_2]A_1^{-1}\\ &= A_1^{-1}\mathbf{A}A_1^{-1}.\end{aligned}$$

On the other hand, for $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2 = 0$, we can obtain

$$\begin{aligned}\overline{(\mathbf{A}^{-1})} &= \overline{\left[A_1^{-1}[\overline{\mathbf{A}} + (A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2)\varepsilon_1\varepsilon_2]A_1^{-1}\right]}\\ &= A_1^{-1}\overline{\overline{\mathbf{A}}}A_1^{-1} = A_1^{-1}\mathbf{A}A_1^{-1}.\end{aligned}$$

Thus, we get

$$\left(\overline{\mathbf{A}}\right)^{-1} = \overline{(\mathbf{A}^{-1})} \text{ if } A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2 = 0.$$

5. Let $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathbf{B} = B_1 + B_2\varepsilon_1 + B_3\varepsilon_2 + B_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. Then

$$\begin{aligned}(\mathbf{AB})^* &= [A_1B_1 + (A_1B_2 + A_2B_1)\varepsilon_1 + (A_1B_3 + A_3B_1)\varepsilon_2 + (A_1B_4 + A_2B_3 + A_4B_1 + A_3B_2)\varepsilon_1\varepsilon_2]^*\\ &= [A_1B_1 - (A_1B_2 + A_2B_1)\varepsilon_1 - (A_1B_3 + A_3B_1)\varepsilon_2 - (A_1B_4 + A_2B_3 + A_4B_1 + A_3B_2)\varepsilon_1\varepsilon_2]^{\mathrm{T}}\\ &= B_1^{\mathrm{T}}A_1^{\mathrm{T}} - (B_2^{\mathrm{T}}A_1^{\mathrm{T}} + B_1^{\mathrm{T}}A_2^{\mathrm{T}})\varepsilon_1 - (B_3^{\mathrm{T}}A_1^{\mathrm{T}} + B_1^{\mathrm{T}}A_3^{\mathrm{T}})\varepsilon_2 - (B_4^{\mathrm{T}}A_1^{\mathrm{T}} + B_3^{\mathrm{T}}A_2^{\mathrm{T}} + B_1^{\mathrm{T}}A_4^{\mathrm{T}} + B_2^{\mathrm{T}}A_3^{\mathrm{T}})\varepsilon_1\varepsilon_2\\ &= (B_1^{\mathrm{T}} - B_2^{\mathrm{T}}\varepsilon_1 - B_3^{\mathrm{T}}\varepsilon_2 - B_4^{\mathrm{T}}\varepsilon_1\varepsilon_2)(A_1^{\mathrm{T}} - A_2^{\mathrm{T}}\varepsilon_1 - A_3^{\mathrm{T}}\varepsilon_2 - A_4^{\mathrm{T}}\varepsilon_1\varepsilon_2)\\ &= \mathbf{B}^*\mathbf{A}^*.\end{aligned}$$

8. Let $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and A_1 is invertible. Then

$$\begin{aligned}(\mathbf{A}^*)^{-1} &= (A_1^{\mathrm{T}} - A_2^{\mathrm{T}}\varepsilon_1 - A_3^{\mathrm{T}}\varepsilon_2 - A_4^{\mathrm{T}}\varepsilon_1\varepsilon_2)^{-1}\\ &= (A_1^{\mathrm{T}})^{-1}[\mathbf{A}^{\mathrm{T}} + (A_2^{\mathrm{T}}(A_1^{\mathrm{T}})^{-1}A_3^{\mathrm{T}} + A_3^{\mathrm{T}}(A_1^{\mathrm{T}})^{-1}A_2^{\mathrm{T}})\varepsilon_1\varepsilon_2](A_1^{\mathrm{T}})^{-1}\\ &= (A_1^{-1})^{\mathrm{T}}[\mathbf{A}^{\mathrm{T}} + (A_2^{\mathrm{T}}(A_1^{-1})^{\mathrm{T}}A_3^{\mathrm{T}} + A_3^{\mathrm{T}}(A_1^{-1})^{\mathrm{T}}A_2^{\mathrm{T}})\varepsilon_1\varepsilon_2](A_1^{-1})^{\mathrm{T}}\\ &= [A_1^{-1}[\mathbf{A} + (A_3^{\mathrm{T}}A_1^{-1}A_2^{\mathrm{T}} + A_2^{\mathrm{T}}A_1^{-1}A_3^{\mathrm{T}})\varepsilon_1\varepsilon_2]A_1^{-1}]^{\mathrm{T}}\end{aligned}$$

$$= \left[A_1^{-1}\left[\overline{\mathbf{A}} + (A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2)\varepsilon_1\varepsilon_2\right]A_1^{-1}\right]^* = (\mathbf{A}^{-1})^*.$$

