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Full Paper

Some special Smarandache ruled surfaces according to alternative frame in E^3

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Abstract: In this study ruled surfaces formed by alternative unit vectors that draw these curves along Smarandache curves obtained from alternative vectors and alternative unit Darboux vectors of a differentiable curve are defined. Then the Gaussian and mean curvatures of each ruled surface are computed. It is also examined whether the surfaces are developable or minimal. Finally, examples of these surfaces are given and their graphics are drawn.

Keywords: Smarandache ruled surfaces, mean curvature, Gaussian curvature

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INTRODUCTION

A function with two real variables in three-dimensional space creates a surface. Surfaces are used in many fields, especially in architecture and engineering, according to their curvature. Research on the curvature of a surface in the field of differential geometry has been done by various mathematicians since the Ancient Greek period. The method of calculating the curvature of a surface was described by Gauss in the 19th century and is therefore called Gaussian curvature. The Gaussian curvature of a surface gives information about whether the surface is developable. The fact that the surface can be developable means that the Gaussian curvature at every point is zero. As for the mean curvature of the surface, if the mean curvature of a surface is zero at each point, the surface is called a minimal surface. Minimal surfaces are the most used surface type in architecture [1-4]. The basic concepts of surfaces and some studies on their geometric properties in various spaces are available [5-10]. Surfaces formed by lines in surface theory are called ruled surfaces. There are many studies on such surfaces [11-17]. On the other hand, a special type of curves, Smarandache curve, is the position vector produced by the Frenet vectors of a regular curve. The first study on this subject was made by Ali [18], who expressed some special Smarandache curves in Euclidean 3-space and introduced the Serret-Frenet elements of a special case. Recently, ruled surfaces along Smarandache curves obtained by using various frames on the curve have been defined and some properties have been examined [19-30]. The aim of this study is to obtain new ruled surfaces by considering the Smarandache curves which have an important place in the literature using the alternative frame of any curve. The obtained ruled surfaces will provide new contribution to the current studies in this field.

PRELIMINARIES

The Frenet vectors $\{T,\ N,\ B\}$, curvatures $\kappa,\ \tau$ and Frenet derivative formulas of unit speed curve $\alpha : I \to E^3$ are given in many resources. The Darboux vector belonging to Frenet vectors is found as $F = \tau T + \kappa B$ [31]. The unit Darboux vector is denoted by

$$W = \sin\varphi T + \cos\varphi B \ , \tag{1}$$

where $\angle(B,F) = \varphi,\ \cos\varphi = \dfrac{\kappa}{\sqrt{\kappa^2+\tau^2}},\ \ \sin\varphi = \dfrac{\tau}{\sqrt{\kappa^2+\tau^2}},\ \varphi' = \left(\dfrac{\tau}{\kappa}\right)'\left(\dfrac{\kappa^2}{\kappa^2+\tau^2}\right)$ [31]. Let define the orthonormal system $\{N,\ C = W \wedge N,\ W\}$ of the curve $\alpha(s)$. Here,

$$C = -\cos\varphi T + \sin\varphi B, \qquad W = \sin\varphi T + \cos\varphi B \ . \tag{2}$$

This system is called alternative frame of $\alpha(s)$. The derivative formulas of this frame are

$$N' = \beta C,\ \ C' = -\beta N + \varphi' W,\ \ \ W' = -\varphi' C, \tag{3}$$

where $\beta = \sqrt{\kappa^2+\tau^2}$. If the alternative unit Darboux vector is denoted by D, then

$$D = pN + qW, \tag{4}$$

where $p = \dfrac{\varphi'}{\sqrt{(\varphi')^2+\beta^2}},\ \ q = \dfrac{\beta}{\sqrt{(\varphi')^2+\beta^2}}$. Or from (1),

$$D = \sin\omega\, N + cos\omega\, W, \tag{5}$$

where $\angle(D,W) = \omega,\ \sin\omega = p,\ \ \cos\omega = q$ [26]. The unit vector linearly dependent on the vectors $N,\ C,\ W$ of a curve can be expressed as

$$\gamma = \frac{fN + gC + hW}{\sqrt{f^2+g^2+h^2}}, \tag{6}$$

where f, g, h are functions. The new curves obtained by taking the vector γ as the position vector are called Smarandache curves [18]. In particular, the Smarandache curves obtained when functions f, g, h are taken as constant numbers are as follows:

i) $h = 0, f = g = 1 \Rightarrow\ NC-$ Smarandache curve $\gamma_1 = \dfrac{N+C}{\sqrt{2}}$ drawn by the vector $\gamma = \dfrac{N+C}{\sqrt{2}}$,

ii) $g = 0, f = h = 1 \Rightarrow\ NW-$ Smarandache curve $\gamma_2 = \dfrac{N+W}{\sqrt{2}}$ drawn by the vector $\gamma = \dfrac{N+W}{\sqrt{2}}$,

iii) $f = 0, g = h = 1 \Rightarrow\ CW-$ Smarandache curve $\gamma_3 = \dfrac{C+W}{\sqrt{2}}$ drawn by the vector $\gamma = \dfrac{C+W}{\sqrt{2}}$,

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iv) $f = g = h = 1 \Rightarrow NCW-$Smarandache curve $\gamma_4 = \dfrac{N+C+W}{\sqrt{3}}$ drawn by the vector $\gamma = \dfrac{N+C+W}{\sqrt{3}}$.

A surface formed by the line moving depending on the parameter of a curve is called ruled surface and its parametric expression is given as

$$\Psi(s,v) = \alpha(s) + vr(s). \tag{7}$$

The normal vector, Gaussian and mean curvatures of $\Psi(s,v)$ are

$$\mathrm{N}_\Psi = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|}, \tag{8}$$

$$K = \frac{eg - f^2}{EG - F^2}, \qquad H = \frac{Eg - 2fF + eG}{2\left(EG - F^2\right)}, \tag{9}$$

respectively [5]. Here, the coefficients of the first and second fundamental forms of the surface are

$$E = \langle \Psi_s, \Psi_s \rangle, \quad F = \langle \Psi_s, \Psi_v \rangle, \quad G = \langle \Psi_v, \Psi_v \rangle, \tag{10}$$

$$e = \langle \Psi_{ss}, N_\Psi \rangle, \quad f = \langle \Psi_{sv}, N_\Psi \rangle, \quad g = \langle \Psi_{vv}, N_\Psi \rangle, \tag{11}$$

respectively. According to Gray et al. [6], the Gauss and mean curvatures of the obtained surfaces

$$\Psi(s,v) = \frac{1}{\sqrt{2}}\left(N(s) + C(s)\right) + vW(s), \qquad \Phi(s,v) = \frac{1}{\sqrt{2}}\left(N(s) + W(s)\right) + vC(s)$$

were calculated when the *NC*- and *NW*-Smarandache curves were taken as base curves.

SMARANDACHE RULED SURFACES ACCORDING TO ALTERNATIVE FRAME

In (6), $ND, CD, WD-$Smarandache curves for the values f, g, h are

i) $g = 0,\ f = 1+p,\ h = q \ \Rightarrow \gamma = \dfrac{N + pN + qW}{\sqrt{2+2p}} = \dfrac{1}{\sqrt{2+2p}}(N+D),$

ii) $f = p,\ g = 1,\ h = p \ \Rightarrow \gamma = \dfrac{1}{\sqrt{2}}(C+D),$

iii) $f = p,\ g = 0,\ h = 1+q \ \Rightarrow \gamma = \dfrac{1}{\sqrt{2+2q}}(W+D),$

respectively.

