Maejo Int. J. Sci. Technol. **2022**, *16*(03), 199-207

Maejo International
Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

New-type tangent indicatrix of involute and ruled surface according to Blaschke frame in dual space

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Received: 29 July 2022 / Accepted: 22 September 2022 / Published: 23 September 2022

Abstract: In this study we first take the tangent vector of an involute curve in dual space as a generating vector and define the dual Blaschke frame and Blaschke invariants of a new type of tangent indicatrix of the involute curve. We then introduce the ruled surfaces generated by the dual Blashke frame using Study's transfer principle and compute the geometric invariants of these surfaces, thus obtaining some important results about them. We illustrate our method by presenting an example.

Keywords: tangent indicatrix, involute curve, ruled surface, Blaschke frame, dual space

INTRODUCTION

According to the Study principle [1], the dual points of a unit dual sphere have a one-to-one correspondence with the oriented lines in E^3. This spherical point geometry allows the construction of oriented lines and surfaces in the line space using dual numbers. As is well known, all studies on the kinematics and differential geometry of ruled surfaces in dual space are based on dual vectorial calculations. Some major work has been written using such calculations [2-5].

Gursoy [6] showed that the dual integral invariant of a closed ruled surface is the dual angle of pitch; it corresponds to the dual spherical surface area described by the dual spherical indicatrix of the closed ruled surface. Yayli and Saracoglu [7] have given the dual angles and lengths of the pitch of closed ruled surfaces using the dual spherical indicatrices of a curve. Recently, Bilici [8] studied the ruled surfaces generated by Frenet vectors $\{R_1, R_2, R_3\}$ of the closed involute curve in dual space. Senyurt and Caliskan [9] expressed the vectorial moments of the alternative vectors in terms of the alternative frame; thus, they examined the parametric equations of the closed ruled surfaces corresponding to the dual curves and the integral invariants of these surfaces.

In previous studies curves and ruled surfaces have been handled many times with the Blaschke approach in dual space. For example, Kahraman et al. [10] defined the dual Smarandache curves on a dual unit sphere according to the Blaschke frame. Sahiner et al. [11] examined the motion of a robot end-effector by modelling the ruled surface generated by a fixed line on that end-effector and applying the Blaschke approach. Abdel-Baky [12] and Yayli et al. [13] studied a ruled surface as a curve on the dual unit sphere based on the Blaschke approach.

Looking at the existing literature, it is seen that the Blaschke frame and Blaschke invariants have yet to be defined for the tangent indicatrix of an involute curve in dual space. In addition, based on the Study principle, dual-ruled surfaces generated by the dual Blashke frame and their integral invariants are among the topics that will be discussed for the first time in this article. In the future, this work may be generalised from a dual curve to a new special type of surface construction using the alternative frames of the new type of spherical curves.

PRELIMINARIES

The set of dual numbers that form a commutative ring is given by

$$\mathbb{D} = \{\boldsymbol{a} + \boldsymbol{\varepsilon a}^* | \boldsymbol{a}, \boldsymbol{a}^* \in \mathbb{R}, \varepsilon^2 = 0\}.$$

The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \{\vec{\boldsymbol{A}} = \vec{\boldsymbol{a}} + \boldsymbol{\varepsilon}\vec{\boldsymbol{a}}^* | \vec{\boldsymbol{a}}, \vec{\boldsymbol{a}}^* \in \mathbb{R}^3, \varepsilon^2 = 0\}$$

is a module that is named dual space or $\mathbb{D}$-module over the ring $\mathbb{D}$ [4].

For any dual vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$, the inner product and the vector product are defined by

$$\langle \vec{\boldsymbol{A}}, \vec{\boldsymbol{B}} \rangle = \langle \vec{\boldsymbol{a}}, \vec{\boldsymbol{b}} \rangle + \varepsilon\big(\langle \vec{\boldsymbol{a}}^*, \vec{\boldsymbol{b}} \rangle + \langle \vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}^* \rangle\big)$$

and

$$\vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{B}} = \vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}} + \varepsilon\big(\vec{\boldsymbol{a}}^* \wedge \vec{\boldsymbol{b}} + \vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}}^*\big) \tag{1}$$

respectively.

