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Technical Note

Palatini approach to R^{-2} gravity

Tritos Ngampitipan

Physics Programme, Faculty of Science, Chandrakasem Rajabhat University, Bangkok 10900, Thailand

E-mail: tritos.ngampitipan@gmail.com

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Abstract: An $f(R)$ gravity, where the Ricci scalar (R) term in the Einstein-Hilbert action of general relativity is generalised to its function $f(R)$, is an alternative to a dark energy model to explain the current cosmic acceleration. In this paper the R^{-2} gravity, in which a term proportional to R^{-2} is added to the R term in the Einstein-Hilbert action, is explored in the Palatini approach. The results show that when the proportionality constant is negative, the R^{-2} gravity can give the de-Sitter universe, where the universe has the exponentially accelerating expansion, after the matter-dominated era without introducing dark energy.

Keywords: current cosmic acceleration, Einstein-Hilbert action, $f(R)$ gravity, Palatini approach, R^{-2} gravity

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INTRODUCTION

Nowadays our universe is expanding with acceleration [1, 2]. However, the mechanism accounting for this acceleration remains unclear. One popular way to explain the cosmic acceleration is to introduce an exotic matter called dark energy [3-5]. In addition to dark energy, another possible way to account for this current acceleration is to modify Einstein's general relativity. An example of such modification is to generalise the Ricci scalar (R) term in the Einstein-Hilbert action of general relativity to its function $f(R)$. The new gravity obtained in this way is called $f(R)$ gravity [6]. It was shown that adding a term proportional to $1/R$ to the R term in the Einstein-Hilbert action, which gives the fourth-order field equations from the metric variation, can explain the current cosmic acceleration without introducing dark energy [7, 8]. However, it was shown that such fourth-order field equations encounter an instability problem [9]. To avoid this problem, there is another variational principle, namely Palatini variational principle [10], which gives the second-order field equation and therefore faces no instability [11, 12]. If the Einstein-Hilbert action is used, the Palatini approach gives the same results as does the metric approach.

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However, for a more general action, the Palatini approach gives different results from the metric variation, and therefore two methods give different physics [13]. The Palatini approach is used on the $1/R$ gravity and can lead to accelerating expansion of the universe [14, 15]. It has been shown, however, that the $1/R$ gravity is not compatible with the data from the cosmological observations [16]. In this paper I add a term proportional to R^{-2} to the R term in the Einstein-Hilbert action and use the Palatini variation.

FIELD EQUATIONS IN *f*(*R*) GRAVITY

The action describing $f(R)$ gravity is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}f(R) + L_m\right], \tag{1}$$

where $\kappa^2 = 8\pi G$, $f(R)$ is a function of the Ricci scalar R, and L_m is a Lagrangian density for matter. The field equation in Palatini formulation is given by

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{2}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ and

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{3}{2}[f'(R)]^{-2}\nabla_\mu f'(R)\nabla_\nu f'(R) - \frac{1}{f'(R)}\nabla_\mu\nabla_\nu f'(R) - \frac{1}{2f'(R)}g_{\mu\nu}\nabla_\alpha\nabla^\alpha f'(R). \tag{3}$$

Here, $\tilde{R}_{\mu\nu}$ is the Ricci tensor defined in the standard general relativity.

Consider the flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{4}$$

In this background the non-vanishing components of the Ricci tensor are given by

$$R_{00} = -3\frac{\ddot{a}}{a} + \frac{3}{2}[f'(R)]^{-2}\left[\frac{d}{dt}f'(R)\right]^2 - \frac{3}{2f'(R)}\frac{d^2}{dt^2}f'(R) - \frac{3}{2f'(R)}\frac{\dot{a}}{a}\frac{d}{dt}f'(R)$$

$$R_{11} = a\ddot{a} + 2\dot{a}^2 + \frac{5a\dot{a}}{2f'(R)}\frac{d}{dt}f'(R) + \frac{a^2}{2f'(R)}\frac{d^2}{dt^2}f'(R)$$

$$R_{22} = r^2\left(a\ddot{a} + 2\dot{a}^2\right) + \frac{5r^2a\dot{a}}{2f'(R)}\frac{d}{dt}f'(R) + \frac{a^2r^2}{2f'(R)}\frac{d^2}{dt^2}f'(R)$$

$$R_{33} = r^2\sin^2\theta\left(a\ddot{a} + 2\dot{a}^2\right) + \frac{5r^2\sin^2\theta a\dot{a}}{2f'(R)}\frac{d}{dt}f'(R) + \frac{a^2r^2\sin^2\theta}{2f'(R)}\frac{d^2}{dt^2}f'(R). \tag{5}$$