Definition 1. The ruled surface formed by the *ND*-vector along the *ND*-Smarandache curve is defined as

$$\Sigma(s,v) = \frac{N+D}{\sqrt{2+2p}} + v\frac{N+D}{\sqrt{2+2p}} = \frac{1+v}{\sqrt{2}}\left(\sqrt{1+p}N + \sqrt{1-p}W\right).$$

Theorem 1. The Gaussian and mean curvatures of the surface $\Sigma(s,v)$ are

$$K_\Sigma = 0, \qquad H_\Sigma = \frac{\left(\left(\dfrac{x_1}{y_1}\right)' y_1^2 - \beta\left(y_1^2 + x_1^2\right) + x_1 z_1 \varphi'\right)\sqrt{1-p} + \left(\left(\dfrac{y_1}{z_1}\right)' z_1^2 - \varphi'\left(y_1^2 + z_1^2\right) + x_1 z_1 \beta\right)\sqrt{1+p}}{\sqrt{2}(1+v)^{-1}\left(x_1^2 + 2y_1^2 + z_1^2 + p\left(z_1^2 - x_1^2\right) - 2x_1 z_1 q\right)^{\frac{3}{2}}},$$

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respectively, where $x_1=\left(\sqrt{1+p}\right)'$, $y_1=\left(\beta\sqrt{1+p}-\varphi'\sqrt{1-p}\right)$, $z_1=\left(\sqrt{1-p}\right)'$.

Proof: If the partial derivatives of the surface $\Sigma(s,v)$ are taken and the necessary operations are performed, then

$$\sqrt{2}\Sigma_s=(1+v)(x_1N+y_1C+z_1W),\quad \sqrt{2}\Sigma_{sv}=x_1N+y_1C+z_1W,\quad \sqrt{2}\Sigma_v=\sqrt{1+p}N+\sqrt{1-p}W,$$

$$\Sigma_{ss}=\frac{1+v}{\sqrt{2}}\left(\left(x_1'-y_1\beta\right)N+\left(x_1\beta+y_1'-z_1\varphi'\right)C+\left(y_1\varphi'+z_1'\right)W\right),\quad \Sigma_{vv}=0,$$

$$\Sigma_s\wedge\Sigma_v=\frac{1+v}{2}\left(y_1\sqrt{1-p}N+\left(z_1\sqrt{1+p}-x_1\sqrt{1-p}\right)C-y_1\sqrt{1+p}W\right),$$

$$\left\|\Sigma_s\wedge\Sigma_v\right\|=\frac{1+v}{2}\sqrt{x_1^2+2y_1^2+z_1^2+p\left(z_1^2-x_1^2\right)-2x_1z_1q}.$$

From (8), the normal vector N_Σ of the surface is

$$N_\Sigma=\frac{y_1\sqrt{1-p}N+\left(z_1\sqrt{1+p}-x_1\sqrt{1-p}\right)C-y_1\sqrt{1+p}W}{\sqrt{x_1^2+2y_1^2+z_1^2+p\left(z_1^2-x_1^2\right)-2x_1z_1q}}.$$

From (10) and (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\Sigma=\frac{(1+v)^2\left(x_1^2+y_1^2+z_1^2\right)}{2},\quad F_\Sigma=\frac{(1+v)\left(x_1\sqrt{1+p}+z_1\sqrt{1-p}\right)}{2},\quad G_\Sigma=1,$$

$$e_\Sigma=\frac{\left(\left(\frac{x_1}{y_1}\right)'y_1^2-\beta\left(y_1^2+x_1^2\right)+x_1z_1\varphi'\right)\sqrt{1-p}+\left(\left(\frac{y_1}{z_1}\right)'z_1^2-\varphi'\left(y_1^2+z_1^2\right)+x_1z_1\beta\right)\sqrt{1+p}}{\sqrt{2}(1+v)^{-1}\sqrt{x_1^2+2y_1^2+z_1^2+p\left(z_1^2-x_1^2\right)-2x_1z_1q}},\quad f_\Sigma=g_\Sigma=0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Corollary 1. The ruled surface $\Sigma(s,v)$ is developable.

Definition 2. The ruled surface formed by the *CD*-vector along the *ND*-Smarandache curve is defined as

$$\Delta(s,v)=\frac{N+D}{\sqrt{2+2p}}+\frac{v}{\sqrt{2}}(C+D)=\frac{1}{\sqrt{2}}\left(\left(\sqrt{1+p}+vp\right)N+vC+\left(\sqrt{1-p}+vq\right)W\right).$$

Theorem 2. The Gaussian and mean curvatures of the surface $\Delta(s,v)$ are

$$K_\Delta=-2\left(\frac{(y_2q-z_2)\left(p'-\beta\right)+(x_2-y_2p)\left(q'+\varphi'\right)}{(y_2q-z_2)^2+(z_2p-x_2q)^2+(x_2-y_2p)^2}\right)^2,$$

$$H_\Delta=\frac{\begin{aligned}&\Big[(y_2q-z_2)\left(x_2'-y_2\beta-\left(p'-\beta\right)(x_2p+y_2+z_2q)\right)+(z_2p-x_2q)(x_2\beta+y_2'-z_2\varphi')\\&+(x_2-y_2p)\left(y_2\varphi'+z_2'-\left(q'+\varphi'\right)(x_2p+y_2+z_2q)\right)\Big]\end{aligned}}{2^{\frac{-1}{2}}\left((y_2q-z_2)^2+(z_2p-x_2q)^2+(x_2-y_2p)^2\right)^{\frac{3}{2}}},$$

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respectively, where $x_2=\left(\sqrt{1+p}+vp\right)'-\beta v,\quad y_2=\beta\sqrt{1+p}-\varphi'\sqrt{1-p},\quad z_2=\left(\sqrt{1-p}+vq\right)'+v\varphi'.$

Proof: If the partial derivatives of the surface $\Delta(s,v)$ are taken and the necessary operations are performed, then

$$\sqrt{2}\Delta_s=x_2N+y_2C+z_2W,\ \sqrt{2}\Delta_{ss}=\left(x_2'-y_2\beta\right)N+\left(x_2\beta+y_2'-z_2\varphi'\right)C+\left(y_2\varphi'+z_2'\right)W,$$

$$\sqrt{2}\Delta_v=pN+C+qW,\ \sqrt{2}\Delta_{sv}=\left(p'-\beta\right)N+\left(q'+\varphi'\right)W,\ \ \Delta_{vv}=0,$$

$$2\Delta_s\wedge\Delta_v=\left(y_2q-z_2\right)N+\left(z_2p-x_2q\right)C+\left(x_2-y_2p\right)W,$$

$$2\left\|\Delta_s\wedge\Delta_v\right\|=\sqrt{\left(y_2q-z_2\right)^2+\left(z_2p-x_2q\right)^2+\left(x_2-y_2p\right)^2}.$$