Real Matrix Representation of Hyper Dual Numbers

Let $\mathbf{A} = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ be a hyper dual matrix. We define the linear map $\mathcal{R}_{\mathbf{A}}$ by

$$\mathcal{R}_{\mathbf{A}} : \mathbb{M}_{n\times n}(\mathbb{HDN}) \longrightarrow \mathbb{M}_{n\times n}(\mathbb{HDN})$$
$$\mathcal{R}_{\mathbf{A}}(\mathbf{B}) = \mathbf{AB}.$$

Using this operator and the basis $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$ of the vector space $\mathbb{M}_{n\times n}(\mathbb{HDN})$, we can write

$$\begin{aligned}\mathcal{R}_{\mathbf{A}}(1) &= \mathbf{A}1 = A_1 + A_2\varepsilon_1 + A_3\varepsilon_2 + A_4\varepsilon_1\varepsilon_2,\\ \mathcal{R}_{\mathbf{A}}(\varepsilon_1) &= \mathbf{A}\varepsilon_1 = A_1\varepsilon_1 + A_3\varepsilon_1\varepsilon_2,\\ \mathcal{R}_{\mathbf{A}}(\varepsilon_2) &= \mathbf{A}\varepsilon_2 = A_1\varepsilon_2 + A_2\varepsilon_1\varepsilon_2,\\ \mathcal{R}_{\mathbf{A}}(\varepsilon_1\varepsilon_2) &= \mathbf{A}\varepsilon_1\varepsilon_2 = A_1\varepsilon_1\varepsilon_2.\end{aligned}$$

Then the following real matrix representation can be found as

$$\mathcal{R}_{\mathbf{A}} = \begin{bmatrix} A_1 & 0_n & 0_n & 0_n \\ A_2 & A_1 & 0_n & 0_n \\ A_3 & 0_n & A_1 & 0_n \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix}_{4n\times 4n}.$$

Furthermore, $\det\mathcal{R}_{\mathbf{A}} = (\det A_1)^4$.

Example 3. The real matrix representations of $\mathbf{1}, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2$ are

$$\mathcal{R}_{\mathbf{1}} = \begin{bmatrix} I_n & 0_n & 0_n & 0_n \\ 0_n & I_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \\ 0_n & 0_n & 0_n & I_n \end{bmatrix} = I_{4n} \qquad \mathcal{R}_{\boldsymbol{\varepsilon}_1} = \begin{bmatrix} 0_n & 0_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & I_n & 0_n \end{bmatrix}_{4n\times 4n}$$

$$\mathcal{R}_{\boldsymbol{\varepsilon}_2} = \begin{bmatrix} 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \\ 0_n & I_n & 0_n & 0_n \end{bmatrix}_{4n\times 4n} \qquad \mathcal{R}_{\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2} = \begin{bmatrix} 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \\ I_n & 0_n & 0_n & 0_n \end{bmatrix}_{4n\times 4n}$$

where $\mathbf{1}, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathcal{R}_{\mathbf{1}}, \mathcal{R}_{\boldsymbol{\varepsilon}_1}, \mathcal{R}_{\boldsymbol{\varepsilon}_2}, \mathcal{R}_{\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. Furthermore, these real representation matrices satisfy:

$$\begin{aligned}\mathcal{R}_{\mathbf{1}}^2 &= I_{4n},\\ \mathcal{R}_{\boldsymbol{\varepsilon}_1}^2 &= \mathcal{R}_{\boldsymbol{\varepsilon}_2}^2 = \mathcal{R}_{\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2}^2 = 0,\\ \mathcal{R}_{\boldsymbol{\varepsilon}_1}\mathcal{R}_{\boldsymbol{\varepsilon}_2} &= \mathcal{R}_{\boldsymbol{\varepsilon}_2}\mathcal{R}_{\boldsymbol{\varepsilon}_1} = \mathcal{R}_{\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2}.\end{aligned}$$