The Ricci scalar is given by

$$R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} - \frac{3}{2}[f'(R)]^{-2}\left[\frac{d}{dt}f'(R)\right]^2 + \frac{3}{f'(R)}\frac{d^2}{dt^2}f'(R) + \frac{9}{f'(R)}\frac{\dot{a}}{a}\frac{d}{dt}f'(R). \tag{6}$$

The independent components of field equation (2) are

$$-3\frac{\ddot{a}}{a}f'(R) + \frac{3}{2f'(R)}\left[\frac{d}{dt}f'(R)\right]^2 - \frac{3}{2}\frac{d^2}{dt^2}f'(R) - \frac{3}{2}\frac{\dot{a}}{a}\frac{d}{dt}f'(R) + \frac{1}{2}f(R) = \kappa^2\rho \tag{7}$$

and

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$$\left(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}\right)f'(R)+\frac{1}{2}\frac{d^2}{dt^2}f'(R)+\frac{5}{2}\frac{\dot{a}}{a}\frac{d}{dt}f'(R)-\frac{1}{2}f(R) = \kappa^2 p. \tag{8}$$

Combining the first equation (7) with the triple of the second equation (8) gives

$$6\frac{\dot{a}^2}{a^2}f'(R)+\frac{3}{2f'(R)}\left[\frac{d}{dt}f'(R)\right]^2+6\frac{\dot{a}}{a}\frac{d}{dt}f'(R)-f(R) = \kappa^2\left(\rho+3p\right). \tag{9}$$

The above equation can be rewritten as

$$6\left[\frac{\dot{a}}{a}f'(R)+\frac{1}{2}\frac{d}{dt}f'(R)\right]^2-f(R)f'(R) = \kappa^2\left(\rho+3p\right)f'(R). \tag{10}$$

This is the modified Friedmann equation which can be reduced to the standard Friedmann equation when $f(R) = R$.

R^{-2} GRAVITY

In this paper the R^{-2} gravity is considered, where

$$f(R) = R+\frac{\alpha}{R^2}, \tag{11}$$

where α is a constant. Equation (10) becomes

$$6\left[H\left(1-\frac{2\alpha}{R^3}\right)+\frac{3\alpha\dot{R}}{R^4}\right]^2-\left(R+\frac{\alpha}{R^2}\right)\left(1-\frac{2\alpha}{R^3}\right) = \kappa^2\left(\rho+3p\right)\left(1-\frac{2\alpha}{R^3}\right), \tag{12}$$

where $H = \dot{a}/a$. The contracted form of the field equation (2) is

$$Rf'(R)-2f(R) = \kappa^2 T. \tag{13}$$

For the R^{-2} gravity, we obtain

$$R+\frac{4\alpha}{R^2} = \kappa^2\left(\rho-3p\right). \tag{14}$$

Therefore,

$$\dot{R}-\frac{8\alpha\dot{R}}{R^3} = \kappa^2\left(\dot{\rho}-3\dot{p}\right). \tag{15}$$

If matter dominates our universe, the above equation is reduced to

$$\dot{R}-\frac{8\alpha\dot{R}}{R^3} = \kappa^2\dot{\rho}_m \tag{16}$$

because $p_m = 0$. From the conservation of energy-momentum,

$$\dot{\rho}_m = -3H\rho_m, \tag{17}$$

we obtain

$$\dot{R}-\frac{8\alpha\dot{R}}{R^3} = -3H\kappa^2\rho_m. \tag{18}$$

Solving for $\dot{R}$ gives

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$$\dot{R} = -\frac{3HR^3\kappa^2\rho_m}{R^3 - 8\alpha}. \tag{19}$$

Substituting (19) in equation (12), we obtain

$$H^2 = \frac{\left(R^3 - 8\alpha\right)^2 R\left(R^3 - 2\alpha\right)\left(\kappa^2\rho_m R^2 + R^3 + \alpha\right)}{6\left[\left(R^3 - 2\alpha\right)\left(R^3 - 8\alpha\right) - 9\alpha R^2\kappa^2\rho_m\right]^2}. \tag{20}$$

Eliminating the R^3 terms using equation (14) gives

$$H^2 = \frac{R\left(\kappa^2\rho_m R^2 - 12\alpha\right)^2\left(\kappa^2\rho_m R^2 - 6\alpha\right)\left(2\kappa^2\rho_m R^2 - 3\alpha\right)}{6\left(\kappa^2\rho_m R^2 - 24\alpha\right)^2\left(\kappa^2\rho_m R^2 - 3\alpha\right)^2}. \tag{21}$$