From (8), the normal vector N_Δ of the surface is

$$N_\Delta=\frac{\left(y_2q-z_2\right)N+\left(z_2p-x_2q\right)C+\left(x_2-y_2p\right)W}{\sqrt{\left(y_2q-z_2\right)^2+\left(z_2p-x_2q\right)^2+\left(x_2-y_2p\right)^2}}.$$

From (10) and (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\Delta=\frac{1}{2}\left(x_2^{\,2}+y_2^{\,2}+z_2^{\,2}\right),\qquad F_\Delta=\frac{1}{2}\left(x_2p+y_2+z_2q\right),\qquad G_\Delta=1,$$

$$E_\Delta G_\Delta-F_\Delta^2=\frac{\left(y_2q-z_2\right)^2+\left(z_2p-x_2q\right)^2+\left(x_2-y_2p\right)^2}{4}$$

$$e_\Delta=\frac{\left(y_2q-z_2\right)\left(x_2'-y_2\beta\right)+\left(z_2p-x_2q\right)\left(x_2\beta+y_2'-z_2\varphi'\right)+\left(x_2-y_2p\right)\left(y_2\varphi'+z_2'\right)}{\sqrt{2}\sqrt{\left(y_2q-z_2\right)^2+\left(z_2p-x_2q\right)^2+\left(x_2-y_2p\right)^2}},$$

$$f_\Delta=\frac{\left(y_2q-z_2\right)\left(p'-\beta\right)+\left(x_2-y_2p\right)\left(q'+\varphi'\right)}{\sqrt{2}\sqrt{\left(y_2q-z_2\right)^2+\left(z_2p-x_2q\right)^2+\left(x_2-y_2p\right)^2}},\qquad g_\Delta=0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Definition 3. The ruled surface formed by the *WD*-vector along the *ND*-Smarandache curve is defined as

$$\Upsilon(s,v)=\frac{N+D}{\sqrt{2+2p}}+v\frac{W+D}{\sqrt{2+2q}}=\frac{1}{\sqrt{2}}\left(\left(\sqrt{1+p}+v\sqrt{1-q}\right)N+\left(\sqrt{1-p}+v\sqrt{1+q}\right)W\right).$$

Theorem 3. The Gaussian and mean curvatures of the surface $\Upsilon(s,v)$ are

$$K_\Upsilon=-2\left(\frac{y_3\omega'+\left(x_3\varphi'+z_3\beta\right)+q\left(x_3\varphi'-z_3\beta\right)-p\left(x_3\beta+z_3\varphi'\right)}{x_3^{\,2}+2y_3^{\,2}+z_3^{\,2}+q\left(x_3^{\,2}-z_3^{\,2}\right)-2pz_3x_3}\right)^2,$$

$$H_\Upsilon=\frac{\begin{array}{l}\Big[\left(y_3x_3'-x_3y_3'-\beta y_3^2-x_3^2\beta+x_3z_3\varphi'-z_3\left(y_3\omega'+x_3\varphi'+2x_3q\varphi'\right)+\left(x_3^{\,2}-z_3^{\,2}\right)\varphi'p\right)\sqrt{1+q}\\ \quad+\left(z_3y_3'-y_3z_3'-\left(z_3^{\,2}+y_3^{\,2}\right)\varphi'+x_3\left(z_3\beta+y_3\omega'+2z_3q\beta\right)+\left(x_3^{\,2}-z_3^{\,2}\right)\beta p\right)\sqrt{1-q}\Big]\end{array}}{2^{\frac{-1}{2}}\left(x_3^{\,2}+2y_3^{\,2}+z_3^{\,2}+q\left(x_3^{\,2}-z_3^{\,2}\right)-2pz_3x_3\right)^{\frac{3}{2}}},$$

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respectively, where $x_3=\left(\sqrt{1+p}+v\sqrt{1-q}\right)'$, $y_3=\beta\sqrt{1+p}-\varphi'\sqrt{1-p}+v\left(\beta\sqrt{1-q}-\varphi'\sqrt{1+q}\right)$

$z_3=\left(\sqrt{1-p}+v\sqrt{1+q}\right)'$.

Proof: If the partial derivatives of the surface $\Upsilon(s,v)$ are taken and the necessary operations are performed, then

$$\Upsilon_s=\frac{1}{\sqrt{2}}\left(x_3N+y_2C+z_3W\right),\quad \Upsilon_v=\frac{1}{\sqrt{2}}\left(\sqrt{1-q}N+\sqrt{1+q}W\right),\quad \Upsilon_{vv}=0,$$

$$\Upsilon_{ss}=\frac{1}{\sqrt{2}}\left(\left(x_3'-\beta y_3\right)N+\left(y_3'+x_3\beta-z_3\varphi'\right)C+\left(z_3'+y_3\varphi'\right)W\right),$$

$$\Upsilon_{sv}=\frac{1}{\sqrt{2}}\left(\left(\sqrt{1-q}\right)'N+\left(\beta\sqrt{1-q}-\varphi'\sqrt{1+q}\right)C+\left(\sqrt{1+q}\right)'W\right),$$

$$\Upsilon_s\wedge\Upsilon_v=\frac{1}{2}\left(y_3\sqrt{1+q}N+\left(z_3\sqrt{1-q}-x_3\sqrt{1+q}\right)C-y_3\sqrt{1-q}W\right),$$

$$\left\|\Upsilon_s\wedge\Upsilon_v\right\|=\frac{1}{2}\sqrt{x_3^2+2y_3^2+z_3^2+q\left(x_3^2-z_3^2\right)-2pz_3x_3}.$$

From (8), the normal vector N_Υ of the surface is

$$N_\Upsilon=\frac{y_3\sqrt{1+q}N+\left(z_3\sqrt{1-q}-x_3\sqrt{1+q}\right)C-y_3\sqrt{1-q}W}{\sqrt{x_3^2+2y_3^2+z_3^2+q\left(x_3^2-z_3^2\right)-2pz_3x_3}}.$$

From (10) and (11), by using (5), the coefficients of the first and second fundamental forms of the surface are

$$E_\Upsilon=\frac{1}{2}\left(x_3^2+y_3^2+z_3^2\right),\quad F_\Upsilon=\frac{1}{2}\left(x_3\sqrt{1-q}+z_3\sqrt{1+q}\right),\quad G_\Upsilon=1,$$

$$E_\Upsilon G_\Upsilon-F_\Upsilon^2=\frac{x_3^2+2y_3^2+z_3^2+q\left(x_3^2-z_3^2\right)-2pz_3x_3}{4},$$

$$e_\Upsilon=\frac{\left(y_3x_3'-x_3y_3'-\beta y_3^2-x_3^2\beta+x_3z_3\varphi'\right)\sqrt{1+q}+\left(z_3y_3'-y_3z_3'-z_3^2\varphi'-y_3^2\varphi'+z_3x_3\beta\right)\sqrt{1-q}}{\sqrt{2}\sqrt{x_3^2+2y_3^2+z_3^2+q\left(x_3^2-z_3^2\right)-2pz_3x_3}},$$

$$f_\Upsilon=\frac{y_3\omega'+\left(x_3\varphi'+z_3\beta\right)+q\left(x_3\varphi'-z_3\beta\right)-p\left(x_3\beta+z_3\varphi'\right)}{\sqrt{2}\sqrt{x_3^2+2y_3^2+z_3^2+q\left(x_3^2-z_3^2\right)-2pz_3x_3}},\quad g_\Upsilon=0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Definition 4. The ruled surface formed by the *CD*-vector along the *CD*-Smarandache curve is defined as

$$\chi(s,v)=\frac{1}{\sqrt{2}}(C+D)+\frac{v}{\sqrt{2}}(C+D)=\frac{1+v}{\sqrt{2}}(pN+C+qW).$$