The inverse of the real matrix representation of hyper dual numbers is found as

$$\mathcal{R}_{\mathbf{A}}^{-1} = \begin{bmatrix} A_1^{-1} & 0_n & 0_n & 0_n \\ -A_1^{-1}A_2A_1^{-1} & A_1^{-1} & 0_n & 0_n \\ -A_1^{-1}A_3A_1^{-1} & 0_n & A_1^{-1} & 0_n \\ A_1^{-1}\left(A_2A_1^{-1}A_3 + A_3A_1^{-1}A_2 - A_4\right)A_1^{-1} & -A_1^{-1}A_3A_1^{-1} & -A_1^{-1}A_2A_1^{-1} & A_1^{-1} \end{bmatrix}.$$

Example 4. Let $\mathbf{A} = \begin{bmatrix} 2 + \varepsilon_1 - 3\varepsilon_1\varepsilon_2 & -1 + 3\varepsilon_1 + 5\varepsilon_2 - 2\varepsilon_1\varepsilon_2 \\ -5 - 2\varepsilon_1 - 4\varepsilon_2 + \varepsilon_1\varepsilon_2 & 3 + \varepsilon_1 - \varepsilon_2 + 4\varepsilon_1\varepsilon_2 \end{bmatrix}$ be a hyper dual number matrix. Then the real matrix representation of $\mathbf{A}$ is

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$$\mathcal{R}_{\mathbf{A}}=\begin{bmatrix}2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ -5 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 3 & 2 & -1 & 0 & 0 & 0 & 0\\ -2 & 1 & -5 & 3 & 0 & 0 & 0 & 0\\ 0 & 5 & 0 & 0 & 2 & -1 & 0 & 0\\ -4 & -1 & 0 & 0 & -5 & 3 & 0 & 0\\ -3 & -2 & 0 & 5 & 1 & 3 & 2 & -1\\ 1 & 4 & -4 & -1 & -2 & 1 & -5 & 3\end{bmatrix}$$

and the inverse of the real matrix representation of **A** is

$$\mathcal{R}_{\mathbf{A}}^{-1}=\begin{bmatrix}3 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\ -53 & -21 & 3 & 1 & 0 & 0 & 0 & 0\\ -88 & -35 & 5 & 2 & 0 & 0 & 0 & 0\\ -58 & -24 & 0 & 0 & 3 & 1 & 0 & 0\\ -91 & -38 & 0 & 0 & 5 & 2 & 0 & 0\\ 2022 & 822 & -58 & -24 & -53 & -21 & 3 & 1\\ 3254 & 1324 & -91 & -38 & -88 & -35 & 5 & 2\end{bmatrix}.$$

Therefore, the inverse of **A** is

$$\begin{aligned}\mathbf{A}^{-1}&=\begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}+\varepsilon_1\begin{bmatrix}-53 & -21\\ -88 & -35\end{bmatrix}+\varepsilon_2\begin{bmatrix}-58 & -24\\ -91 & -38\end{bmatrix}+\varepsilon_1\varepsilon_2\begin{bmatrix}2022 & 822\\ 3254 & 1324\end{bmatrix}\\ &=\begin{bmatrix}3-53\varepsilon_1-58\varepsilon_2+2022\varepsilon_1\varepsilon_2 & 1-21\varepsilon_1-24\varepsilon_2+822\varepsilon_1\varepsilon_2\\ 5-88\varepsilon_1-91\varepsilon_2+3254\varepsilon_1\varepsilon_2 & 2-35\varepsilon_1-38\varepsilon_2+1324\varepsilon_1\varepsilon_2\end{bmatrix}.\end{aligned}$$

Also, $\det\mathcal{R}_{\mathbf{A}}=(\det A_1)^4=1$.

DUAL ADJOINT MATRIX OF HYPER DUAL NUMBER MATRICES

In this section we define the dual adjoint matrix of a hyper dual number matrix. Next, we give some relations between hyper dual number matrices and their dual adjoint matrices and explain these relations with example.

Definition 2. Let $\mathbf{A}=D_1+\varepsilon^*D_2\in\mathbb{M}_{n\times n}(\mathbb{HDN})$, where D_1 and D_2 are dual matrices. We define the $2n\times 2n$ dual matrices **A** as

$$\mathcal{X}_{\mathbf{A}}=\begin{bmatrix}D_1 & 0_n\\ D_2 & D_1\end{bmatrix}.$$

This matrix $\mathcal{X}_{\mathbf{A}}$ is called the dual adjoint matrix of the hyper dual number matrix **A**.

