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Full Paper

Complex-type Narayana sequence and its application

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Abstract: We define the complex-type Narayana sequence and give miscellaneous properties of this sequence by using the matrix method. Also, we study the complex-type Narayana sequence modulo m. In addition, we describe the complex-type Narayana sequence in a 3-generator group and investigate the sequence in finite groups in detail. Finally, we give the lengths of the periods of the complex-type Narayana sequences in polyhedral groups $(n,2,2)$, $(2,n,2)$ and $(2,2,n)$ with respect to the generating triple (x,y,z) as application of the results produced.

Keywords: complex-type Narayana sequence, recurrence sequences, matrix method, generator group

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INTRODUCTION

Recurrence sequences are widely utilised to solve some problems in various scientific fields. In the literature many interesting properties and applications of the recurrence sequences relevant to this paper have been studied by many authors [1-5]. Especially, the authors defined new sequences using quaternions and complex numbers and gave miscellaneous properties and many applications of the sequences defined [6-11]. In the second part of this paper, firstly a new sequence, the so-called complex-type Narayana sequence, is defined, and by the aid of the matrix method, miscellaneous properties of the sequence are obtained.

Wall [12] started research on linear recurrence sequences modulo m by investigating the periods of ordinary Fibonacci sequences modulo m. Recently, the theory was extended to some special linear recurrence sequences [13, 14]. In this sense, we consider the complex-type Narayana sequence modulo m and derive some interesting results on the periods of the complex-type Narayana sequence for any m. Also, we produce cyclic groups using multiplicative orders of generating matrices of complex-type Narayana number when reading modulo m. Then we give connections between periods of complex-type Narayana sequence modulo m and orders of the cyclic groups produced.

The study of recurrence sequences in groups began with the work of Wall [12]. Later, Wilcox [15] studied Fibonacci sequences in abelian groups. The theory was expanded to some finite simple groups by Campbell et al. [16]. They defined Fibonacci length of the Fibonacci orbit and basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group. The concept of Fibonacci length for more than two generators has also been considered [17, 18]. Knox [19] signified that a k-nacci (k-step Fibonacci) sequence in a finite group is periodic. Many properties of recurrence sequences in algebraic structures have been studied and the concept extended to complex numbers and quaternions [20-29], after which a variety of properties and numerous applications for the sequences developed were provided [7, 8, 30]. In the third part of this paper we give the definition of the complex-type Narayana sequence in 3-generator groups and investigate these sequences in finite groups in detail. Finally, we obtain periods of the sequence in polyhedral groups $(n,2,2), (2,n,2), (2,2,n)$ as application of the results produced.

The Narayana sequence $\{N_n\}$ is defined [31] by third-order linear, homogeneous recurrence relation:

$$N_{n+1} = N_n + N_{n-2}$$

for $n \geq 2$ and with initial conditions $N_0 = 0$, $N_1 = 1$ and $N_2 = 1$. The complex Fibonacci sequence $\{F_n^*\}$ is given [32] for $n \geq 0$:

$$F_n^* = F_n + iF_{n+1}$$

where $i = \sqrt{-1}$ is an imaginary unit and F_n is n^{th} Fibonacci number (cf. [33, 34]).

Suppose that $\{c_j\}_{j=0}^{k-1}$, $(k \geq 2)$ is a sequence of real numbers such that $c_{k-1} \neq 0$. The k-generalised Fibonacci sequence $\{a_n\}_{n=0}^{+\infty}$ is defined as

$$a_{n+k} = c_{k-1}a_{n+k-1} + c_{k-2}a_{n+k-2} + \cdots + c_0 a_n$$

for $n \geq 0$ and where $a_0, a_1, \ldots, a_{k-1}$ are specified by the initial conditions.

Number sequences can be derived from a matrix representation, as demonstrated by Kalman [35] who, by using the companion matrix method, arrived at the following closed-form formulas for the generalised sequence:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he proved that

$$\left(A_k\right)^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

We recall when a sequence is composed only of repetition of a fixed subsequence. A sequence is periodic if after a certain point it consists only of repetition of a fixed subsequence. We refer to the number of members in the shortest repeating subsequence as the period of the sequence. For instance, when a sequence with the terms $x, y, z, t, y, z, t, y, z, t, \ldots$ is considered, one would say it is periodic after the initial term k and it has period 3. Also, when the first r terms in a sequence form a repeating subsequence, then it is said to be simply periodic with period r. For instance, when a sequence with the terms $x, y, z, t, x, y, z, t, x, y, z, t, \ldots$ is considered, one would say it is simply periodic with period 4.

For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_u = a_{u+1}$, $0 \leq u \leq n-1$, $x_{n+u} = \prod_{v=1}^{n} x_{u+v-1}$, $u \geq 0$ is called the Fibonacci orbit of G with respect to the generating set A, denoted as $F_A(G)$ [18].