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Theorem 4. The Gaussian and mean curvatures of the surface $\chi(s,v)$ are

$$K_\chi=0,\qquad H_\chi=\frac{\begin{bmatrix}(p'-\beta)(\varphi''+q''+p\varphi'\beta-q\varphi'^2)-(\varphi'+q')(p''-\beta'-p\beta^2+qp''+\beta'\varphi')\\ +(p\varphi'+pq'-qp'+q\beta)(p'\beta-\varphi'q'-\beta^2-\varphi'^2)\end{bmatrix}}{(2+2v)^{-1}\left((\varphi'+q')^2+(p'-\beta)^2+(p\varphi'+pq'-qp'+q\beta)^2\right)^{\frac{3}{2}}},$$

respectively.

Proof: If the partial derivatives of the surface $\chi(s,v)$ are taken and the necessary operations are performed, then

$$\chi_s=\frac{1+v}{\sqrt{2}}\left((p'-\beta)N+(p\beta-q\varphi')C+(\varphi'+q')W\right),\qquad \chi_v=\frac{1}{\sqrt{2}}(pN+C+qW),$$

$$\chi_{sv}=\frac{1}{\sqrt{2}}\left((p'-\beta)N+(p\beta-q\varphi')C+(\varphi'+q')W\right),\quad \chi_{vv}=0,$$

$$\chi_{ss}=\frac{1+v}{\sqrt{2}}\left((p''-\beta'-p\beta^2+qp''+\beta'\varphi')N+(p'\beta-\varphi'q'-\beta^2-\varphi'^2)C+(\varphi''+q''+p\varphi'\beta-q\varphi'^2)W\right),$$

$$\chi_s\wedge\chi_v=\frac{1+v}{2}\left(-(\varphi'+q')N+(p\varphi'+pq'-qp'+q\beta)C+(p'-\beta)W\right),$$

$$\|\chi_s\wedge\chi_v\|=\frac{1+v}{2}\sqrt{(\varphi'+q')^2+(p'-\beta)^2+(p\varphi'+pq'-qp'+q\beta)^2}.$$

From (8), the normal vector N_χ of the surface is

$$N_\chi=\frac{-(\varphi'+q')N+(p\varphi'+pq'-qp'+q\beta)C+(p'-\beta)W}{\sqrt{(\varphi'+q')^2+(p'-\beta)^2+(p\varphi'+pq'-qp'+q\beta)^2}}.$$

From (5), (10), (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\chi=\frac{(1+v)^2}{2}\left((\varphi'+q')^2+(p'-\beta)^2+(p\beta-q\varphi')^2\right),\quad F_\chi=0,\quad G_\chi=1,$$

$$e_\chi=\frac{\begin{bmatrix}-(\varphi'+q')(p''-\beta'-p\beta^2+qp''+\beta'\varphi')+(p'-\beta)(\varphi''+q''+p\varphi'\beta-q\varphi'^2)\\ +(p\varphi'+pq'-qp'+q\beta)(p'\beta-\varphi'q'-\beta^2-\varphi'^2)\end{bmatrix}}{(1+v)^{-1}\sqrt{2}\sqrt{(\varphi'+q')^2+(p'-\beta)^2+(p\varphi'+pq'-qp'+q\beta)^2}},\quad f_\chi=g_\chi=0.$$

If these expressions are substituted in (9), Gaussian and mean curvatures are found.

Corollary 2. The ruled surface $\chi(s,v)$ is developable.

Definition 5. The ruled surface formed by the *ND*-vector along the *CD*-Smarandache curve is defined as

$$\delta(s,v)=\frac{1}{\sqrt{2}}(C+D)+v\frac{N+D}{\sqrt{2+2p}}=\frac{1}{\sqrt{2}}\left(\left(p+v\sqrt{1+p}\right)N+C+\left(q+v\sqrt{1-p}\right)W\right).$$

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Theorem 5. The Gaussian and mean curvatures of the surface $\delta(s,v)$ are

$$K_\delta=-2\left(\frac{y_4\omega'+(x_4\varphi'+z_4\beta)+p(z_4\beta-x_4\varphi')-q(x_4\beta+z_4\varphi')}{x_4^2+2y_4^2+z_4^2+(z_4^2-x_4^2)p-2qz_4x_4}\right)^2,$$

$$H_\delta=\frac{\begin{pmatrix}y_4x_4'-x_4y_4'-\beta y_4^2-x_4^2\beta+x_4z_4\varphi'\\-z_4\left(y_4\omega'+(x_4\varphi'+z_4\beta)+p(z_4\beta-x_4\varphi')-q(x_4\beta+z_4\varphi')\right)\end{pmatrix}\sqrt{1-p}+\begin{pmatrix}z_4y_4'-y_4z_4'-z_4^2\varphi'-y_4^2\varphi'+z_4x_4\beta\\-x_4\left(y_4\omega'+(x_4\varphi'+z_4\beta)+p(z_4\beta-x_4\varphi')-q(x_4\beta+z_4\varphi')\right)\end{pmatrix}\sqrt{1+p}}{2^{\frac{-1}{2}}\left(x_4^2+2y_4^2+z_4^2+(z_4^2-x_4^2)p-2qz_4x_4\right)^{\frac{3}{2}}},$$

respectively, where $x_4=\left(p+v\sqrt{1+p}\right)'-\beta,\quad y_4=v\left(\beta\sqrt{1+p}-\varphi'\sqrt{1-p}\right),\quad z_4=\varphi'+\left(q+v\sqrt{1-p}\right)'.$

Proof: If the partial derivatives of the surface $\delta(s,v)$ are taken and the necessary operations are performed, then

$$\sqrt{2}\delta_s=\left(\left(p+v\sqrt{1+p}\right)'-\beta\right)N+v\left(\beta\sqrt{1+p}-\varphi'\sqrt{1-p}\right)C+\left(\varphi'+\left(q+v\sqrt{1-p}\right)'\right)W,$$

$$\sqrt{2}\delta_s=x_4N+y_4C+z_4W,\qquad \sqrt{2}\delta_v=\sqrt{1+p}N+\sqrt{1-p}W,\qquad \delta_{vv}=0,$$

$$\sqrt{2}\delta_{ss}=\left(x_4'-\beta y_4\right)N+\left(y_4'+x_4\beta-z_4\varphi'\right)C+\left(z_4'+y_4\varphi'\right)W,$$

$$\sqrt{2}\delta_{sv}=\left(\sqrt{1+p}\right)'N+\left(\beta\sqrt{1+p}-\varphi'\sqrt{1-p}\right)C+\left(\sqrt{1-p}\right)'W,$$

$$2\delta_s\wedge\delta_v=y_4\sqrt{1-p}N+\left(z_4\sqrt{1+p}-x_4\sqrt{1-p}\right)C-y_4\sqrt{1+p}W,$$

$$2\left\|\delta_s\wedge\delta_v\right\|=\sqrt{x_4^2+2y_4^2+z_4^2+(z_4^2-x_4^2)p-2qz_4x_4}.$$