Theorem 8. Let $\mathbf{A},\mathbf{B}\in\mathbb{M}_{n\times n}(\mathbb{HDN})$ be a hyper dual regular matrix. Then the following hold:

1. $\mathcal{X}_{I_n}=I_{2n}$;

2. $\mathcal{X}_{\mathbf{A+B}}=\mathcal{X}_{\mathbf{A}}+\mathcal{X}_{\mathbf{B}}$;

3. $\mathcal{X}_{\mathbf{AB}}=\mathcal{X}_{\mathbf{A}}\mathcal{X}_{\mathbf{B}}$;

4. $\mathcal{X}_{\mathbf{A}^{-1}}=(\mathcal{X}_{\mathbf{A}})^{-1}$ if D_1 exists;

5. $\mathcal{X}_{\mathbf{A}^*}=(\mathcal{X}_{\mathbf{A}})^*$ if $D_2=0$.

Proof. The proof can be shown easily.

Definition 3. Let $\mathbf{A}\in\mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\boldsymbol{\lambda}\in\mathbb{HDN}$. If $\boldsymbol{\lambda}$ holds in the equation $\mathbf{Ax}=\boldsymbol{\lambda}\mathbf{x}$ for some non-zero hyper dual number column vector **x**, then $\boldsymbol{\lambda}$ is called the eigenvalue of **A**. The set of the eigenvalues

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$$\sigma(\mathbf{A}) = \{\boldsymbol{\lambda} \in \mathbb{HDN} : \mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}, \text{for some } \mathbf{x} \neq 0\}$$

is called spectrum of **A**.

Since the characteristic polynomial of matrix $\mathcal{X}_{\mathbf{A}}$ is equal to the characteristic polynomial of matrix $(\mathcal{X}_{\mathbf{A}})^{\mathrm{T}}$, it can be clearly seen that $\sigma(\mathcal{X}_{\mathbf{A}}) = \sigma((\mathcal{X}_{\mathbf{A}})^{\mathrm{T}})$.

Theorem 9. Let $\mathbf{A} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. If $\sigma(\mathbf{A}) \cap \mathbb{D} \neq \emptyset$, then

$$\sigma(\mathbf{A}) \cap \mathbb{D} = \sigma(\mathcal{X}_{\mathbf{A}}),$$

where $\sigma(\mathcal{X}_{\mathbf{A}}) = \{\boldsymbol{\mu} \in \mathbb{D} : \mathcal{X}_{\mathbf{A}}\mathbf{y} = \boldsymbol{\mu}\mathbf{y}, \text{for some } \mathbf{y} \neq 0\}$ is spectrum of the dual adjoint matrix of **A**.

Proof. Let $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$, where $D_1, D_2 \in \mathbb{D}$ and $\boldsymbol{\lambda} \in \mathbb{D}$ are eigenvalues of **A**. That is, let $\sigma(\mathbf{A}) \cap \mathbb{D} \neq \emptyset$. Therefore, there exists a non-zero column vector $\mathbf{x} = d_1 + \varepsilon^* d_2$, where d_1, d_2 are dual column vectors such that $\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$. This implies

$$(D_1 + \varepsilon^* D_2)(d_1 + \varepsilon^* d_2) = \boldsymbol{\lambda}(d_1 + \varepsilon^* d_2),$$
$$D_1 d_1 + \varepsilon^* D_1 d_2 + \varepsilon^* D_2 d_1 = \boldsymbol{\lambda} d_1 + \varepsilon^* \boldsymbol{\lambda} d_2,$$
$$D_1 d_1 + \varepsilon^* (D_1 d_2 + D_2 d_1) = \boldsymbol{\lambda} d_1 + \varepsilon^* \boldsymbol{\lambda} d_2.$$

Thus, we obtain the following equations:

$$\begin{cases} D_1 d_1 = \boldsymbol{\lambda} d_1, \\ D_1 d_2 + D_2 d_1 = \boldsymbol{\lambda} d_2. \end{cases}$$

Using these equations, we find

$$\begin{bmatrix} D_1 & 0_n \\ D_2 & D_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \boldsymbol{\lambda} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

Therefore, dual eigenvalue of the hyper dual number matrix A is equivalent to the eigenvalue of the dual adjoint matrix $\mathcal{X}_{\mathbf{A}}$, that is

$$\sigma(\mathbf{A}) \cap \mathbb{D} = \sigma(\mathcal{X}_{\mathbf{A}}).$$

Example 5. Let $\mathbf{A} = \begin{bmatrix} 3 - 2\varepsilon & 1 + 2\varepsilon - 2\varepsilon^* - \varepsilon\varepsilon^* \\ 1 + 2\varepsilon + 2\varepsilon^* + \varepsilon\varepsilon^* & 3 + 2\varepsilon \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{HDN})$. Then dual adjoint matrix of **A** is

$$\mathcal{X}_{\mathbf{A}} = \begin{bmatrix} 3 - 2\varepsilon & 1 + 2\varepsilon & 0 & 0 \\ 1 + 2\varepsilon & 3 + 2\varepsilon & 0 & 0 \\ 0 & -2 - \varepsilon & 3 - 2\varepsilon & 1 + 2\varepsilon \\ 2 + \varepsilon & 0 & 1 + 2\varepsilon & 3 + 2\varepsilon \end{bmatrix}.$$

