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Statistically order continuous operators on Riesz spaces

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Abstract: Some kinds of statistical convergence have been studied and investigated on Riesz spaces with respect to order convergence recently. In this paper we introduce the concept of statistically continuous and bounded operators with statistically order convergent sequences on Riesz spaces. Moreover, we give some relations with other kinds of operators.

Keywords: vector lattice, statistically continuous operator, statistically bounded operator, statistical order convergence

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INTRODUCTION

Statistical convergence is a natural and efficient tool in the theory of functional analysis, and it was introduced by Fast [1] and Steinhaus [2] independently. Also, several applications and generalisations of the statistical convergence of sequences have been investigated by several authors [3-7]. On the other hand, Riesz space (or vector lattice) introduced by Riesz [8] is an ordered vector space which has many applications in measure theory, operator theory, and economics [9-12]. It is well known that order, unbounded order, relatively uniform, and various multiplicative order convergences in Riesz algebras are not topological in general [13]. However, even without using any topological structure, several kinds of continuous operators can be defined [14]. As far as we know, there is no comprehensive study of operator theory with statistical convergence. Our aim in this paper is to put forth and study statistically σ-order continuous and bounded operators on Riesz spaces with respect to the statistical order convergence. The results obtained in the settings of statistical convergence on Riesz spaces will shed light on the case of operators on Riesz spaces and more general settings [15,16]. Throughout the paper, we always assume that all Riesz spaces are real and operators are linear.

It is recalled that a sequence (x_n) on a Riesz space E is *order convergent* (or, shortly, *o-convergent*) to $x \in E$ if there exists another sequence $y_n \downarrow \theta$ in E such that $|x_n - x| \leq y_n$ for all

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 1-9

$n \in \mathbb{N}$. In this case we write $x_n \xrightarrow{o} x$. We refer to an alternative type of order convergence [17]. On the other hand, an operator T between Riesz spaces E and F is called

1) *order bounded* if T(A) is order bounded in F for each order bounded subset A of E;
2) *order continuous* if $Tx_\alpha \xrightarrow{o} Tx$ whenever $x_\alpha \xrightarrow{o} x$;
3) *σ-order continuous* if $Tx_n \xrightarrow{o} Tx$ whenever $x_n \xrightarrow{o} x$.

The collection $L_b(E,F)$ denotes the set of all order bounded operators between Riesz spaces E and F. It is clear that $L_b(E,F)$ is a vector space. It is also well known that if F is a Dedekind complete Riesz space, then $L_b(E,F)$ is also a Dedekind complete Riesz space; see for example Theorem 1.18 [10]. In this case the collection $L_c(E,F)$ of all σ-order continuous operators from E to F is a band in $L_b(E,F)$. Note that every order bounded operator is σ-order continuous whenever it is order continuous [10], yet the converse need not be true in general. To see this, consider Example 1.55. [10].

Now we recall some basic properties of the concepts related to statistical convergence. The *natural density* of a subset K of $\mathbb{N}$ is defined (if exists) by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where $|A|$ stands for the cardinality of a set A. It is easy to see that $\delta(K)$ does not exist for a set $K = \cup_{n=1}^{\infty} ([2^{2n}, 2^{2n+1}] \cap \mathbb{N})$. We refer to an exposition on the natural density of sets [18-24]. In the same way a sequence $x = (x_k)$ is called *statistically convergent* to L provided that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. Then it is written as $st\text{-}\lim x_k = L$. A sequence (x_n) on a Riesz space is called

1) *statistically order decreasing* to zero if there exists a set $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that (x_{k_n}) is decreasing and $\inf_{k_n \in K}(x_{k_n}) = \theta$, and so it is abbreviated as $x_n \downarrow^{st_o} \theta$;
2) *statistically order convergent* to x if there exists a sequence $q_n \downarrow^{st_o} \theta$ with a set K such that $\delta(K) = 1$ and $|x_{k_n} - x| \leq q_{k_n}$ for every $k_n \in K$, and so we write it as $x_n \xrightarrow{st_o} x$.