From (8), the normal vector N_δ of the surface is

$$N_\delta=\frac{y_4\sqrt{1-p}N+\left(z_4\sqrt{1+p}-x_4\sqrt{1-p}\right)C-y_4\sqrt{1+p}W}{\sqrt{x_4^2+2y_4^2+z_4^2+(z_4^2-x_4^2)p-2qz_4x_4}}.$$

From (5), (10), (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\delta=\frac{1}{2}\left(x_4^2+y_4^2+z_4^2\right),\qquad F_\delta=\frac{1}{2}\left(x_4\sqrt{1+p}+z_4\sqrt{1-p}\right),\qquad G_\delta=1,$$

$$E_\delta G_\delta-F_\delta^2=\frac{x_4^2+2y_4^2+z_4^2+p\left(z_4^2-x_4^2\right)-2qx_4z_4}{4},$$

$$e_\delta=\frac{\left(y_4x_4'-x_4y_4'-\beta y_4^2-x_4^2\beta+x_4z_4\varphi'\right)\sqrt{1-p}+\left(z_4y_4'-y_4z_4'-z_4^2\varphi'-y_4^2\varphi'+z_4x_4\beta\right)\sqrt{1+p}}{\sqrt{2}\sqrt{x_4^2+2y_4^2+z_4^2+(z_4^2-x_4^2)p-2qz_4x_4}},$$

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$$f_\delta = \frac{y_4\omega' + (x_4\varphi' + z_4\beta) + p(z_4\beta - x_4\varphi') - q(x_4\beta + z_4\varphi')}{\sqrt{2}\sqrt{x_4^2 + 2y_4^2 + z_4^2 + (z_4^2 - x_4^2)p - 2qz_4x_4}}, \qquad g_\delta = 0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Definition 6. The ruled surface formed by the *WD*-vector along the *CD*-Smarandache curve is defined as

$$\eta(s,v) = \frac{1}{\sqrt{2}}(C+D) + v\frac{W+D}{\sqrt{2+2q}} = \frac{1}{\sqrt{2}}\left(\left(p + v\sqrt{1-q}\right)N + C + \left(q + v\sqrt{1+q}\right)W\right).$$

Theorem 6. The Gaussian and mean curvatures of the surface $\eta(s,v)$ are respectively

$$K_\eta = -2\left(\frac{y_5\omega' + z_5\beta - x_5\varphi' - (z_5\beta + x_5\varphi')q + (z_5\varphi' - x_5\beta)p}{x_5^2 + 2y_5^2 + z_5^2 + q(x_5^2 - z_5^2) - 2x_5z_5p}\right)^2,$$

$$H_\eta = \frac{\begin{pmatrix} y_5x_5' - x_5y_5' - (y_5^2 + x_5^2 - z_5)\beta + (x_5z_5 - x_5)\varphi' \\ -z_5y_5\omega' - (z_5\beta + x_5\varphi')q + (z_5\varphi' - x_5\beta)p \end{pmatrix}\sqrt{1+q} + \begin{pmatrix} z_5y_5' - y_5z_5' - (z_5^2 + y_5^2 + x_5)\varphi' + (z_5x_5 - z_5)\beta \\ -x_5y_5\omega' + (z_5\beta + x_5\varphi')q - (z_5\varphi' - x_5\beta)p \end{pmatrix}\sqrt{1-q}}{2^{\frac{-1}{2}}\left(x_5^2 + 2y_5^2 + z_5^2 + q(x_5^2 - z_5^2) - 2x_5z_5p\right)^{\frac{3}{2}}},$$

where $x_5 = -\beta + \left(p + v\sqrt{1-q}\right)'$, $y_5 = \beta\left(p + v\sqrt{1-q}\right) - \varphi'\left(q + v\sqrt{1+q}\right)$, $z_5 = \varphi' + \left(q + v\sqrt{1+q}\right)'$.

Proof: If the partial derivatives of the surface $\eta(s,v)$ are taken and the necessary operations are performed, then

$$\eta_s = \frac{1}{\sqrt{2}}(x_5N + y_5C + z_5W), \qquad \eta_v = \frac{1}{\sqrt{2}}\left(\sqrt{1-q}N + \sqrt{1+q}W\right), \qquad \eta_{vv} = 0$$

$$\eta_{ss} = \frac{1}{\sqrt{2}}\left(\left(x_5' - y_5\beta\right)N + \left(y_5' + x_5\beta - z_5\varphi'\right)C + \left(z_5' + y_5\varphi'\right)W\right),$$

$$\eta_{sv} = \frac{1}{\sqrt{2}}\left(\left(\sqrt{1-q}\right)' N + \left(\beta\sqrt{1-q} - \varphi'\sqrt{1+q}\right)C + \left(\sqrt{1+q}\right)' W\right),$$

$$\eta_s \wedge \eta_v = \frac{1}{2}\left(y_5\sqrt{1+q}N + \left(z_5\sqrt{1-q} - x_5\sqrt{1+q}\right)C - y_5\sqrt{1-q}W\right),$$

$$\|\eta_s \wedge \eta_v\| = \frac{1}{2}\sqrt{x_5^2 + 2y_5^2 + z_5^2 + q\left(x_5^2 - z_5^2\right) - 2x_5z_5p}.$$

From (8), the normal vector N_η of the surface is

$$N_\eta = \frac{y_5\sqrt{1+q}N + \left(z_5\sqrt{1-q} - x_5\sqrt{1+q}\right)C - y_5\sqrt{1-q}W}{\sqrt{x_5^2 + 2y_5^2 + z_5^2 + q\left(x_5^2 - z_5^2\right) - 2x_5z_5p}}.$$

From (5), (10), (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\eta = \frac{1}{2}\left(x_5^{\,2} + y_5^{\,2} + z_5^{\,2}\right), \quad F_\eta = \frac{1}{2}\left(x_5\sqrt{1-q} + z_5\sqrt{1+q}\right), \quad G_\eta = 1,$$

$$e_\eta = \frac{\left(y_5 x_5' - x_5 y_5' - \left(y_5^{\,2} + x_5^{\,2}\right)\beta + x_5 z_5 \varphi'\right)\sqrt{1+q} + \left(z_5 y_5' - y_5 z_5' - \left(z_5^{\,2} + y_5^{\,2}\right)\varphi' + z_5 x_5 \beta\right)\sqrt{1-q}}{\sqrt{2}\sqrt{x_5^{\,2} + 2y_5^{\,2} + z_5^{\,2} + q\left(x_5^{\,2} - z_5^{\,2}\right) - 2x_5 z_5 p}},$$

$$f_\eta = \frac{y_5\omega' + z_5\beta - x_5\varphi' - \left(z_5\beta + x_5\varphi'\right)q + \left(z_5\varphi' - x_5\beta\right)p}{\sqrt{2}\sqrt{x_5^{\,2} + 2y_5^{\,2} + z_5^{\,2} + q\left(x_5^{\,2} - z_5^{\,2}\right) - 2x_5 z_5 p}}, \quad g_\eta = 0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Definition 7. The ruled surface formed by the *WD*-vector along the *WD*-Smarandache curve is defined as

$$\mathrm{F}(s,v) = \frac{W+D}{\sqrt{2+2q}} + v\frac{W+D}{\sqrt{2+2q}} = \frac{1+v}{\sqrt{2}}\left(\sqrt{1-q}N + \sqrt{1+q}W\right).$$