A k-nacci (k-step Fibonacci) sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ in which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \ldots x_{n-1} & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \ldots x_{n-1} & \text{for } n \geq k. \end{cases}$$

We also require that the initial elements of the sequence x_0, x_1, x_2, $\ldots$, x_{j-1} generate a group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group G generated by $x_0, x_1, x_2, \ldots, x_{j-1}$, is denoted by $F_k\left(G; x_0, x_1, x_2, \ldots, x_{j-1}\right)$ [19]. Note also that the orbit of a k-generated group is a k-nacci sequence.

COMPLEX-TYPE NARAYANA SEQUENCE

Now we define complex-type Narayana sequence by the following homogeneous linear recurrence relation:

$$\grave{u}_{n+3} = i \cdot \grave{u}_{n+2} - i \cdot \grave{u}_{n} \tag{1}$$

for $n \geq 0$, where $\grave{u}_0 = 0$, $\grave{u}_1 = 1$, $\grave{u}_2 = 1$ and $i = \sqrt{-1}$.

By definition of complex-type Narayana numbers, we can write the following vector recurrence relation:

$$\begin{bmatrix} \grave{u}_{n+3} \\ \grave{u}_{n+2} \\ \grave{u}_{n+1} \end{bmatrix} = N \cdot \begin{bmatrix} \grave{u}_{n+2} \\ \grave{u}_{n+1} \\ \grave{u}_{n} \end{bmatrix}, \tag{2}$$

where N is a companion matrix:

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$$N=\begin{bmatrix} i & 0 & -i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Matrix N is said to be a complex-type Narayana matrix.

By mathematical induction on n, we find the relationship between the elements of the sequence $\{\grave{u}_n\}$ and the matrix N as follows.

i. If n is odd

$$(N)^n=\begin{bmatrix} \operatorname{Im}(\grave{u}_{n+2}) & \operatorname{Re}(\grave{u}_{n+2}) & \operatorname{Im}(\grave{u}_{n+3}) \\ \operatorname{Re}(\grave{u}_{n+1}) & \operatorname{Im}(\grave{u}_{n+1}) & \operatorname{Re}(\grave{u}_{n+2}) \\ \operatorname{Im}(\grave{u}_{n}) & \operatorname{Re}(\grave{u}_{n}) & \operatorname{Im}(\grave{u}_{n+1}) \end{bmatrix}. \tag{3}$$

ii. If n is even

$$(N)^n=\begin{bmatrix} \operatorname{Re}(\grave{u}_{n+2}) & \operatorname{Im}(\grave{u}_{n+2}) & \operatorname{Re}(\grave{u}_{n+3}) \\ \operatorname{Im}(\grave{u}_{n+1}) & \operatorname{Re}(\grave{u}_{n+1}) & \operatorname{Im}(\grave{u}_{n+2}) \\ \operatorname{Re}(\grave{u}_{n}) & \operatorname{Im}(\grave{u}_{n}) & \operatorname{Re}(\grave{u}_{n+1}) \end{bmatrix}. \tag{4}$$

Generalising Eq. (2), we derive

$$\begin{bmatrix} i & 0 & -i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} \grave{u}_{n+2} \\ \grave{u}_{n+1} \\ \grave{u}_{n} \end{bmatrix} = \begin{bmatrix} \grave{u}_{2n+2} \\ \grave{u}_{2n+1} \\ \grave{u}_{2n} \end{bmatrix}.$$

Now we consider permanental representations for complex-type Narayana numbers.

Definition 1. A $u\times v$ real matrix $M=\left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if it contains exactly two non-zero entries.

Suppose that $\beta_1,\beta_2,\ldots,\beta_u$ are row vectors of matrix M. If M is contractible in the k^{th} column such that $m_{i,k}\neq 0, m_{j,k}\neq 0$ and $i\neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ is obtained from M by replacing the i^{th} row with $m_{i,k}x_j+m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called a contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

Lemma 1 [36]**.** If M is a non-negative integral matrix of order $\alpha>1$ and N is a contraction of M, then $per(M)=per(N)$.

Now we concentrate on finding relationships among the complex-type Narayana numbers and the permanents of certain matrices that are obtained by using the generating matrix of complex-type Narayana numbers. Let $A_u=\left[a^u_{jk}\right]$ be $u\times u$ super-diagonal matrix, defined by

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$$A_u = \begin{array}{c} \begin{array}{cccccccc} & & \text{3rd} & & & & & \\ & & \downarrow & & & & & \end{array} \\ \begin{bmatrix} i & 0 & -i & 0 & \cdots & 0 & 0 & 0 \\ 1 & i & 0 & -i & 0 & \cdots & 0 & 0 \\ 0 & 1 & i & 0 & -i & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & i & 0 & -i & 0 \\ 0 & 0 & \cdots & 0 & 1 & i & 0 & -i \\ 0 & 0 & 0 & \cdots & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & i \end{bmatrix} \end{array}.$$

For complex-type Narayana numbers, we can then provide a permanental representation.