It is well known that order convergence implies statistical order convergence [16, p.7].

STATISTICAL CONTINUOUS OPERATORS

In this section we introduce the concepts of statistically order bounded and statistically σ-order continuous operators. We begin with the following basic notion of this present paper.

Definition 1. Let (x_n) be a sequence on a Riesz space E. Then it is called *statistically order bounded sequence* whenever there exist a positive vector $w \in E$ and an index set $K = \{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $|x_{k_n}| \leq w$ for every $k_n \in K$, and so we say that (x_n) is st_o-bounded.

It is clear that every order bounded sequence is st_o-bounded. Moreover, a statistically order convergent sequence is also st_o-bounded. But their converse need not be true in general. To see these, we give the following two examples.

Example 1. Let us consider the Riesz space c_0 of all real null sequences and we define a sequence (x_n) in c_0 by

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 1-9

$$x_n := \begin{cases} e_n, & \text{if } n = m^3 \text{ for some } m \in \mathbb{N} \\ \theta, & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$, where θ is the real zero sequence $(0,0,\cdots)$ and e_n is the standard unit vector. Then it is clear that (x_n) is st_o-bounded, but not order bounded in c_0.

Example 2. Consider the Riesz space E consisting of sequences that have only a finite different value. We take the sequence $(x_n) = (x_1^k, x_2^k, \cdots)$ in E defined by

$$x_n^k := (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{n}, 0, 0, \cdots)$$

for all $n \in \mathbb{N}$ and every $k \in \mathbb{N}$. Thus, we get $\theta \leq x_n^k \leq \mathbb{1}$ for all n and k, where $\mathbb{1}$ denotes the sequence identically equal to 1. Therefore, the sequence (x_n) is order bounded, and so it is st_o-bounded. But it is not statistically order convergent.

Definition 2. Let E and F be two Riesz spaces and $T: E \to F$ be an operator. Then T is called

1) *statistically σ-order continuous* if $Tx_n \xrightarrow{st_o} Tx$ holds in F for every $x_n \xrightarrow{st_o} x$ in E;
2) *statistically order bounded* if (Tx_n) is a statistically order bounded sequence in F for every statistically order bounded sequence (x_n) in E.

Note that the notation σ is used for sequences on Riesz spaces in general. Now let B be a projection band on a Riesz space E. Thus, $E = B \oplus B^d$ holds, and so every vector $x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. Then a projection $P_B: E \to E$ is defined via the formula $P_B(x) = x_1$, which is called *a band projection*. Thus, the band projection is associated with the *projection band B*.

Example 3. If P_B is the band projection corresponding to a projection band B on a Riesz space E, then P_B is a statistically σ-order continuous and statistically order bounded operator. Indeed, assume that $x_n \xrightarrow{st_o} x$ holds in E. Then there exists a sequence $q_n \downarrow^{st_o} \theta$ with index set $\delta(K) = 1$ such that $|x_{k_n} - x| \leq q_{k_n}$ holds for all $k_n \in K$. On the other hand, it is well known that P_B is a lattice homomorphism satisfying $\theta \leq P_B \leq I$ [9]. Then it follows from the inequality

$$|P_B x_{k_n} - P_B x| = P_B(|x_{k_n} - x|) \leq |x_{k_n} - x| \leq q_{k_n}$$

for all $k_n \in K$ that we obtain $P_B x_{k_n} \xrightarrow{o} P_B x$ on K. Therefore, we have $P_B x_n \xrightarrow{st_o} P_B x$ and P_B is a statistically σ-order continuous operator as desired. The statistically order bounded case is analogous.

Proposition 1. Every order bounded operator is statistically order bounded.

Proof: Let $T: E \to F$ be an order bounded operator between Riesz spaces E and F. If (x_n) is an st_o-bounded sequence in E, then there exist a positive vector $w \in E_+$ and an index set $\delta(K) = 1$ such that $|x_{k_n}| \leq w$ for every $k_n \in K$. Thus, (Tx_{k_n}) is an order bounded sequence in F because T is an order bounded operator. Therefore, (Tx_n) is a statistically order bounded sequence in F as desired.