Theorem 7. The Gaussian and mean curvatures of the surface $\mathrm{F}(s,v)$ are

$$K_\mathrm{F} = 0,$$

$$H_\mathrm{F} = \frac{\sqrt{2}(1+v)\left(\left(y_6 x_6' - y_6^2\beta - x_6 y_6' - x_6^2\beta + x_6 z_6\varphi'\right)\sqrt{1+q} + \left(z_6 y_6' + z_6 x_6\beta - z_6^2\varphi' - y_6 z_6' - y_6^2\varphi'\right)\sqrt{1-q}\right)}{\left(x_6^{\,2} + 2y_6^{\,2} + z_6^{\,2} + q\left(x_6^{\,2} - z_6^{\,2}\right) - 2x_6 z_6 p\right)^{\frac{3}{2}}},$$

respectively, where $x_6 = \omega'\sqrt{1+q}, \quad y_6 = 2\left(\beta\sqrt{1-q} - \varphi'\sqrt{1+q}\right), \quad z_6 = -\omega'\sqrt{1-q}.$

Proof: If the partial derivatives of the surface $\mathrm{F}(s,v)$ are taken and the necessary operations are performed, then

$$\mathrm{F}_s = \frac{1+v}{\sqrt{2}}\left(x_6 N + y_6 C + z_6 W\right), \quad \mathrm{F}_v = \frac{1}{\sqrt{2}}\left(\sqrt{1-q}N + \sqrt{1+q}W\right), \quad \mathrm{F}_{sv} = \frac{1}{\sqrt{2}}\left(x_6 N + y_6 C + z_6 W\right), \quad \mathrm{F}_{vv} = 0,$$

$$\mathrm{F}_{ss} = \frac{1+v}{\sqrt{2}}\left(\left(x_6' - y_6\beta\right)N + \left(y_6' + x_6\beta - z_6\varphi'\right)C + \left(z_6' + y_6\varphi'\right)W\right)$$

$$\mathrm{F}_s \wedge \mathrm{F}_v = \frac{1+v}{2}\left(y_6\sqrt{1+q}N + \left(z_6\sqrt{1-q} - x_6\sqrt{1+q}\right)C - y_6\sqrt{1-q}W\right),$$

$$\left\|\mathrm{F}_s \wedge \mathrm{F}_v\right\| = \frac{1+v}{2}\sqrt{x_6^{\,2} + 2y_6^{\,2} + z_6^{\,2} + q\left(x_6^{\,2} - z_6^{\,2}\right) - 2x_6 z_6 p}.$$

From (8), the normal vector N_F of the surface is

$$N_\mathrm{F} = \frac{y_6\sqrt{1+q}N + \left(z_6\sqrt{1-q} - x_6\sqrt{1+q}\right)C - y_6\sqrt{1-q}W}{\sqrt{x_6^{\,2} + 2y_6^{\,2} + z_6^{\,2} + q\left(x_6^{\,2} - z_6^{\,2}\right) - 2x_6 z_6 p}}.$$

From (10) and (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\mathrm{F} = \frac{(1+v)^2}{2}\left(x_6^{\,2} + y_6^{\,2} + z_6^{\,2}\right), \quad F_\mathrm{F} = f_\mathrm{F} = g_\mathrm{F} = 0, \quad G_\mathrm{F} = 1,$$

$$e_{\mathrm{F}}=\frac{(1+v)\left(\left(y_6x_6'-y_6^2\beta-x_6y_6'-x_6^2\beta+x_6z_6\varphi'\right)\sqrt{1+q}+\left(z_6y_6'+z_6x_6\beta-z_6^2\varphi'-y_6z_6'-y_6^2\varphi'\right)\sqrt{1-q}\right)}{\sqrt{2}\sqrt{x_6^2+2y_6^2+z_6^2+q\left(x_6^2-z_6^2\right)-2x_6z_6p}}.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Corollary 3. The ruled surface $\mathrm{F}(s,v)$ is developable.

Definition 8. The ruled surface formed by the *ND*-vector along the *WD*-Smarandache curve is defined as

$$\Gamma(s,v)=\frac{W+D}{\sqrt{2+2q}}+v\frac{N+D}{\sqrt{2+2p}}=\frac{1}{\sqrt{2}}\left(\left(\sqrt{1-q}+v\sqrt{1+p}\right)N+\left(\sqrt{1+q}+v\sqrt{1-p}\right)W\right).$$

Theorem 8. The Gaussian and mean curvatures of the surface $\Gamma(s,v)$ are

$$K_{\Gamma}=-2\left(\frac{y_7\omega'+z_7\beta+x_7\varphi'+\left(z_7\beta-x_7\varphi'\right)p-\left(z_7\varphi'+x_7\beta\right)q}{x_7^2+2y_7^2+z_7^2+p\left(z_7^2-x_7^2\right)-2x_7z_7q}\right)^2,$$

$$H_{\Gamma}=\frac{\begin{pmatrix}y_7x_7'-x_7y_7'-\left(y_7^2+x_7^2\right)\beta+x_7z_7\varphi'-z_7y_7\omega'-z_7^2\beta\\-x_7z_7\varphi'-\left(z_7^2\beta-x_7z_7\varphi'\right)p+\left(z_7^2\varphi'+x_7z_7\beta\right)q\end{pmatrix}\sqrt{1-p}+\begin{pmatrix}z_7y_7'-y_7z_7'-\left(z_7^2\varphi'+y_7^2\right)\varphi'+z_7x_7\beta-x_7y_7\omega'\\-x_7z_7\beta-x_7^2\varphi'-\left(x_7z_7\beta-x_7^2\varphi'\right)p+\left(x_7z_7\varphi'+x_7^2\beta\right)q\end{pmatrix}\sqrt{1+p}}{2^{\frac{-1}{2}}\left(x_7^2+2y_7^2+z_7^2+p\left(z_7^2-x_7^2\right)-2x_7z_7q\right)^{\frac{3}{2}}}$$

respectively, where $x_7=\left(\sqrt{1-q}+v\sqrt{1+p}\right)'$, $y_7=\beta\left(\sqrt{1-q}+v\sqrt{1+p}\right)-\varphi'\left(\sqrt{1+q}+v\sqrt{1-p}\right)$, $z_7=\left(\sqrt{1+q}+v\sqrt{1-p}\right)'$.