Theorem 1. *i.* If u is an odd integer such that $u \geq 3$, then

$$perA_u = \operatorname{Im}\grave{u}_{u+2}.$$

ii. If u is an even integer such that $u \geq 3$, then

$$perA_u = \operatorname{Re}\grave{u}_{u+2}.$$

Proof. For Case i, consider the matrix A_u and let the equation hold for $u \geq 3$. We prove by induction on u. Then we show that the equation holds for $u+2$. If we expand A_u by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perA_{u+2} = i \cdot perA_{u+1} - i \cdot perA_{u-1}.$$

Since $perA_{u+1} = \operatorname{Im}\grave{u}_{u+3}$ and $perA_{u-1} = \operatorname{Im}\grave{u}_{u+1}$, it is clear that $perA_{u+2} = \operatorname{Im}\grave{u}_{u+4}$. So the proof is complete. There is a similar proof for Case ii. □

The $u \times u$ matrix $B_u = \left[b_{jk}^u\right]$ is defined as

$$b_{jk}^u = \begin{cases} i, & \text{if } j = \eta \text{ and } k = \eta \text{ for } 1 \leq \eta \leq u-1, \\ -i, & \text{if } j = \eta \text{ and } k = \eta + 2 \text{ for } 1 \leq \eta \leq u-2, \\ 1, & \begin{array}{c} \text{if } j = \eta \text{ and } k = \eta \text{ for } \eta = u \\ \text{and} \\ k = \eta + 1 \text{ and } j = \eta \text{ for } 1 \leq \eta \leq u-2, \end{array} \\ 0, & \text{otherwise.} \end{cases}$$

For the complex-type Narayana numbers, we can then provide another permanental representation.

Theorem 2. *i.* If u is an odd integer such that $u \geq 3$, then

$$perB_u = \operatorname{Im}\grave{u}_{u+1}.$$

ii. If u is an even integer such that $u \geq 3$, then

$$perB_u = \operatorname{Re}\grave{u}_{u+1}.$$

Proof. For Case ii, consider the matrix B_u and let the equation hold for $u \geq 3$. We prove by induction on u. Then we show that the equation holds for $u+2$. If we expand B_u by the Laplace expansion of permanent with respect to the first row, then we obtain

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$$perB_{u+2} = i \cdot perB_{u+1} - i \cdot perB_{u-1}.$$

Since $perB_{u+1} = \operatorname{Re}\grave{u}_{u+2}$ and $perB_{u-1} = \operatorname{Re}\grave{u}_{u}$, it is clear that $perA_{u+2} = \operatorname{Re}\grave{u}_{u+3}$. So the proof is complete. There is a similar proof for Case i. □

Suppose that $u \times u$ matrix $D_u = \left[d^u_{jk}\right]$ is indicated by:

∗ If u is an odd integer such that $u > 3$, then

$$\begin{array}{cc} & \begin{array}{cccccccccc} & & & & & (u-4)\text{th} & & & & \\ & & & & & \downarrow & & & & \end{array} \\ D_u = & \begin{bmatrix} -1 & 0 & -1 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 \\ 1 & & & & & & & & & \\ 0 & & & & & & & & & \\ \vdots & & & & & B_{u-1} & & & & \\ 0 & & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}. \end{array}$$

∗ If u is an even integer such that $u > 3$, then

$$\begin{array}{cc} & \begin{array}{cccccccccc} & & & & & (u-3)\text{th} & & & & \\ & & & & & \downarrow & & & & \end{array} \\ D_u = & \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 1 & & & & & & & & & \\ 0 & & & & & & & & & \\ \vdots & & & & & B_{u-1} & & & & \\ 0 & & & & & & & & & \\ 0 & & & & & & & & & \end{bmatrix}. \end{array}$$

Then we have the following results:

Theorem 3. *i*. If u is an odd integer such that $u > 3$, then

$$perD_u = -\operatorname{Im}\sum_{a=5}^{u} \grave{u}_{a}.$$

ii. If u is an even integer such that $u > 3$, then

$$perD_u = \operatorname{Re}\sum_{a=4}^{u} \grave{u}_{a}.$$

Proof. For Case ii, if we extend $perD_u$ with respect to the first row, we write

$$perD_u = perD_{u-2} + perB_{u-1}.$$

Thus, by the results and an inductive argument, the proof is easily seen. There is a similar proof for Case i. □