It is well known that every order continuous operator is order bounded; see for example Lemma 1.54 [10]. Thus, Proposition 1 implies that every order continuous operator is statistically order bounded. Also, it is clear that the converse of Proposition 1 need not be true in general.

Proposition 2. Every σ-order continuous operator is statistically σ-order continuous.

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 1-9

Proof: Let T be a σ-order continuous operator between Riesz spaces E and F. Assume that $x_n \xrightarrow{st_o} x$ holds in E. Then there exists a sequence $q_n \downarrow^{st_o} \theta$ with an index set $\delta(K) = 1$ such that $|x_{k_n} - x| \leq q_{k_n}$ for all $k_n \in K$ (that is, $x_{k_n} \xrightarrow{o} x$ holds on K). Thus, by using the σ-order continuity of T, we have $Tx_{k_n} \xrightarrow{o} Tx$ in F. Therefore, we get $Tx_n \xrightarrow{st_o} Tx$ because order convergence implies statistical order convergence.

It is clear that the converse of Proposition 2 does not satisfy in general because statistical order convergence does not imply order convergence; see Example 3 [16].

Example 4. Consider the Riesz spaces $E = c$ and $F = \mathbb{R}$, where c is the set of all convergent real sequences. We define the order bounded operator $T: E \to F$ by $T\alpha = \lim_{k\to\infty}\alpha_k$ for every element $(\alpha_k) \in c$. So T is not statistically σ-order continuous. Indeed, $\alpha_n = 1_{m\geq n} \xrightarrow{o} 0$ and hence $\alpha_n \xrightarrow{st_o} 0$, yet $T\alpha_n \equiv 1 \xrightarrow{st_o} 1 \neq 0$.

Proposition 3. Let E and F be Riesz spaces. Then the set $St_c(E, F)$ of all statistically σ-order continuous operators from E to F is a vector space.

Proof: Let $S, T \in St_c(E, F)$ and $\beta \in \mathbb{R}$. Suppose that $x_n \xrightarrow{st_o} x$ holds in E. It follows that $Sx_n \xrightarrow{st_o} Sx$ and $Tx_n \xrightarrow{st_o} Tx$. Then there exist sequences $p_n \downarrow^{st_o} \theta$ and $q_n \downarrow^{st_o} \theta$ with densities $\delta(K) = \delta(M) = 1$ of index sets such that

$$|Sx_{m_n} - Sx| \leq q_{m_n} \text{ and } |Tx_{k_n} - Tx| \leq p_{k_n}$$

hold for all $m_n \in M$ and $k_n \in K$. Thus, we have the following inequality:

$$|(\beta S + T)x_{j_n} - (\beta S + T)x| \leq |\beta||Sx_{j_n} - Sx| + |Tx_{j_n} - Tx| \leq |\beta|q_{j_n} + p_{j_n}$$

for all $j_n \in J := M \cap K$. Since $\delta(J) = 1$ and $(|\beta|q_{j_n} + p_{j_n}) \downarrow \theta$ on the index set J, we get $(\beta S + T)x_n = \beta Sx_n + Tx_n \xrightarrow{st_o} \beta Sx + Tx = (\beta S + T)x$ as desired.

We show that $St_c(E, F) = \{\theta\}$ is possible in the next example.

Example 5. Let $T: C[0,1] \to L_2[0,1]$ be a positive statistically σ-order continuous operator. Then for any fixed $0 \leq h \in L_2[0,1]$, we define a function G from $C[0,1]$ to $\mathbb{R}$ by

$$G(f) := \int_0^1 h(x)[Tf(x)]dx$$

for each $f \in C[0,1]$. Then G is statistically σ-order continuous. Thus, by applying Example 1.58 [10], we obtain $\int_0^1 h(x)[Tf(x)]dx = 0$ for every $h \in L_2[0,1]$ and all $f \in C[0,1]$. Hence we get $T = \theta$ as desired.