The scalar triple product (mixed product) of the dual vector $\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}, \vec{\boldsymbol{C}}$ is a transformation of the form

$$f: \mathbb{D}^3 \times \mathbb{D}^3 \times \mathbb{D}^3 \longrightarrow \mathbb{D}$$

and is defined as

$$f\big(\vec{\boldsymbol{A}}, \vec{\boldsymbol{B}}, \vec{\boldsymbol{C}}\big) = \langle \vec{\boldsymbol{A}} \wedge \vec{\boldsymbol{B}}, \vec{\boldsymbol{C}} \rangle$$
$$= \langle \vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}} \rangle + \varepsilon\big(\langle \vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}^* \rangle + \langle \vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}}^*, \vec{\boldsymbol{c}} \rangle + \langle \vec{\boldsymbol{a}}^* \wedge \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}} \rangle\big).$$

Let $\vec{\boldsymbol{A}} \neq (\boldsymbol{0}, \vec{\boldsymbol{a}}^*)$ be the dual vector; the norm is defined by

$$\|\vec{\boldsymbol{A}}\| = \sqrt{\langle \vec{\boldsymbol{A}}, \vec{\boldsymbol{A}} \rangle} = \|\vec{\boldsymbol{a}}\| + \varepsilon \frac{\langle \vec{\boldsymbol{a}}, \vec{\boldsymbol{a}}^* \rangle}{\|\vec{\boldsymbol{a}}\|}. \tag{2}$$

If $\vec{\boldsymbol{A}}$ is a unit dual vector and $\vec{\boldsymbol{a}} \neq 0$, then $\|\vec{\boldsymbol{a}}\| = 1, \langle \vec{\boldsymbol{a}}, \vec{\boldsymbol{a}}^* \rangle = 0$.

In dual space the unit sphere is defined as

$$\mathrm{K} = \{\vec{\boldsymbol{A}} = \vec{\boldsymbol{a}} + \boldsymbol{\varepsilon}\vec{\boldsymbol{a}}^* | \|\vec{\boldsymbol{A}}\| = 1 + \varepsilon 0, \vec{\boldsymbol{a}}, \vec{\boldsymbol{a}}^* \in \mathbb{R}^3, \varepsilon^2 = 0\}.$$

Study's transfer principle says that "There exists a one-to-one transformation between the dual points on the unit dual sphere and the oriented lines in $\mathbb{R}^3$." According to Study principle, we can say that a differentiable closed curve on the dual unit sphere represents a closed ruled surface in $\mathbb{R}^3$.

Let $\vec{\alpha}: \mathrm{I} \rightarrow \mathbb{R}^3$ be the unit speed curve with Frenet frame $\{\vec{\boldsymbol{t}}, \vec{\boldsymbol{n}}, \vec{\boldsymbol{b}}\}$; $\vec{\boldsymbol{t}}, \vec{\boldsymbol{n}}, \vec{\boldsymbol{b}}$ are the unit tangent, principal normal and binormal vectors respectively. If $\vec{\beta}: \mathrm{I} \rightarrow \mathbb{R}^3$ is the involute of $\vec{\alpha}$, then we have

$$\vec{\beta}(s) = \vec{\alpha}(s) + \lambda\vec{\boldsymbol{t}}(s), \ \ \lambda = c - s, \ \ c = \text{constant}. \tag{3}$$

With the assistance of $\vec{\alpha}, \vec{\beta}$ we can define two dual curves in $\mathbb{D}^3$. First, let define a dual curve $\tilde{\alpha}$. In the next section, we will define the involute curve $\tilde{\beta}$ of the dual curve $\tilde{\alpha}$. So let us have a closed dual curve $\tilde{\alpha}$ of class C^1 in $\mathbb{D}^3$. Denoted by the unit tangent, the normal and binormal vectors of $\tilde{\alpha}$ are $\vec{\boldsymbol{T}} = \vec{\boldsymbol{t}} + \varepsilon\vec{\boldsymbol{t}}^*, \vec{\boldsymbol{N}} = \vec{\boldsymbol{n}} + \varepsilon\vec{\boldsymbol{n}}^*, \vec{\boldsymbol{B}} = \vec{\boldsymbol{b}} + \varepsilon\vec{\boldsymbol{b}}^*$ respectively. The functions $\bar{\kappa} = \kappa + \varepsilon\vec{\kappa}^*$ and $\bar{\tau} = \tau + \varepsilon\vec{\tau}^*$ are called dual curvature and dual torsion of $\tilde{\alpha}$ respectively. Then the dual Frenet formulas may be expressed as