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K. Now we give relationships among the complex-type Narayana numbers and the determinants of certain matrices which are obtained by using the matrix A_u, B_u and D_u. Let $u > 3$ and let H be the $u \times u$ matrix defined by

$$H = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 1. *i.* If u is an odd integer such that $u > 3$, then

$$\det(A_u \circ H) = \mathrm{Im}\grave{u}_{u+2},$$
$$\det(B_u \circ H) = \mathrm{Im}\grave{u}_{u+1}$$

and

$$\det(D_u \circ H) = -\mathrm{Im}\sum_{a=5}^{u}\grave{u}_{a}.$$

ii. If u is an even integer such that $u > 3$, then

$$\det(A_u \circ H) = \mathrm{Re}\grave{u}_{u+2},$$
$$\det(B_u \circ H) = \mathrm{Re}\grave{u}_{u+1}$$

and

$$\det(D_u \circ H) = \mathrm{Re}\sum_{a=4}^{u}\grave{u}_{a}.$$

Proof. Since $perA_u = \det(A_u \circ H)$, $perB_u = \det(B_u \circ H)$ and $perD_u = \det(D_u \circ H)$ for cases i and ii, by Theorems 1-3, we have the conclusion. □

COMPLEX-TYPE NARAYANA SEQUENCE IN GROUPS

If we reduce the sequence $\{\grave{u}_n\}$ by a modulo m, taking the least non-negative residues, then we get a repeating sequence, denoted by

$$\{\grave{u}_n(m)\} = \{\grave{u}_1(m), \grave{u}_2(m), \ldots, \grave{u}_j(m), \ldots\},$$

where $\grave{u}_j(m)$ is used to mean the n^{th} element of the complex-type Narayana sequence when reading modulo m. We note here that sequence $\{\grave{u}_n(m)\}$ has the same recurrence relation as in sequence $\{\grave{u}_n\}$.

Theorem 4. $\{\grave{u}_n(m)\}$ forms a simple periodic sequence for every positive integer m.

Proof. Consider the set

$$A = \{(z_1, z_2, z_3) \mid z_k\text{'s are complex numbers } x_k + iy_k \text{ where } x_k \text{ and } y_k \text{ are integers such that } 0 \le x_k, y_k \le m-1 \text{ and } k \in \{1,2,3\}\}.$$

Let $|A|$ be the cardinality of set A. Since set A is finite, there are $|A|$ distinct 3-tuples of complex-type Narayana sequence modulo m. Thus, it is clear that at least one of these 3-tuples appears twice in sequence $\{\grave{u}_n(m)\}$. Thus, the subsequence following this 3-tuple repeats; that is, sequence $\{\grave{u}_n(m)\}$ is periodic. Therefore, if $\grave{u}_{u+2}(m)\equiv \grave{u}_{v+2}(m)$, $\grave{u}_{u+1}(m)\equiv \grave{u}_{v+1}(m)$, $\grave{u}_u(m)\equiv \grave{u}_v(m)$ and $u>v$, then $u\equiv v(\bmod 4)$. By the recurrence relation of complex-type Narayana sequence $\{\grave{u}_n\}$, we can easily derive that

$$\grave{u}_u(m)\equiv \grave{u}_v(m), \grave{u}_{u-1}(m)\equiv \grave{u}_{v-1}(m),\ldots, \grave{u}_{u-v+2}(m)\equiv \grave{u}_2(m), \grave{u}_{u-v+1}(m)\equiv \grave{u}_1(m).$$

Thus, it is verified that sequence $\{\grave{u}_n(m)\}$ is simply periodic. □

We denote the lengths of periods of sequence $\{\grave{u}_n(m)\}$ by $h_{\grave{u}_n(m)}$. Given an integer matrix $X=[x_{ij}]$, $X(\bmod m)$ means that all entries of A are modulo m, that is, $X(\bmod m)=(x_{ij}(\bmod m))$. Let us consider set $\langle X\rangle_m=\{(X)^n(\bmod m) \mid n\geq 0\}$. If $(\det X, m)=1$, then set $\langle X\rangle_m$ is a cyclic group; if $(\det X, m)\neq 1$, then set $\langle X\rangle_m$ is a semi-group. From companion matrices, we can easily obtain $\det N=-i$. Then it is clear that the set $\langle N\rangle_m$ is a cyclic group for every positive integer $m\geq 2$. From (3) and (4), it is easy to see that $h_{\grave{u}_n(m)}=|\langle N\rangle_m|$. Thus, for $m\geq 3$ we obtain $h_{\grave{u}_n(m)}=4\alpha$, $(\alpha\in \mathrm{N})$.

Theorem 5. Let ρ be a prime and ε be the largest positive integer such that $|\langle N\rangle_\rho|=|\langle N\rangle_{\rho^\varepsilon}|$. Then $|\langle N\rangle_{\rho^\tau}|=\rho^{\tau-\varepsilon}|\langle N\rangle_\rho|$ for every $\tau\geq\varepsilon$. In particular, if $|\langle N\rangle_\rho|\neq|\langle N\rangle_{\rho^2}|$, then $|\langle N\rangle_{\rho^\tau}|=\rho^{\tau-1}|\langle N\rangle_\rho|$ for every $\tau\geq\varepsilon$.