It is recalled that every $T \in L_b(E, F)$ has modulus $|T|$ whenever F is Dedekind complete; see for example Theorem 1.67 [9]. It is clear from the formula $|T(x)| \leq |T|(|x|)$ that if an operator T has a statistically σ-order continuous modulus $|T|$ (correspondingly statistically order bounded modulus $|T|$), then T is also statistically σ-order continuous (correspondingly statistically order bounded).

Proposition 4. If $T: E \to F$ is a statistically order bounded operator which has the modulus $|T|$, then $|T|$ is also statistically order bounded.

Proof: Let (x_n) be a statistically order bounded sequence in E. Then there exist a positive vector $w \in E$ and an index set $\delta(K) = 1$ such that $|x_{k_n}| \leq w$ for every $k_n \in K$. It follows from Lemma 1.6. [10] that

$$||T|(x_{k_n})| \leq |T|(|x_{k_n}|) \leq |T|(w) \in F_+$$

holds for each $k_n \in K$ because $|T|$ is a positive operator. Therefore, we obtain the desired result.

Question 1. Let $T: E \to F$ be a statistically σ-order continuous operator which has the modulus $|T|$. Is $|T|$ statistically σ-order continuous?

Recall that an operator $S: E \to F$ between Riesz spaces is said to be *dominated* if there is a positive operator $T: E \to F$ satisfying $|Sx| \leq T|x|$ for all $x \in E$. Then T is called a *dominant* for S.

Proposition 5. Let $T: E \to F$ be a statistically σ-order continuous (correspondingly statistically order bounded) positive operator and it is dominant for an operator $S: E \to F$. Then S is statistically σ-order continuous (correspondingly statistically order bounded).

Proof: Suppose that $x_n \xrightarrow{st_o} x$ holds in E. Then there exists a sequence $q_n \downarrow^{st_o} \theta$ with an index set $\delta(K) = 1$ such that $|x_{k_n} - x| \leq q_{k_n}$ for every $k_n \in K$. Therefore, we obtain the desired result by the inequality $|Sx_{k_n} - Sx| \leq T|x_{k_n} - x|$. The order bounded case is also similar.

MAIN RESULTS

We have a partial answer to Question 1 in the following theorem.

Theorem 1. Let $T: E \to \mathbb{R}$ be an order bounded linear functional. Then the following statements are equivalent:

(i) T is statistically σ-order continuous;
(ii) T^+ is statistically σ-order continuous;
(iii) T^- is statistically σ-order continuous;
(iv) |T| is statistically σ-order continuous.

Proof: $(i) \Longrightarrow (ii)$ Assume that $x_n \xrightarrow{st_o} \theta$ holds in E. By applying Theorem 1.18 [10], we have $T^+x = \sup\{Ty: \theta \leq y \leq x\}$ because $\mathbb{R}$ is a Dedekind complete Riesz space. Now take a sequence $t_n \downarrow \theta$ in $\mathbb{R}$. Thus, for each $n \in \mathbb{N}$, we can find an element $y_n \in E$ such that $\theta \leq y_n \leq x_n$ and $T^+(x_n) - t_n \leq Ty_n$. Hence we have $T^+x_n \leq t_n + Ty_n$ for each n. On the other hand, by using $x_n \xrightarrow{st_o} \theta$, we get $y_n \xrightarrow{st_o} \theta$. It follows from the statistical σ-order continuity of T that $Ty_n \xrightarrow{st_o} \theta$, and so we get $(t_n + Ty_n) \xrightarrow{st_o} \theta$ because $t_n \downarrow \theta$ implies $t_n \xrightarrow{st_o} \theta$. Thus, we have $T^+x_n \xrightarrow{st_o} \theta$. Therefore, T^+ is statistically σ-order continuous as desired.

$(ii) \Longrightarrow (iii)$ We can get the statistical order continuity of T^- by the formula $T^- = (-T)^+$.