By theorem 9, the dual eigenvalues of **A** are equivalent to the eigenvalues of $\mathcal{X}_{\mathbf{A}}$, and the set of these eigenvalues is

$$\sigma(\mathcal{X}_{\mathbf{A}}) = \sigma(\mathbf{A}) \cap \mathbb{D} = \{3 + \sqrt{1 + 4\varepsilon}, 3 - \sqrt{1 + 4\varepsilon}\}.$$

Definition 4. Let $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathcal{X}_{\mathbf{A}}$ be the dual adjoint matrix of **A**. We define the determinant of **A** and $\mathcal{X}_{\mathbf{A}}$ as follows:

$$\det\mathbf{A} = (\det D_1)\left(1 + \varepsilon^* \mathrm{tr}(D_1^{-1} D_2)\right) \quad \text{and} \quad \det\mathcal{X}_{\mathbf{A}} = (\det D_1)^2,$$

where $\mathrm{tr}(D_1^{-1} D_2)$ is the trace of $D_1^{-1} D_2$.

Theorem 10. Let $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. The following are equivalent:

1. **A** is invertible;

2. $\mathbf{Ax} = 0$ has a unique solution 0;

3. $\det\mathcal{X}_{\mathbf{A}} \neq 0$, i.e. $\mathcal{X}_{\mathbf{A}}$ is invertible;

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4. $\mathbf{A}$ has no zero eigenvalue. More precisely, if $\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$ for some hyper dual number $\boldsymbol{\lambda}$ and some hyper dual number vector $\mathbf{x} \neq 0$, then $\boldsymbol{\lambda} \neq 0$.

Proof. $(1) \Rightarrow (2)$: This part is clear.
$(2) \Rightarrow (3)$: Let $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$, $\mathbf{x} = d_1 + \varepsilon^* d_2$, where D_1, D_2 are dual matrices and d_1, d_2 are dual column vectors. Then

$$\begin{aligned}\mathbf{Ax} &= (D_1 + \varepsilon^* D_2)(d_1 + \varepsilon^* d_2)\\ &= D_1 d_1 + \varepsilon^* D_1 d_2 + \varepsilon^* D_2 d_1\\ &= D_1 d_1 + \varepsilon^*(D_1 d_2 + D_2 d_1)\end{aligned}$$

Since $\mathbf{Ax} = 0$, we get

$$D_1 d_1 = 0$$

and

$$D_1 d_2 + D_2 d_1 = 0.$$

So we have

$$\mathbf{Ax} = 0 \text{ if and only if } \begin{bmatrix} D_1 & 0_n \\ D_2 & D_1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0.$$

From here, $\mathcal{X}_{\mathbf{A}}(d_1, d_2)^{\mathrm{T}} = 0$. Since $\mathbf{Ax} = 0$ has a unique solution, $\mathcal{X}_{\mathbf{A}}(d_1, d_2)^{\mathrm{T}} = 0$ has a unique solution. Therefore, since $\mathcal{X}_{\mathbf{A}}$ is a dual matrix, $\mathcal{X}_{\mathbf{A}}$ is invertible.
$(2) \Leftrightarrow (4)$: Let $\mathbf{x} = 0$ be a unique solution of $\mathbf{Ax} = 0$ for $\mathbf{A} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. Suppose $\mathbf{A}$ has a zero eigenvalue. Then for some hyper dual number vector $\mathbf{x} \neq 0$, the equation $\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$ has a zero eigenvalue. So $\mathbf{Ax} = 0$, and this is a contradiction to our assumption of $\mathbf{x} = 0$. Now suppose that $\mathbf{A}$ has no eigenvalue of zero. So if $\mathbf{Ax} = 0 = \boldsymbol{\lambda}\mathbf{x}$, then by our assumption, $\mathbf{x} = 0$.
$(3) \Rightarrow (1)$: Let $\mathcal{X}_{\mathbf{A}}$ be invertible. So because $\mathcal{X}_{\mathbf{A}}$ is invertible, there is a dual matrix