Proof. Since $h_{\grave{u}_n(m)}=|\langle N\rangle_m|$, we have a positive integer μ such that $(N)^{h_{\grave{u}_n(\rho^{\mu+1})}}\equiv I(\bmod \rho^{\mu+1})$. Then it is clear that $(N)^{h_{\grave{u}_n(\rho^{\mu+1})}}\equiv I(\bmod \rho^{\mu})$, where I is a 3×3 identity matrix. Thus, we get that $h_{h_{\grave{u}_n(\rho^\mu)}}$ divides $h_{h_{\grave{u}_n(\rho^{\mu+1})}}$. On the other hand, by writing $(N)^{h_{h_{\grave{u}_n(\rho^\mu)}}}\equiv I+(n_{j,k}^{(\mu)}\cdot\rho^\mu)$, by the binomial theorem we obtain

$$(N)^{h_{h_{\grave{u}_n(\rho^\mu)}}\cdot\rho}=\left(I+\left(n_{j,k}^{(\mu)}\cdot\rho^\mu\right)\right)^\rho=\sum_{n=0}^{\rho}\binom{\rho}{n}\left(n_{j,k}^{(\mu)}\cdot\rho^\mu\right)^n\equiv I(\bmod \rho^{\mu+1}),$$

which implies that $h_{h_{\grave{u}_n(\rho^{\mu+1})}}$ divides $h_{h_{\grave{u}_n(\rho^\mu)}}\cdot\rho$. According to these results, it is seen that $h_{h_{\grave{u}_n(\rho^{\mu+1})}}=h_{h_{\grave{u}_n(\rho^\mu)}}$ or $h_{h_{\grave{u}_n(\rho^{\mu+1})}}=h_{h_{\grave{u}_n(\rho^\mu)}}\cdot\rho$, and the latter holds if and only if there is $n_{j,k}^{(\mu)}$ which is not divisible by ρ. Since $h_{h_{\grave{u}_n(\rho^{\varepsilon+1})}}\neq h_{h_{\grave{u}_n(\rho^\varepsilon)}}$, there is $n_{j,k}^{(\varepsilon)}$ which is not divisible by ρ. This shows that $h_{h_{\grave{u}_n(\rho^{\varepsilon+2})}}\neq h_{h_{\grave{u}_n(\rho^{\varepsilon+1})}}$. So the proof is complete. □

Theorem 6. Suppose that m_1 and m_2 are positive integers with $m_1,m_2\geq 3$, then $|\langle N\rangle_{\mathrm{lcm}[m_1,m_2]}|=\mathrm{lcm}\left[|\langle N\rangle_{m_1}|,|\langle N\rangle_{m_2}|\right]$.

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Proof. Let $\mathrm{lcm}[m_1, m_2] = m$. Clearly, $(N)^{h_{h_{\grave{u}}\, n(m_1)}} \equiv I(\bmod\, m_1)$ and $(N)^{h_{h_{\grave{u}}\, n(m_2)}} \equiv I(\bmod\, m_2)$. Using the least common multiple operation, this implies that $(N)^{h_{h_{\grave{u}}\, n(m)}} \equiv I(\bmod\, m_1)$ and $(N)^{h_{h_{\grave{u}}\, n(m)}} \equiv I(\bmod\, m_2)$. So we get $\left|\langle N\rangle_{m_1}\right| \mid \left|\langle N\rangle_{m}\right|$ and $\left|\langle N\rangle_{m_2}\right| \mid \left|\langle N\rangle_{m}\right|$, which means that $\mathrm{lcm}\left[\left|\langle N\rangle_{m_1}\right|, \left|\langle N\rangle_{m_2}\right|\right]$ divides $\left|\langle N\rangle_{\mathrm{lcm}[m_1, m_2]}\right|$. Now we consider $\mathrm{lcm}\left[\left|\langle N\rangle_{m_1}\right|, \left|\langle N\rangle_{m_2}\right|\right] = q$. Then we can write $N^q \equiv I(\bmod\, m_1)$ and $N^q \equiv I(\bmod\, m_2)$, which yields that $N^q \equiv I(\bmod\, m)$. Thus, it is seen that $\mathrm{lcm}\left[\left|\langle N\rangle_{m_1}\right|, \left|\langle N\rangle_{m_2}\right|\right]$ is divisible by $\left|\langle N\rangle_{\mathrm{lcm}[m_1, m_2]}\right|$. So the proof is complete. □

Now we take into account the complex-type Narayana sequence in groups. Assume G is a finite k-generator group and suppose that X is the subset of $\underbrace{G \times G \times \cdots \times G}_{k \text{ times}}$ such that $(x_1, x_2, \ldots, x_k) \in X$ if and only if G is generated by $x_1, x_2, \ldots, x_k$. Here $(x_1, x_2, \ldots, x_k)$ is said to be a generating k-tuple for G.