$(iii) \Longrightarrow (iv)$ It follows from $|T| = T^+ + T^-$ and $T^+ = (-T)^-$ that we obtain the desired result.

$(iv) \Longrightarrow (i)$ Since $|T|$ is a positive operator and a dominant for T, it follows from Proposition 5 that T is statistically σ-order continuous whenever $|T|$ is statistically σ-order continuous.

Proposition 6. Let T: E → F be a positive and statistically σ-order continuous operator. Then T is σ-order continuous.

Let us give the following remark before the proof of the proposition.

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 1-9

Remark 1. Any monotone $\mathrm{st_o}$-convergent sequence order converges to its $\mathrm{st_o}$-limit on Riesz spaces. Indeed, assume that $x_n \downarrow$ and $x_n \xrightarrow{st_o} x$ on a Riesz space E and fix an arbitrary $m \in \mathbb{N}$. So we have $x_m - x_n \in E_+$ for all $n \geq m$. It follows from Theorem 6 [16] that $x_m - x_n \xrightarrow{st_o} x_m - x \in E_+$ (that is, $x_m \geq x$). Thus, x is a lower bound of (x_n) because m is arbitrary. If $x_n \geq y$ for all n, again by using Theorem 6 [16], we have $x_n - y \xrightarrow{st_o} x - y \in E_+$ (that is, $x \geq y$). Thus, we obtain $x_n \downarrow x$ as desired.

Proof: Assume that $T: E \to F$ is a positive operator and $x_n \downarrow \theta$ holds in E. Then we have $x_n \xrightarrow{st_o} \theta$ [16]. Now we get $Tx_n \xrightarrow{st_o} \theta$ by assumption. Thus, by applying Remark 1, it follows from $Tx_n \downarrow$ that we have $Tx_n \downarrow \theta$. Thus, T is σ-order continuous [9].

Theorem 2. Let $T: E \to F$ be a positive operator and F be a Dedekind complete Riesz space. Then the operator $\tilde{T}$ from E to F defined by

$$\tilde{T}(x) := \inf\{\sup T(x_n) : \theta \leq x_n \uparrow \text{ and } x_n \xrightarrow{st_o} x\}$$

for each $x \in E_+$ is statistically σ-order continuous.

Proof: Suppose that $\theta \leq x_n \uparrow$ and $x_n \xrightarrow{st_o} x$ holds in E. Then there exists a sequence $q_n \downarrow^{st_o} \theta$ with an index set $\delta(K) = 1$ such that $|x_{k_n} - x| \leq q_{k_n}$ for all $k_n \in K$ (that is, $x_{k_n} \xrightarrow{o} x$ on K). Take an arbitrary $\varepsilon > 0$ and fix $k_n \in K$. It follows from Theorem 1.66 [9] that we can define an operator $T_{k_n}: E^+ \to F^+$ by

$$T_{k_n}(u) = \sup\{Tv : v \in I_{k_n} \text{ and } \theta \leq v \leq u\}$$

for every $u \in E_+$ and for each $k_n \in K$. Thus, it coincides with T on the ideal I_{k_n} generated by $(\varepsilon x - x_{k_n})^+$, and so it vanishes on $(\varepsilon x - x_{k_n})^-$. Hence it is clear that $\theta \leq T_{k_n} \downarrow \leq T$ and $T_{k_n}(x_{k_n} - \varepsilon x)^+ = \theta$ holds for each $k_n \in K$. Now assume that $T_n \xrightarrow{st_o} G$ (that is, $T_{k_n} \downarrow G$ holds in $\mathcal{L}_b(E,F)$. Then, by considering $\theta \leq (x_{k_n} - \varepsilon x)^+ \uparrow (x - \varepsilon x) = (1-\varepsilon)x$ and $G(x_{k_n} - \varepsilon x)^+ = \theta$ for every $k_n \in K$, we have $\tilde{G}(x) = \inf\{\sup G(x_{k_n}) : \theta \leq x_n \uparrow \text{ and } x_n \xrightarrow{st_o} x\} = \theta$. On the other hand, it follows from the inequality $\theta \leq x - x_{k_n} \leq (1-\varepsilon)x + (\varepsilon x - x_{k_n})^+$ that

$$\theta \leq \tilde{T}(x - x_{k_n}) \leq (1-\varepsilon)\tilde{T}x + \tilde{T}(\varepsilon x - x_{k_n})^+. \qquad (*)$$