$$\frac{d}{ds}\begin{bmatrix}\vec{\boldsymbol{T}}\\ \vec{\boldsymbol{N}}\\ \vec{\boldsymbol{B}}\end{bmatrix} = \begin{bmatrix}0 & \bar{\kappa} & 0\\ -\bar{\kappa} & 0 & \bar{\tau}\\ 0 & -\bar{\tau} & 0\end{bmatrix}\begin{bmatrix}\vec{\boldsymbol{T}}\\ \vec{\boldsymbol{N}}\\ \vec{\boldsymbol{B}}\end{bmatrix}. \tag{4}$$

BLASCHKE ELEMENTS OF TANGENT INDICATRIX OF INVOLUTE

In this section we firstly compute the Blaschke vectors $\{\vec{\boldsymbol{E}}_1, \vec{\boldsymbol{E}}_2, \vec{\boldsymbol{E}}_3\}$ and invariants which are defined for the tangent indicatrix of the involute curve in dual space. Secondly, using the Study principle, the ruled surfaces generated by the dual Blashke vectors and their integral invariants are introduced. For these purposes, let us first make the following preliminary preparations.

The derivative formulas among the dual Frenet vectors of a dual curve are given by [14]:

$$\begin{cases}\vec{\boldsymbol{t}}' = \boldsymbol{\kappa}\vec{\boldsymbol{n}}\\ \vec{\boldsymbol{n}}' = -\boldsymbol{\kappa}\vec{\boldsymbol{t}} + \boldsymbol{\tau}\vec{\boldsymbol{b}}\cdot\\ \vec{\boldsymbol{b}}' = -\boldsymbol{\tau}\vec{\boldsymbol{n}}\end{cases} \tag{5}$$

Since the vectors $\vec{\boldsymbol{t}}^*, \vec{\boldsymbol{n}}^*, \vec{\boldsymbol{b}}^*$ are vectorial moments of the vectors $\vec{\boldsymbol{t}}, \vec{\boldsymbol{n}}, \vec{\boldsymbol{b}}$, we can write the following expressions:

$$\begin{cases}\vec{\boldsymbol{t}}^* = \vec{\alpha} \wedge \vec{\boldsymbol{t}}\\ \vec{\boldsymbol{n}}^* = \vec{\alpha} \wedge \vec{\boldsymbol{n}}\\ \vec{\boldsymbol{b}}^* = \vec{\alpha} \wedge \vec{\boldsymbol{b}}\end{cases}, \quad \begin{cases}\vec{\boldsymbol{t}}^{*\prime} = \boldsymbol{\kappa}\vec{\boldsymbol{n}}^*\\ \vec{\boldsymbol{n}}^{*\prime} = \vec{\boldsymbol{b}} - \boldsymbol{\kappa}\vec{\boldsymbol{t}}^* + \boldsymbol{\tau}\vec{\boldsymbol{b}}^*\cdot\\ \vec{\boldsymbol{b}}^{*\prime} = -\vec{\boldsymbol{n}} - \boldsymbol{\tau}\vec{\boldsymbol{n}}^*\end{cases} \tag{6}$$

The vector products of dual vectors $\vec{\boldsymbol{T}}, \vec{\boldsymbol{N}}, \vec{\boldsymbol{B}}$ can be given as