Definition 2. Let G be a 3-generator group. For generating triple $(x_1, x_2, x_3) \in G$, we define the complex-type Narayana orbit by

$$a_0 = x_1, a_1 = x_3 x_2, a_2 = x_3, a_n = (a_{n-3})^{-i} (a_{n-1})^{i}$$

for $n \geq 3$. For generating triple (x_1, x_2, x_3), the complex-type Narayana orbit is denoted by $N^{(i)}_{(x_1, x_2, x_3)}(G)$.

The following rules establish the terms of a complex-type sequence that is specified by group elements [7]: For each elements x, y of the group G,

(i) Let e be the identity of G and consider $z = a + ib$, where a, b are integers. Then

* $x^z \equiv x^{a(\bmod |x|) + ib(\bmod |x|)} = x^{a(\bmod |x|)} x^{ib(\bmod |x|)} = x^{ib(\bmod |x|)} x^{a(\bmod |x|)} = x^{ib(\bmod |x|) + a(\bmod |x|)}$,

* $x^{ia} = (x^i)^a = (x^a)^i$,

* $e^u = e$,

* $x^{0+i0} = e$.

(ii) Given $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, b_1, a_2 and b_2 are integers, $y^{-z_2} x^{-z_1} = \left(x^{z_1} y^{z_2}\right)^{-1}$.

(iii) If $yx \neq xy$, then $y^i x^i \neq x^i y^i$.

(iv) $y^i x^i = (xy)^i$ and $x^{-1} y^{-1} = \left(x^i y^i\right)^i$.

(v) $y^i x = x y^i$ and so $x^i y^{-1} = \left(x y^i\right)^i$ and $x^{-1} y^i = \left(x^i y\right)^i$.

It is important to note that we obtain the terms of the complex-type Narayana orbit according to the above rules.

Theorem 7. If G is a finite group, then the complex-type Narayana orbit of G is a periodic.

Proof. Consider the set

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$$Q=\left\{\left((q_1)^{a_1(\mathrm{mod}|q_1|)+ib_1(\mathrm{mod}|q_1|)},(q_2)^{a_2(\mathrm{mod}|q_2|)+ib_2(\mathrm{mod}|q_2|)},(q_3)^{a_3(\mathrm{mod}|q_3|)+ib_3(\mathrm{mod}|q_3|)}\right):\right.$$
$$\left. q_1,q_2,q_3\in G \text{ and } a_n,b_n\in Z \text{ such that } 1\le n\le 3\right\}.$$

G is a finite set and therefore Q is a finite set. Then for any $\upsilon\ge 0$, there exists $\varepsilon\ge\upsilon+3$ such that $a_{\upsilon+1}=a_{\varepsilon+1}$, $a_{\upsilon+2}=a_{\varepsilon+2}$ and $a_{\upsilon+3}=a_{\varepsilon+3}$. Because of the repeating, for all generating 3-tuple, the sequence $N^{(i)}_{(x_1,x_2,x_3)}(G)$ is periodic. □

Now, consider that the length of the period of the orbit $N^{(i)}_{(x_1,x_2,x_3)}(G)$ is denoted by $LN^{(i)}_{(x_1,x_2,x_3)}(G)$. By definition of the orbit $N^{(i)}_{(x_1,x_2,x_3)}(G)$, it is easy to see that the chosen generating set and the order in which assignments of x_1,x_2,x_3 are made determine the length of the period of this sequence in a finite group.

The triangle group (polyhedral group) (l,m,n) for $l,m,n>1$ is defined by the presentation

$$\langle x,y,z : x^l=y^m=z^n=xyz=e\rangle$$

or

$$\langle x,y : x^l=y^m=(xy)^n=e\rangle.$$

The triangle group (l,m,n) is finite if and only if the number

$$\mu=lmn\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}-1\right)=mn+nl+lm-lmn$$

is positive, i.e. in the cases $(2,2,n)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. Its order is $2lmn\,|\,\mu$. Using Tietze transformations, we may show that $(l,m,n)\cong(m,n,l)\cong(n,l,m)$.

We now address the periods of complex-type Narayana orbits of polyhedral groups $(n,2,2)$, $(2,n,2)$ and $(2,2,n)$. Consider the sequences defined as follows:

$$u_n=\begin{cases} i\cdot u_{n-1}-i\cdot u_{n-3} & \text{for } n\equiv 1,4,5,7,8,10\,(\mathrm{mod}\,14),\\ i\cdot\overline{u_{n-1}}-i\cdot\overline{u_{n-3}} & \text{for } n\equiv 0,2,6,12\,(\mathrm{mod}\,14),\\ -i\cdot\overline{u_{n-1}}-i\cdot u_{n-3} & \text{for } n\equiv 3,9,11,13\,(\mathrm{mod}\,14)\end{cases}$$

for $n\ge 3$, where $u_0=1$, $u_1=1$ and $u_2=0$;