If

$$\alpha < 0 \text{ or } \alpha > \frac{\kappa^6\rho_m^3}{27}, \tag{22}$$

then solving equation (14) for R and substituting in equation (21), we obtain

$$H^2 = \frac{R_m\left(\kappa^2\rho_m R_m^2 - 12\alpha\right)^2\left(\kappa^2\rho_m R_m^2 - 6\alpha\right)\left(2\kappa^2\rho_m R_m^2 - 3\alpha\right)}{6\left(\kappa^2\rho_m R_m^2 - 24\alpha\right)^2\left(\kappa^2\rho_m R_m^2 - 3\alpha\right)^2}, \tag{23}$$

where

$$R_m = \sqrt[3]{\frac{\kappa^6\rho_m^3}{27} - 2\alpha + \sqrt{4\alpha^2 - \frac{4\alpha\kappa^6\rho_m^3}{27}}} + \sqrt[3]{\frac{\kappa^6\rho_m^3}{27} - 2\alpha - \sqrt{4\alpha^2 - \frac{4\alpha\kappa^6\rho_m^3}{27}}} + \frac{\kappa^2\rho_m}{3} \tag{24}$$

If the universe continues to expand, ρ_m goes to zero. For $\alpha < 0$, equation (23) is reduced to

$$H^2 = \frac{\Lambda}{3}, \tag{25}$$

where

$$\Lambda = \frac{\sqrt[3]{-4\alpha}}{4} \tag{26}$$

is the cosmological constant, which describes the de-Sitter universe. This means that after the matter-dominated era, the universe can, with no more assumptions, enter the cosmological-constant-dominated era in which the universe has the exponentially accelerating expansion.

Solving equation (17), we obtain

$$\rho_m(t) = \frac{\rho_{m0}}{a^3(t)}, \tag{27}$$

where $\rho_{m0} = \rho_m(t_0)$ is the present matter density, t_0 is the present time and we set $a(t_0) = 1$. Substituting ρ_m in equation (23) and solving it for $a(t)$, we obtain the evolution of the matter-dominated universe followed by the cosmological-constant-dominated one as shown in the left panel of Figure 1. The result obtained from general relativity is shown in the right panel of Figure 1.

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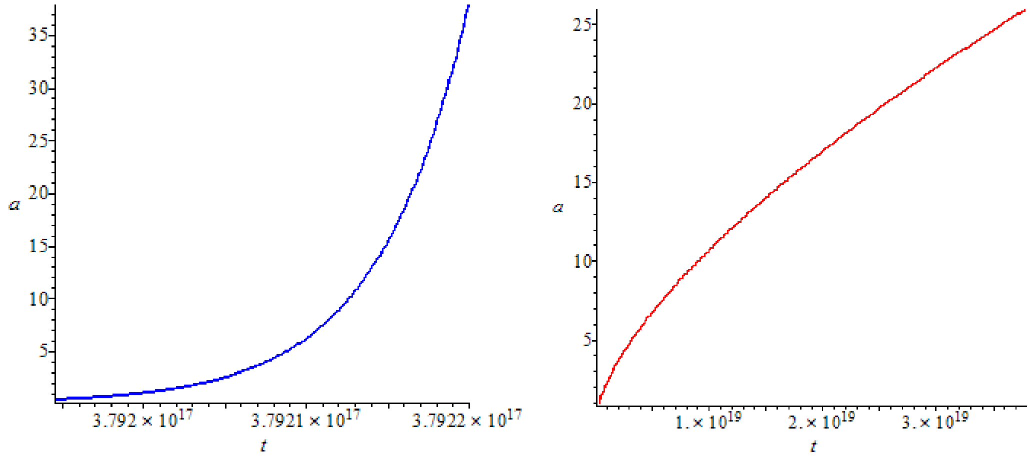

Figure 1. Evolution of matter-dominated universe in R^{-2} gravity with α = -1.55 × 10^{-74} s^{-6} (left) and in general relativity with α = 0 (right)

We can see from Figure 1 that in R^{-2} gravity the matter-dominated universe has the accelerating expansion (left panel) whereas in general relativity the matter-dominated universe has decelerating expansion (right panel). To account for the current cosmic acceleration in general relativity, the cosmological constant is introduced to enter the acceleration phase of the cosmological-constant-dominated era, whereas the R^{-2} gravity can automatically enter the cosmological-constant-dominated era after the matter-dominated era without introducing any assumption.

CONCLUSIONS

In this paper I add the R^{-2} term to the R term in the Einstein-Hilbert action in order to modify the standard general relativity. I derive the modified Friedmann equation using the Palatini formalism. The results show that when the proportionality constant α of the R^{-2} term is negative, the cosmological constant becomes $\Lambda = \sqrt[3]{-4\alpha}\,/\,4$. Moreover, the R^{-2} gravity can give the exponentially accelerating expansion of the universe, which is also called the de-Sitter universe, after the matter-dominated era without the introduction of dark energy.

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