$$\begin{aligned}\vec{\boldsymbol{T}} \wedge \vec{\boldsymbol{T}} &= (\vec{\boldsymbol{t}} + \varepsilon\vec{\boldsymbol{t}}^*) \wedge (\vec{\boldsymbol{t}} + \varepsilon\vec{\boldsymbol{t}}^*)\\ &= \vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{t}} + \varepsilon(\vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{t}}^* + \vec{\boldsymbol{t}}^* \wedge \vec{\boldsymbol{t}}) = 0\\ &\Rightarrow \vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{t}} = 0, \ \vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{t}}^* + \vec{\boldsymbol{t}}^* \wedge \vec{\boldsymbol{t}} = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}\vec{\boldsymbol{T}} \wedge \vec{\boldsymbol{N}} &= (\vec{\boldsymbol{t}} + \varepsilon\vec{\boldsymbol{t}}^*) \wedge (\vec{\boldsymbol{n}} + \varepsilon\vec{\boldsymbol{n}}^*)\\ &= \vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{n}} + \varepsilon(\vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{n}}^* + \vec{\boldsymbol{t}}^* \wedge \vec{\boldsymbol{n}}) = \vec{\boldsymbol{b}} + \varepsilon\vec{\boldsymbol{b}}^*\\ &\Rightarrow \vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{n}} = \vec{\boldsymbol{b}}, \ \vec{\boldsymbol{t}} \wedge \vec{\boldsymbol{n}}^* + \vec{\boldsymbol{t}}^* \wedge \vec{\boldsymbol{n}} = \vec{\boldsymbol{b}}^*,\end{aligned} \tag{8}$$

$$\begin{aligned}\vec{\boldsymbol{N}} \wedge \vec{\boldsymbol{B}} &= (\vec{\boldsymbol{n}} + \varepsilon\vec{\boldsymbol{n}}^*) \wedge (\vec{\boldsymbol{b}} + \varepsilon\vec{\boldsymbol{b}}^*)\\ &= \vec{\boldsymbol{n}} \wedge \vec{\boldsymbol{b}} + \varepsilon(\vec{\boldsymbol{n}} \wedge \vec{\boldsymbol{b}}^* + \vec{\boldsymbol{n}}^* \wedge \vec{\boldsymbol{b}}) = \vec{\boldsymbol{t}} + \varepsilon\vec{\boldsymbol{t}}^*\\ &\Rightarrow \vec{\boldsymbol{n}} \wedge \vec{\boldsymbol{b}} = \vec{\boldsymbol{t}}, \ \vec{\boldsymbol{n}} \wedge \vec{\boldsymbol{b}}^* + \vec{\boldsymbol{n}}^* \wedge \vec{\boldsymbol{b}} = \vec{\boldsymbol{t}}^*\end{aligned} \tag{9}$$

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$$\vec{\boldsymbol{B}} \wedge \vec{\boldsymbol{T}} = (\vec{\boldsymbol{b}} + \varepsilon \vec{\boldsymbol{b}}^*) \wedge (\vec{\boldsymbol{t}} + \varepsilon \vec{\boldsymbol{t}}^*)$$
$$= \vec{\boldsymbol{b}} \wedge \vec{\boldsymbol{t}} + \varepsilon(\vec{\boldsymbol{b}} \wedge \vec{\boldsymbol{t}}^* + \vec{\boldsymbol{b}}^* \wedge \vec{\boldsymbol{t}}) = \vec{\boldsymbol{n}} + \varepsilon \vec{\boldsymbol{n}}^*$$
$$\Longrightarrow \vec{\boldsymbol{b}} \wedge \vec{\boldsymbol{t}} = \vec{\boldsymbol{n}},\ \vec{\boldsymbol{b}} \wedge \vec{\boldsymbol{t}}^* + \vec{\boldsymbol{b}}^* \wedge \vec{\boldsymbol{t}} = \vec{\boldsymbol{n}}^*. \quad (10)$$

On the other hand, let $\tilde{\beta}: I \to \mathbb{D}^3$ be the dual involute of $\tilde{\alpha}$ in dual space. Senyurt et al. [15] defined the involute curve $\tilde{\beta}$ as

$$\tilde{\beta}(s) = \tilde{\alpha}(s) + \mu \vec{\boldsymbol{T}}(s),\ \ \mu = c - s + \varepsilon d,\ \ c, d \epsilon \mathbb{R}. \quad (11)$$