$$\upsilon_n=\begin{cases} -i\cdot\overline{\upsilon_{n-1}}+i\cdot\upsilon_{n-3} & \text{for } n\equiv 0,1,2,3,6,12\,(\mathrm{mod}\,14),\\ i\cdot\upsilon_{n-1}-i\cdot\upsilon_{n-3} & \text{for } n\equiv 4,5,8,9,10\,(\mathrm{mod}\,14),\\ -i\cdot\overline{\upsilon_{n-1}}+i\cdot\overline{\upsilon_{n-3}} & \text{for } n\equiv 7,13\,(\mathrm{mod}\,14)\\ i\cdot\upsilon_{n-1}-i\cdot\overline{\upsilon_{n-3}} & \text{for } n\equiv 11\,(\mathrm{mod}\,14)\end{cases}$$

for $n\ge 3$, where $\upsilon_0=-1$, $\upsilon_1=1$ and $\upsilon_2=0$; and

$$\omega_n=\begin{cases} i\cdot\overline{\omega_{n-1}}-i\cdot\overline{\omega_{n-3}} & \text{for } n\equiv 0,1,8,10\,(\mathrm{mod}\,14),\\ i\cdot\omega_{n-1}-i\cdot\omega_{n-3} & \text{for } n\equiv 2,4,9,13\,(\mathrm{mod}\,14),\\ -i\cdot\overline{\omega_{n-1}}-i\cdot\omega_{n-3} & \text{for } n\equiv 3,5,7,12\,(\mathrm{mod}\,14)\\ -i\cdot\omega_{n-1}-i\cdot\overline{\omega_{n-3}} & \text{for } n\equiv 6,11\,(\mathrm{mod}\,14)\end{cases}$$

for $n\ge 3$, where $\omega_0=1$, $\omega_1=1$ and $\omega_2=1$.

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Reducing the sequences $\{u_n\}$, $\{\upsilon_n\}$ and $\{\omega_n\}$ by a modulus m, we get the repeating sequences, respectively, denoted by

$$\{u_n(m)\} = \{u_0(m), u_1(m), \ldots, u_\varepsilon(m), \ldots\},$$
$$\{\upsilon_n(m)\} = \{\upsilon_0(m), \upsilon_1(m), \ldots, \upsilon_\varepsilon(m), \ldots\}$$

and

$$\{\omega_n(m)\} = \{\omega_0(m), \omega_1(m), \ldots, \omega_\varepsilon(m), \ldots\}.$$

We note here that the sequences $\{u_n(m)\}$, $\{\upsilon_n(m)\}$ and $\{\omega_n(m)\}$ have the same recurrence relations as in definitions of the sequences $\{u_n\}$, $\{\upsilon_n\}$ and $\{\omega_n\}$.

Theorem 8. $\{u_n(m)\}$, $\{\upsilon_n(m)\}$ and $\{\omega_n(m)\}$ are simply periodic sequences.

Proof. Consider sequence $\{\omega_n(m)\}$. Suppose that $H = \{h_0, h_1, \ldots, h_k \mid 0 \le k \le m-1\}$. Since the order of set H is m^{14}, H is finite. Also, there are m^{14} distinct 14-tuples of elements Z_m. So the sequence repeats because there are only a finite number of terms possible, and the recurrence of m^{14} term results in the recurrence of all following terms. Consequently, sequence $\{\omega_n(m)\}$ is periodic. So if

$$\omega_a(m) = \omega_b(m), \omega_{a+1}(m) = \omega_{b+1}(m), \ldots, \omega_{a+13}(m) = \omega_{b+13}(m)$$

such that $a > b$, then $a \equiv b(\text{mod}14)$. From the defined recurrence relation of sequence $\{\omega_n(m)\}$, we can easily get that

$$\omega_{a-1}(m) = \omega_{b-1}(m), \omega_{a-2}(m) = \omega_{b-2}(m), \ldots, \omega_{a-b}(m) = \omega_0(m),$$

which implies that $\{\omega_n(m)\}$ is a simply periodic sequence. The proofs for $\{u_n(m)\}$ and $\{\upsilon_n(m)\}$ are similar to the above and are omitted. □

Let notations $h_{u_n(m)}$, $h_{\upsilon_n(m)}$ and $h_{\omega_n(m)}$ denote the smallest period of sequences $\{u_n(m)\}$, $\{\upsilon_n(m)\}$ and $\{\omega_n(m)\}$ respectively.

Theorem 9. For $n \ge 2$, $LN^{(i)}_{(x,y,z)}((n,2,2)) = LN^{(i)}_{(x,y,z)}((2,n,2)) = LN^{(i)}_{(x,y,z)}((2,2,n)) = lcm[14, h_{\grave{u}_n(m)}]$.