Proof: If the partial derivatives of the surface $\Gamma(s,v)$ are taken and the necessary operations are performed, then

$$\Gamma_s=\frac{1}{\sqrt{2}}\left(x_7N+y_7C+z_7W\right),\qquad \Gamma_v=\frac{1}{\sqrt{2}}\left(\sqrt{1+p}N+\sqrt{1-p}W\right),\qquad \Gamma_{vv}=0,$$

$$\Gamma_{ss}=\frac{1}{\sqrt{2}}\left(\left(x_7'-y_7\beta\right)N+\left(y_7'+x_7\beta-z_7\varphi'\right)C+\left(z_7'+y_7\varphi'\right)W\right),$$

$$\Gamma_{sv}=\frac{1}{\sqrt{2}}\left(\left(\sqrt{1+p}\right)'N+\left(\beta\sqrt{1+p}-\varphi'\sqrt{1-p}\right)C+\left(\sqrt{1-p}\right)'W\right),$$

$$\Gamma_s\wedge\Gamma_v=\frac{1}{2}\left(y_7\sqrt{1-p}N+\left(z_7\sqrt{1+p}-x_7\sqrt{1-p}\right)C-y_7\sqrt{1+p}W\right),$$

$$\left\|\Gamma_s\wedge\Gamma_v\right\|=\frac{1}{2}\sqrt{x_7^2+2y_7^2+z_7^2+p\left(z_7^2-x_7^2\right)-2x_7z_7q}.$$

From (8), the normal vector N_{Γ} of the surface is

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$$N_{\Gamma} = \frac{y_7\sqrt{1-p}N + \left(z_7\sqrt{1+p} - x_7\sqrt{1-p}\right)C - y_7\sqrt{1+p}W}{\sqrt{x_7^2 + 2y_7^2 + z_7^2 + p\left(z_7^2 - x_7^2\right) - 2x_7z_7q}}.$$

From (5), (10), (11), the coefficients of the first and second fundamental forms of the surface are

$$E_{\Gamma} = \frac{1}{2}\left(x_7^2 + y_7^2 + z_7^2\right), \quad F_{\Gamma} = \frac{1}{2}\left(x_7\sqrt{1+p} + z_7\sqrt{1-p}\right), \quad G_{\Gamma} = 1,$$

$$E_{\Gamma}G_{\Gamma} - F_{\Gamma}^2 = \frac{1}{4}\left(x_7^2 + 2y_7^2 + z_7^2 + p\left(z_7^2 - x_7^2\right) - 2x_7z_7q\right),$$

$$e_{\Gamma} = \frac{\left(y_7x_7' - x_7y_7' - \left(y_7^2 + x_7^2\right)\beta + x_7z_7\varphi'\right)\sqrt{1-p} + \left(z_7y_7' - y_7z_7' - \left(z_7^2\varphi' + y_7^2\right)\varphi' + z_7x_7\beta\right)\sqrt{1+p}}{\sqrt{2}\sqrt{x_7^2 + 2y_7^2 + z_7^2 + p\left(z_7^2 - x_7^2\right) - 2x_7z_7q}},$$

$$f_{\Gamma} = \frac{y_7\omega' + z_7\beta + x_7\varphi' + \left(z_7\beta - x_7\varphi'\right)p - \left(z_7\varphi' + x_7\beta\right)q}{\sqrt{2}\sqrt{x_7^2 + 2y_7^2 + z_7^2 + p\left(z_7^2 - x_7^2\right) - 2x_7z_7q}}, \quad g_{\Gamma} = 0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Definition 9. The ruled surface formed by the *CD*-vector along the *WD*-Smarandache curve is defined as

$$\zeta(s,v) = \frac{W+D}{\sqrt{2+2q}} + \frac{v}{\sqrt{2}}(C+D) = \frac{1}{\sqrt{2}}\left(\left(\sqrt{1-q} + vp\right)N + vC + \left(\sqrt{1+q} + vq\right)W\right).$$

Theorem 9. The Gaussian and mean curvatures of the surface $\zeta(s,v)$ are

$$K_{\zeta} = -2\left(\frac{(y_8q - z_8)(p' - \beta) + (x_8 - y_8p)(q' + \varphi')}{(y_8q - z_8)^2 + (z_8p - x_8q)^2 + (x_8 - y_8p)^2}\right)^2,$$

$$H_{\zeta} = \frac{\begin{array}{c}\Big[(y_8q - z_8)\left(x_8' - y_8\beta - (x_8p + y_8 + z_8q)(p' - \beta)\right) + (z_8p - x_8q)\left(y_8' + x_8\beta - z_8\varphi'\right) \\ + (x_8 - y_8p)\left(z_8' + y_8\varphi' - (x_8p + y_8 + z_8q)(q' + \varphi')\right)\Big]\end{array}}{2^{\frac{-1}{2}}\left((y_8q - z_8)^2 + (z_8p - x_8q)^2 + (x_8 - y_8p)^2\right)^{\frac{3}{2}}},$$

respectively, where $x_8 = \left(\sqrt{1-q} + vp\right)' - v\beta$, $\quad y_8 = \beta\left(\sqrt{1-q} + vp\right) - \varphi'\left(\sqrt{1+q} + vq\right)$,

$z_8 = \left(\sqrt{1+q} + vq\right)' + v\varphi'$.

Proof: If the partial derivatives of the surface $\zeta(s,v)$ are taken and the necessary operations are performed, then

$$\zeta_s = \frac{1}{\sqrt{2}}\left(x_8N + y_8C + z_8W\right), \quad \zeta_{sv} = \frac{1}{\sqrt{2}}\left((p' - \beta)N + (q' + \varphi')W\right),$$

$$\zeta_v = \frac{1}{\sqrt{2}}\left(pN + C + qW\right), \quad \zeta_{ss} = \frac{1}{\sqrt{2}}\left(\left(x_8' - y_8\beta\right)N + \left(y_8' + x_8\beta - z_8\varphi'\right)C + \left(z_8' + y_8\varphi'\right)W\right), \quad \zeta_{vv} = 0,$$

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$$\zeta_s \wedge \zeta_v = \frac{1}{2}\left(\left(y_8 q - z_8\right)N + \left(z_8 p - x_8 q\right)C + \left(x_8 - y_8 p\right)W\right),$$

$$\left\|\zeta_s \wedge \zeta_v\right\| = \frac{1}{2}\sqrt{\left(y_8 q - z_8\right)^2 + \left(z_8 p - x_8 q\right)^2 + \left(x_8 - y_8 p\right)^2}.$$

From (8), the normal vector N_ζ of the surface is

$$N_\zeta = \frac{\left(y_8 q - z_8\right)N + \left(z_8 p - x_8 q\right)C + \left(x_8 - y_8 p\right)W}{\sqrt{\left(y_8 q - z_8\right)^2 + \left(z_8 p - x_8 q\right)^2 + \left(x_8 - y_8 p\right)^2}}.$$

From (10) and (11), the coefficients of the first and second fundamental forms of the surface are

$$E_\zeta = \frac{1}{2}\left(x_8^{\,2} + y_8^{\,2} + z_8^{\,2}\right), \quad F_\zeta = \frac{1}{2}\left(x_8 p + y_8 + z_8 q\right), \quad G_\zeta = 1,$$

$$E_\zeta G_\zeta - F_\zeta^2 = \frac{1}{4}\left(\left(y_8 q - z_8\right)^2 + \left(z_8 p - x_8 q\right)^2 + \left(x_8 - y_8 p\right)^2\right),$$

$$e_\zeta = \frac{\left(y_8 q - z_8\right)\left(x_8' - y_8\beta\right) + \left(z_8 p - x_8 q\right)\left(y_8' + x_8\beta - z_8\varphi'\right) + \left(x_8 - y_8 p\right)\left(z_8' + y_8\varphi'\right)}{\sqrt{2}\sqrt{\left(y_8 q - z_8\right)^2 + \left(z_8 p - x_8 q\right)^2 + \left(x_8 - y_8 p\right)^2}},$$

$$f_\zeta = \frac{\left(y_8 q - z_8\right)\left(p' - \beta\right) + \left(x_8 - y_8 p\right)\left(q' + \varphi'\right)}{\sqrt{2}\sqrt{\left(y_8 q - z_8\right)^2 + \left(z_8 p - x_8 q\right)^2 + \left(x_8 - y_8 p\right)^2}}, \quad g_\zeta = 0.$$

If these expressions are substituted in (9), the Gaussian and mean curvatures are found.