Therefore, we have

$$\begin{aligned}\tilde{T}(\varepsilon x - x_{k_n})^+ &= \inf\{\sup(Tz_n) : \theta \leq z_n \uparrow \text{ and } z_n \xrightarrow{st_o} (\varepsilon x - x_{k_n})^+\} \\ &= \inf\{\sup(T_{k_n} z_n) : \theta \leq z_n \uparrow \text{ and } z_n \xrightarrow{st_o} (\varepsilon x - x_{k_n})^+\} \\ &= \tilde{T}_{k_n}(\varepsilon x - x_{k_n})^+ \\ &\leq \tilde{T}_{k_n} x\end{aligned}$$

because $\theta \leq z \leq (\varepsilon x - x_{k_n})^+$ implies $Tz = T_{k_n} z$. Hence the equation $(*)$ implies the inequality

$$\theta \leq \tilde{T}(x - x_{k_n}) \leq (1-\varepsilon)\tilde{T}x + \tilde{T}_{k_n} x. \qquad (**)$$

By considering Theorem 1.59 [10], the operator $T \to \tilde{T}$ is a σ-order continuous operator. Thus, $T_{k_n} \downarrow G$ implies that $\tilde{T}_{k_n} \downarrow \tilde{G}$. So it follows from Theorem VII. 2.3. [25] that $\tilde{T}_{k_n}(x) \downarrow \tilde{G}(x) = \theta$.

Maejo Int. J. Sci. Technol. **2023**, *17*(01), 1-9

Therefore, the inequality (∗∗) implies that $\theta \leq \inf_{k_n}\{\tilde{T}(x - x_{k_n})\} \leq (1-\varepsilon)\tilde{T}x$ holds. Hence we obtain $\tilde{T}(x - x_{k_n}) \downarrow \theta$. That is, $\tilde{T}(x - x_n) \xrightarrow{st_o} \theta$ holds as desired.

Recall that an order ideal A of a Riesz space is said to be a *σ-ideal* whenever a sequence (x_n) in A satisfying $0 \leq x_n \uparrow x$ implies $x \in A$. On the other hand, an operator T between Riesz spaces E and F is said to be a *lattice homomorphism* if $T(x \vee y) = T(x) \vee T(y)$ holds for all vectors x and y in E.

Theorem 3. Let $T: E \to F$ be an onto lattice homomorphism. Then T is a statistically σ-order continuous operator if and only if the subset $Ker(T) = \{x \in E: Tx = \theta\}$ of E is a σ-ideal in E.

Proof: Assume that (x_n) is a sequence in $Ker(T)$ with $\theta \leq x_n \uparrow x$. Thus, we have $x_n \xrightarrow{st_o} x$. It is well known that the kernel of a lattice homomorphism is an order ideal; see for example Theorem 1.31 [9]. So it follows from the statistical σ-order continuity of T that we have $T(x_n) \xrightarrow{st_o} T(x)$. That is, $T(x_{k_n}) \xrightarrow{o} T(x)$. It follows that $T(x) = \theta$, and so we obtain $x \in Ker(T)$. Therefore, $Ker(T)$ is a σ-ideal in E.