The unit tangent of the involute curve $\tilde{\beta}$ is $\vec{\boldsymbol{E}}(s) = \vec{\boldsymbol{e}}_{\mathbf{1}}(s) + \varepsilon \vec{\boldsymbol{e}}_{\mathbf{1}}^*(s)$, where the real part $\vec{\boldsymbol{e}}_{\mathbf{1}}$ is the unit tangent vector of β and the dual part $\vec{\boldsymbol{e}}_{\mathbf{1}}^* = \vec{\beta} \wedge \vec{\boldsymbol{e}}_{\mathbf{1}}$ is the vectorial moment. The geometric location of this vector draws a curve on the unit dual sphere **K**. This curve is called a dual tangent indicatrix $(\vec{\boldsymbol{E}})$ of the involute curve with the equation $\tilde{\beta}_{\vec{E}} = \vec{\boldsymbol{E}}$. The ruled surface $[\vec{\boldsymbol{E}}] = \boldsymbol{\xi}_{\vec{E}}(\boldsymbol{s}, \boldsymbol{v})$ corresponds to the dual spherical curve $(\vec{\boldsymbol{E}})$, which is the generator vector $\vec{\boldsymbol{E}}$ shown in Figure 1.

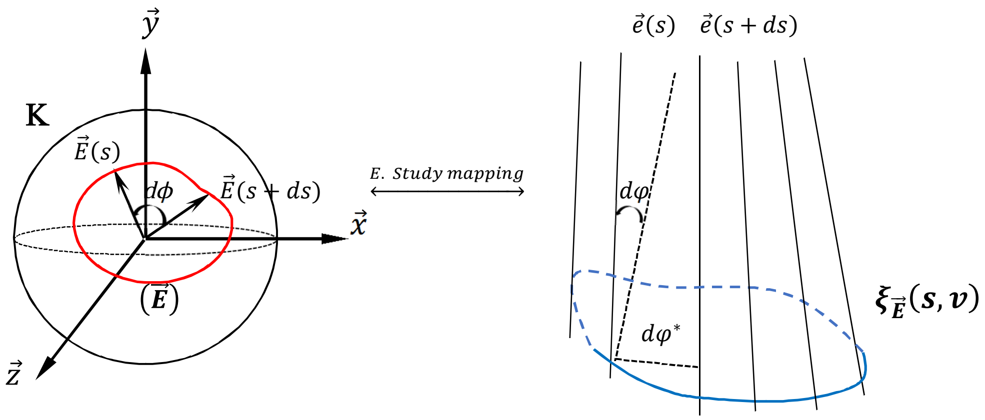

Figure 1. Ruled surface corresponding to tangent indicatrix of involute

The dual angle between two adjacent unit dual vectors $\vec{E}(s)$ and $\vec{E}(s+ds)$ is represented by $d\boldsymbol{\phi} = d\varphi + \varepsilon d\varphi^*$, where $d\varphi$ is the angle between the directional lines $\vec{e}(s)$ and $\vec{e}(s+ds)$ corresponding to the unit dual vectors $\vec{E}(s)$ and $\vec{E}(s+ds)$ and $d\varphi^*$ is the shortest distance between the directional lines corresponding to the unit dual vectors $\vec{E}(s)$ and $\vec{E}(s+ds)$. In other words, this is the dual spherical distance between the endpoints of the two unit dual vectors on the unit dual sphere **K**.

Now we define Blaschke frame along $\tilde{\beta}_{\vec{E}}$ as follows. Let now **K** be a moving dual unit sphere generated by the frame

$$\left\{\vec{\boldsymbol{E}}_{\mathbf{1}} = \vec{\boldsymbol{E}}, \vec{\boldsymbol{E}}_{\mathbf{2}} = \frac{\vec{\boldsymbol{E}}'}{\|\vec{\boldsymbol{E}}'\|}, \vec{\boldsymbol{E}}_{\mathbf{3}} = \vec{\boldsymbol{E}}_{\mathbf{1}} \wedge \vec{\boldsymbol{E}}_{\mathbf{2}}\right\},\ \ \vec{\boldsymbol{E}}_{\boldsymbol{i}} = \vec{\boldsymbol{e}}_{\boldsymbol{i}} + \varepsilon \vec{\boldsymbol{e}}_{\boldsymbol{i}}^*,\ \ i = 1, 2, 3\ ,$$

which is called Blaschke frame along the tangent indicatrix curve $(\vec{\boldsymbol{E}})$, and $\mathbf{K}'$ be a fixed dual unit sphere with the same centre. $\boldsymbol{P} = p + \varepsilon p^* = \|\vec{\boldsymbol{E}}_{\mathbf{1}}'\|, \boldsymbol{Q} = q + \varepsilon q^* = \frac{det(\vec{\boldsymbol{E}}, \vec{\boldsymbol{E}}', \vec{\boldsymbol{E}}'')}{\boldsymbol{P}^2}$ are called the Blaschke invariants.