Example 1. The Frenet vectors $\{T, N, B\}$, the unit Darboux vector C and the curvatures κ, τ of Viviani's curve $\alpha(s) = \left(\cos^2(s), \cos(s)\sin(s), \sin(s)\right)$ are

$$\begin{cases} T(s) = \dfrac{2\left(-\sin(2s), \cos(2s), \cos(s)\right)}{\sqrt{2\cos(2s)+6}}, \\ N(s) = -\dfrac{\left(\cos(4s)+12\cos(2s)+3, \sin(4s)+12\sin(2s), 4\sin(s)\right)}{\sqrt{6\cos(4s)+88\cos(2s)+162}}, \\ B(s) = \dfrac{\left(\sin(3s)+3\sin(s), -\cos(3s)-3\cos(s), 4\right)}{\sqrt{6\cos(2s)+26}}, \end{cases}$$

$$C(s) = \frac{\begin{pmatrix} -3\sin(5s)-7\sin(3s)+36\sin(s), 3\cos(5s)+7\cos(3s)-42\cos(s), \\ 6\cos(4s)+72\cos(2s)+146 \end{pmatrix}}{\sqrt{2\left(9\cos(8s)+207\cos(6s)+2034\cos(4s)+10593\cos(2s)+12757\right)}},$$

$$\kappa(s) = \frac{2\sqrt{3cos(2s)+13}}{\left(cos(2s)+3\right)^{\frac{3}{2}}}, \qquad \tau(s) = \frac{12cos(s)}{3cos(2s)+13}.$$

For $s \in [-\pi, \pi]$ and $v \in [-1, 1]$, the ruled surfaces $\Sigma(s,v)$, $\Delta(s,v)$, $\Upsilon(s,v)$, $\chi(s,v)$, $\delta(s,v)$, $\eta(s,v)$, $\mathrm{F}(s,v)$, $\Gamma(s,v)$ and $\zeta(s,v)$ are drawn in Figure 1.

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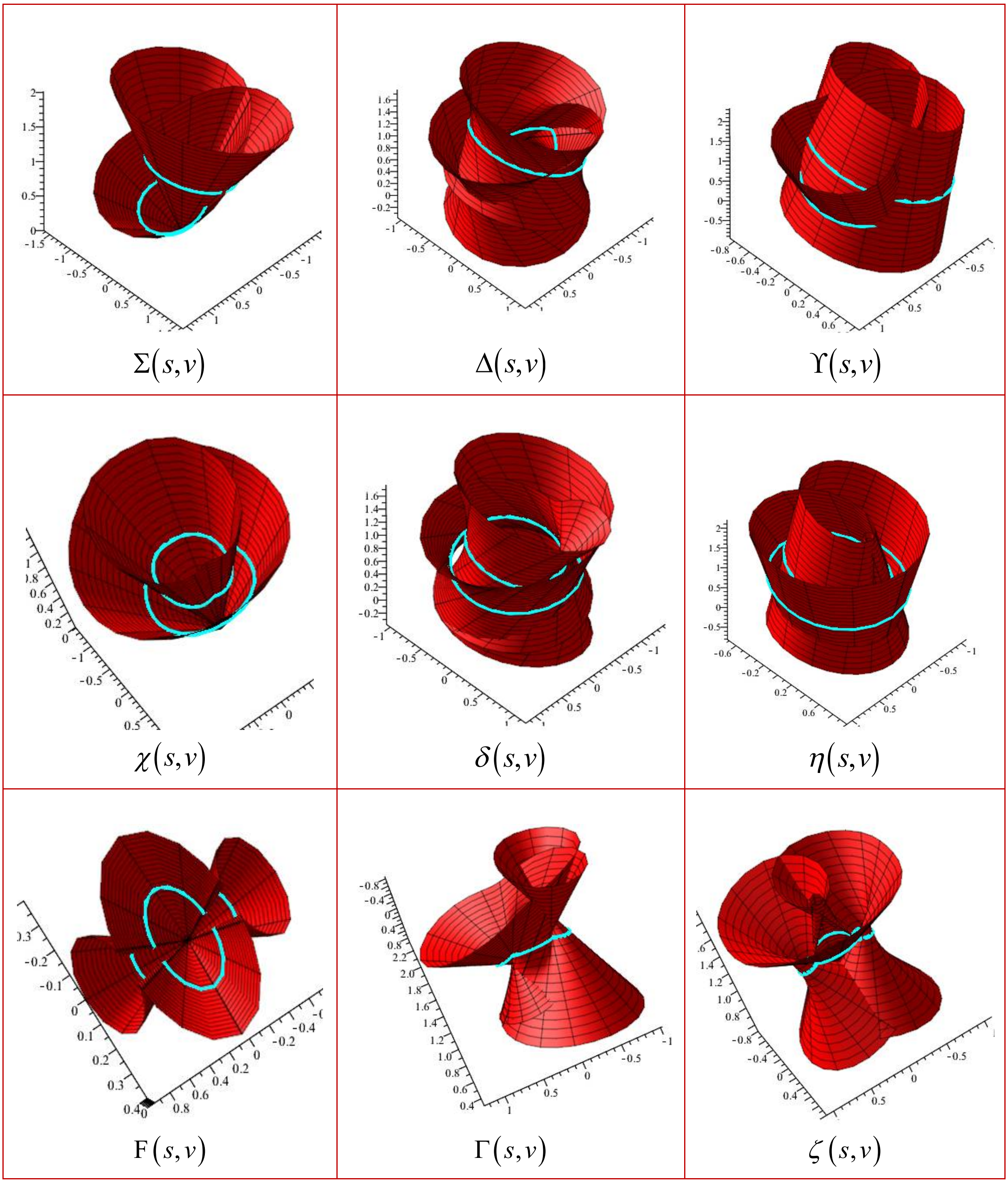

Figure 1. Ruled surfaces $\Sigma(s,v)$, $\Delta(s,v)$, $\Upsilon(s,v)$, $\chi(s,v)$, $\delta(s,v)$, $\eta(s,v)$, $\mathrm{F}(s,v)$, $\Gamma(s,v)$ and $\zeta(s,v)$

CONCLUSIONS

Smarandache curves formed from alternative unit vectors and Darboux vector of any curve have been defined. Then considering the direction vectors obtained from these vectors, the Gaussian and mean curvatures of the ruled surfaces formed along these curves have been examined.

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