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

for $\mathbf{A} = D_1 + \varepsilon^* D_2$ such that

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}\begin{bmatrix} D_1 & 0_n \\ D_2 & D_1 \end{bmatrix} = \begin{bmatrix} I_n & 0_n \\ 0_n & I_n \end{bmatrix}.$$

From the last matrix equation, we can write

$$K_3 D_1 + K_4 D_2 = 0_n \quad \text{and} \quad K_4 D_1 = I_n.$$

Using these equations, we get

$$K_4 D_1 + \varepsilon^*(K_3 D_1 + K_4 D_2) = I_n.$$

That is

$$\mathbf{BA} = I_n$$

for $\mathbf{B} = K_4 + \varepsilon^* K_3$. So $\mathbf{A}$ is an invertible hyper dual number matrix by Theorem 4.

Theorem 11. Let $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{n\times n}(\mathbb{HDN})$. Then
1. $\mathbf{A}$ is invertible $\Leftrightarrow$ $\det\mathbf{A} \neq 0$;
2. $\det(\mathbf{AB}) = \det\mathbf{A}\det\mathbf{B}$;
3. $(\det\mathbf{A})^n = (\det D_1)^n\left(1 + n\varepsilon^* \mathrm{tr}(D_1^{-1}D_2)\right)$, where $\mathrm{tr}(D_1^{-1}D_2) = \mathrm{trace}(D_1^{-1}D_2)$.

Proof. Let $\mathbf{A} = D_1 + \varepsilon^* D_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$ and $\mathbf{B} = K_1 + \varepsilon^* K_2 \in \mathbb{M}_{n\times n}(\mathbb{HDN})$.
1. It is clear from theorem 10.
2. Since

$$\mathbf{AB} = D_1 K_1 + \varepsilon^*(D_1 K_2 + D_2 K_1),$$

we get

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$$
\begin{aligned}
\det(\mathbf{AB}) &= \det(D_1K_1)\left(1+\varepsilon^*\mathrm{tr}(K_1^{-1}K_2+K_1^{-1}D_1^{-1}D_2K_1)\right)\\
&= \det D_1 \det K_1\left(1+\varepsilon^*\mathrm{tr}(K_1^{-1}K_2)+\varepsilon^*\mathrm{tr}(K_1^{-1}D_1^{-1}D_2K_1)\right)\\
&= \det D_1 \det K_1\left(1+\varepsilon^*\mathrm{tr}(K_1^{-1}K_2)+\varepsilon^*\mathrm{tr}(D_1^{-1}D_2)\right)\\
&= \det D_1 \det K_1\left(1+\varepsilon^*\mathrm{tr}(K_1^{-1}K_2)\right)\left(1+\varepsilon^*\mathrm{tr}(D_1^{-1}D_2)\right)\\
&= \det D_1\left(1+\varepsilon^*\mathrm{tr}(K_1^{-1}K_2)\right)\det K_1\left(1+\varepsilon^*\mathrm{tr}(D_1^{-1}D_2)\right)\\
&= \det\mathbf{A}\det\mathbf{B}.
\end{aligned}
$$

3. For $x \in \mathbb{D}$ and $n \in \mathbb{N}$, it is clear that $(1+x\varepsilon^*)^n = 1+nx\varepsilon^*$. Therefore, if $x = \mathrm{tr}(D_1^{-1}D_2)$ is taken, it is seen that

$$
\begin{aligned}
(\det\mathbf{A})^n &= (\det D_1)^n\left(1+\varepsilon^*\mathrm{tr}(D_1^{-1}D_2)\right)^n\\
&= (\det D_1)^n\left(1+n\varepsilon^*\mathrm{tr}(D_1^{-1}D_2)\right).
\end{aligned}
$$

CONCLUSIONS

In this study we have given hyper dual number matrices. Due to the fact that hyper dual number matrices have dual number matrix representation, the investigation of this matrix algebra has brought some advantages. Defining concepts such as determinant, inverse, conjugate and transpose of hyper dual number matrices in terms of dual matrix are some of these advantages. In addition, it was interesting to give the special types of these matrices in terms of some properties provided by dual matrices. Finally, the concept of dual adjoint matrix was defined by the block dual matrix. Thus, properties related to eigenvalue, eigenvector and linear equation systems could be mentioned. It is also very important in terms of practice to find the n^{th}-order power of the determinant of a matrix with the help of adjoint matrix. For further study, by considering the present approach, one can carry out a similar study for more involved structures.

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