Proof. Consider the orbit $N^{(i)}_{(x,y,z)}((n,2,2))$. We prove this by direct calculation. We note here that, in the polyhedral group $(n,2,2)$ defined by $\langle x,y,z : x^n = y^2 = z^2 = xyz = e \rangle$, $zy = x$. We have the sequence

$$\begin{aligned}
&x, x, z, x^{-i}z^{i}, x^{-i+1}z, x^{i+1}, x^{i+2}z,\\
&x^{i-2}, x^{-3i}, x^{-2i+4}z^{i}, x^{6i+3}z, x^{-3i-9}z^{i}, x^{-13i-1}, x^{-2i+19}z^{i},\\
&x^{28i+1}, x^{2i-41}, x^{-60i+4}z, x^{-5i+88}z^{i}, x^{129i+7}z, x^{3i-189}, x^{-277i+8}z,\\
&x^{i+406}, x^{595i+2}, x^{-10i-872}z^{i}, x^{-1278i+11}z, x^{-13i+1873}z^{i}, x^{2745i-3}, x^{-8i-4023}z^{i},\\
&x^{-5896i+5}, x^{8i+8641}, x^{12664i+16}z, \ldots.
\end{aligned}$$

Using the above, the orbit becomes

$$\begin{aligned}
&a_o = x, a_1 = x, a_2 = z, \ldots,\\
&a_{14} = x^{28i+1}, a_{15} = x^{2i-41}, a_{16} = x^{-60i+4} z, \ldots,\\
&a_{28} = x^{-5896i+5}, a_{29} = x^{8i+8641}, a_{30} = x^{12664i+16} z, \ldots.
\end{aligned}$$

It is easy to see that the orbit $N^{(i)}_{(x,y,z)}\left((n,2,2)\right)$ conforms to the following pattern:

$$\begin{aligned}
&a_{14n} = x^{u_{14n}}, a_{14n+1} = x^{u_{14n+1}}, a_{14n+2} = x^{u_{14n+2}} z,\\
&a_{14n+3} = x^{u_{14n+3}} z^i, a_{14n+4} = x^{u_{14n+4}} z, a_{14n+5} = x^{u_{14n+5}},\\
&a_{14n+6} = x^{u_{14n+6}} z, a_{14n+7} = x^{u_{14n+7}}, a_{14n+8} = x^{u_{14n+8}},\\
&a_{14n+9} = x^{u_{14n+9}} z^i, a_{14n+10} = x^{u_{14n+10}} z, a_{14n+11} = x^{u_{14n+11}} z^i,\\
&a_{14n+12} = x^{u_{14n+12}}, a_{14n+13} = x^{u_{14n+13}} z^i, \ldots.
\end{aligned}$$

Then it is clear that the length of the orbit $N^{(i)}_{(x,y,z)}\left((n,2,2)\right)$ is $\mathrm{lcm}\left[14, h_{u_n(m)}\right]$.

By mathematical induction on n, then we derive the following relationships between the elements of the sequence $\{u_n\}$ and the matrix $(N)^n$:

i. If n is odd,

$$(N)^n = \begin{bmatrix} i\cdot\mathrm{Re}(u_{n+4}) & \mathrm{Re}(u_{n+2}) & \mathrm{Im}(u_{n+3}) \\ i\cdot\mathrm{Im}(u_{n+3}) & \mathrm{Im}(u_{n+1}) & \mathrm{Re}(u_{n+2}) \\ i\cdot\mathrm{Re}(u_{n+2}) & \mathrm{Re}(u_n) & \mathrm{Im}(u_{n+1}) \end{bmatrix}. \tag{5}$$

ii. If n is even,

$$(N)^n = \begin{bmatrix} i\cdot\mathrm{Im}(u_{n+4}) & \mathrm{Im}(u_{n+2}) & \mathrm{Re}(u_{n+3}) \\ i\cdot\mathrm{Re}(u_{n+3}) & \mathrm{Re}(u_{n+1}) & \mathrm{Im}(u_{n+2}) \\ i\cdot\mathrm{Im}(u_{n+2}) & \mathrm{Im}(u_n) & \mathrm{Re}(u_{n+1}) \end{bmatrix}. \tag{6}$$

It can be easily seen from equations (5) and (6) that $h_{u_n(m)} = \left|\langle N\rangle_m\right|$. Since $h_{u_n(m)} = \left|\langle N\rangle_m\right|$, we get $h_{\grave{u}_n(m)} = h_{u_n(m)}$, which implies that $LN^{(i)}_{(x,y,z)}\left((n,2,2)\right) = \mathrm{lcm}\left[14, h_{\grave{u}_n(m)}\right]$.

A routine induction shows that $h_{\upsilon_n(m)} = \left|\langle N\rangle_m\right|$ and $h_{\omega_n(m)} = \left|\langle N\rangle_m\right|$. Using a similar method that was used for the case $(n,2,2)$, it is possible to find that $LN^{(i)}_{(x,y,z)}\left((2,n,2)\right) = \mathrm{lcm}\left[14, h_{\upsilon_n(m)}\right] = LN^{(i)}_{(x,y,z)}\left((2,2,n)\right) = \mathrm{lcm}\left[14, h_{\omega_n(m)}\right] = \mathrm{lcm}\left[14, h_{\grave{u}_n(m)}\right]$. □

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