In order to examine the Blaschke elements of ruled surface $[\vec{\boldsymbol{E}}]$ that represents the dual spherical curve $(\vec{\boldsymbol{E}})$ geometrically, we must perform the following operations. We know that we can construct the Blaschke frame as follows:

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$$\vec{E}_1 = \vec{E} = \vec{e}_1 + \varepsilon\vec{e}_1^{*}$$
$$\vec{E}_2 = \frac{\vec{E}\,'}{\|\vec{E}\,'\|} = \vec{e}_2 + \varepsilon\vec{e}_2^{*}$$
$$\vec{E}_3 = \vec{E}_1 \wedge \vec{E}_2 = \vec{e}_3 + \varepsilon\vec{e}_3^{*}\ .$$

To find the elements of this frame one by one, as the first step, if β is the involute of α, we can write

$$\beta = \alpha + \lambda\vec{t}(s),\ \lambda = c - s,\ c\epsilon\mathbb{R}. \tag{12}$$

By taking the derivative of equation (12) with respect to s and applying the Frenet formulas (5), we have

$$\beta' = \frac{d\,\beta}{ds^*}\frac{ds^*}{ds} = \lambda\boldsymbol{\kappa}\vec{n}$$
$$\beta' = \vec{e}_1\frac{ds^*}{ds} = \lambda\boldsymbol{\kappa}\vec{n}$$

and

$$\|\beta'\| = \lambda\boldsymbol{\kappa} = \frac{ds^*}{ds}.$$

Thus, the tangent vector of the involute curve β is found as $\vec{e}_1 = \vec{n}$. This is the real part of the tangent indicatrix of dual involute curve $\tilde{\beta}$. On the other hand, the dual part $\vec{e}_1^{*}$ can be given by

$$\vec{e}_1^{*} = \vec{\beta} \wedge \vec{e}_1 = \left(\vec{\alpha} + \lambda\vec{t}\right) \wedge \vec{n} = \vec{n}^* + \lambda\vec{b}.$$

So we have

$$\vec{E}_1 = \vec{E} = \vec{n} + \varepsilon\left(\vec{n}^* + \lambda\vec{b}\right). \tag{13}$$

The derivative of $\vec{E}$ using equations (5) and (6) is found as

$$\vec{E}' = -\boldsymbol{\kappa}\vec{t} + \boldsymbol{\tau}\vec{b} + \varepsilon\left(-\boldsymbol{\kappa}\vec{t}^* + \boldsymbol{\tau}\vec{b}^* - \boldsymbol{\tau}\vec{n}(c-s)\right), \tag{14}$$

and the norm of $\vec{E}'$ from equation (2) is

$$\left\|\vec{E}'\right\| = \sqrt{\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2}. \tag{15}$$

Then using equations (14) and (15), the vector $\vec{E}_2$ is found as

$$\vec{E}_2 = \frac{\vec{E}\,'}{\|\vec{E}\,'\|} = \frac{-\boldsymbol{\kappa}\vec{t} + \boldsymbol{\tau}\vec{b} + \varepsilon\left(-\boldsymbol{\kappa}\vec{t}^* + \boldsymbol{\tau}\vec{b}^* - \boldsymbol{\tau}\vec{n}(c-s)\right)}{\sqrt{\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2}}. \tag{16}$$

Since $\vec{E}_3 = \vec{E}_1 \wedge \vec{E}_2$ and using equations (8) and (9), we have

$$\vec{E}_3 = \frac{\boldsymbol{\kappa}\vec{b} + \boldsymbol{\tau}\vec{t} + \varepsilon\left(\boldsymbol{\kappa}\vec{b}^* + \boldsymbol{\tau}\vec{t}^* - \boldsymbol{\kappa}\vec{n}(c-s)\right)}{\sqrt{\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2}}\ . \tag{17}$$

Blaschke's invariants of the dual curve $\left(\vec{E}\right)$ can be given by

$$\begin{aligned} \boldsymbol{P} &= \left\|\vec{E}_1{}'\right\| \\ &= \sqrt{\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2} \\ &= p + \varepsilon p^*,\ (p^* = 0). \end{aligned} \tag{18}$$