For the converse, suppose that $Ker(T)$ is a σ-ideal in E. We show that T is a statistically σ-order continuous operator. Take a sequence $x_n \xrightarrow{st_o} \theta$ in E. Then there exists a sequence $q_n \downarrow^{st_o} \theta$ with an index set $\delta(K) = 1$ such that $|x_{k_n}| \leq q_{k_n}$ for all $k_n \in K$, and so $|Tx_{k_n}| = T|x_{k_n}| \leq Tq_{k_n}$ holds for all $k_n \in K$ because T is positive. Assume that $\theta \leq w \leq Tq_{k_n}$ holds for every $k_n \in K$. Then we have $Tz = w$ for some $z \in E_+$ because T is an onto lattice homomorphism. So we have $\theta \leq (z - q_{k_n})^+ \xrightarrow{st_o} z$, and so we obtain $T(z - q_{k_n})^+ = (Tz - Tq_{k_n})^+ = (w - Tq_{k_n})^+ = \theta$. Hence we get $(z - q_{k_n})^+ \in Ker(T)$ for every $k_n \in K$. Hence we obtain $(z - q_{k_n})^+ \in Ker(T)$ for every $k_n \in K$. Now, by using the σ-idealness of $Ker(T)$, we have $z \in Ker(T)$. Thus, we get $w = Tz = \theta$. Therefore, $Tq_{k_n} \xrightarrow{o} \theta$, and so $Tx_n \xrightarrow{st_o} \theta$.

It is reminded that the adjoint of order bounded operator between two Riesz spaces is an order bounded and order continuous operator; see for example Theorem 1.73 [10]. Also, the vector space $E^\sim$ of all order bounded linear functionals on E is called the *order dual* of E. For a Riesz space E, we denote

$$E_{st}^\sim := \{f: E \to \mathbb{R} : f \text{ is statistically } \sigma - \text{order continuous}\}$$

as the statistically σ-order continuous dual of E. The following theorem deals with a similar situation with the adjoint of order bounded operator.

Theorem 4. If $T: E \to F$ is an order bounded and statistically σ-order continuous operator, then the operator $T^\sim: F^\sim \to E_{st}^\sim$ defined by $T^\sim(f)(x) := f(Tx)$ is statistically σ-order continuous, where $F^\sim$ is the σ-order continuous dual of F.

Proof: Let us firstly show that $T^\sim(f) \in E_{st}^\sim$ for each $f \in F^\sim$. Suppose that $x_n \xrightarrow{st_o} \theta$ holds in E. It follows from the statistical order continuity of T that we have $Tx_n \xrightarrow{st_o} \theta$ in F. Since f is σ-order continuous, $f(Tx_n) \xrightarrow{st_o} \theta$ or $(f \circ T)(x_n) \xrightarrow{st_o} \theta$. Thus, we get $f \circ T \in E_{st}^\sim$. Now suppose that $\theta \leq f_n \xrightarrow{st_o} \theta$ in $F^\sim$. We show $T^\sim f_n \xrightarrow{st_o} \theta$ or $f_n \circ T \xrightarrow{st_o} \theta$. Let $x \in E_+$ be fixed; thus, for some $u \in F_+$, we have $|Ty| \leq u$ for all $|y| \leq x$. Hence we obtain $[|T^\sim|(f)](x)) \leq f(u)$ or $f(|T|(x)) \leq f(u)$ holds for all positive $f \in F^{\sim}$. Therefore, it follows that $f_n(|T|(x)) \leq f_n(u)$ holds for all n. On the other hand, there exists a sequence $g_n \downarrow^{st_o} \theta$ in $F^{\sim}$ with a set $\delta(K) = 1$ such that $f_{k_n} \leq g_{k_n}$ for all $k_n \in K$ (that is, $f_{k_n} \xrightarrow{o} \theta$ on K) because of $f_n \xrightarrow{st_o} \theta$. Thus, by applying Theorem $VIII.2.3$ [25], we have $g_{k_n}(u) \downarrow \theta$. Therefore, we get $f_{k_n}(|T|(x)) \downarrow \theta$ or $[|T^{\sim}|f_{k_n}](x) \downarrow \theta$. That is, $|T^{\sim}|$ is statistically σ-order continuous. So it follows from Theorem 1 that $T^{\sim}$ is also statistically σ-order continuous.

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Maejo Int. J. Sci. Technol. **2023**, *17*(01), 1-9

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