The derivative of $\vec{E}_1{}'$ using equations (5) and (6) can be found as

$$\begin{aligned} \vec{E}_1{}'' &= -\kappa'\vec{t} - (\kappa^2 + \tau^2)\vec{n} + \tau'\vec{b} \\ &\quad + \varepsilon\left(\tau\kappa(c-s)\vec{t} - \tau'(c-s)\vec{n} - \tau^2(c-s)\vec{b} - \kappa'\vec{t}^* + (-\kappa^2 - \tau^2)\vec{n}^* + \tau'\vec{b}^*\right). \end{aligned} \tag{19}$$

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Using the formula Q and the properties of the mixed product, we find that

$$\boldsymbol{Q} = \frac{det(\vec{E},\vec{E}\prime,\vec{E}\prime\prime)}{P^2} = \frac{\kappa\tau' - \kappa'\tau}{\kappa^2+\tau^2} = q + \varepsilon q^* \; (q^* = 0). \quad (20)$$

INTEGRAL INVARIANTS OF RULED SURFACES $[\vec{\boldsymbol{E}}_1]$, $[\vec{\boldsymbol{E}}_2]$ and $[\vec{\boldsymbol{E}}_3]$

The integral invariants of the closed ruled surfaces $\vec{\boldsymbol{E}}_1, \vec{\boldsymbol{E}}_2$ and $\vec{\boldsymbol{E}}_3$ are found as follows. The dual vector $\boldsymbol{D} = \oint \boldsymbol{\psi}$ is called the dual Steiner vector of the dual spherical motion $\mathrm{K/K'}$, where $\boldsymbol{\psi} = \boldsymbol{Q}\vec{\boldsymbol{E}}_1 + \boldsymbol{P}\vec{\boldsymbol{E}}_3$ is called the instantaneous dual Pfaff vector. From the definition [6] of the angle of closed ruled surface, we can write

$$\Lambda_{\vec{E}_1} = -\langle \boldsymbol{D}, \vec{\boldsymbol{E}}_1 \rangle = \lambda_{\vec{E}_1} - \varepsilon L_{\vec{E}_1},$$

$$\Lambda_{\vec{E}_1} = -\oint \boldsymbol{Q}\, ds, = -\oint q\, ds - \varepsilon \oint q^*\, ds. \quad (21)$$

Substituting the values of q and q^* in (20) into equation (21), we get

$$\Lambda_{\vec{E}_1} = \oint \frac{\kappa'\tau - \kappa\tau'}{\kappa^2+\tau^2} ds.$$

Corollary 1. The angle of pitch and the pitch of closed ruled surface $[\vec{\boldsymbol{E}}_1]$ are $\lambda_{\vec{E}_1} = \oint \frac{\kappa'\tau - \kappa\tau'}{\kappa^2+\tau^2} ds$ and $L_{\vec{E}_1} = 0$ respectively.

The dual angle of the pitch of closed ruled surface $[\vec{\boldsymbol{E}}_2]$ is

$$\Lambda_{\vec{E}_2} = -\langle \boldsymbol{D}, \vec{\boldsymbol{E}}_2 \rangle = 0.$$

Corollary 2. The angle of pitch and the pitch of closed ruled surface $[\vec{\boldsymbol{E}}_2]$ are $\lambda_{\vec{E}_2} = 0$ and $L_{\vec{E}_2} = 0$ respectively.

The dual angle of the pitch of closed ruled surface $[\vec{\boldsymbol{E}}_3]$ is

$$\Lambda_{\vec{E}_3} = -\langle \boldsymbol{D}, \vec{\boldsymbol{E}}_3 \rangle = 0$$

$$\Lambda_{\vec{E}_3} = -\oint \boldsymbol{P}\, ds = -\oint p\, ds - \varepsilon \oint p^*\, ds. \quad (22)$$

Substituting the values of p and p^* in (18) into equation (22), we have

$$\Lambda_{\vec{E}_3} = -\oint \sqrt{\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2}\, ds.$$

Corollary 3. The angle of pitch and the pitch of closed ruled surface $[\vec{\boldsymbol{E}}_3]$ are $\lambda_{\vec{E}_3} = -\oint \sqrt{\boldsymbol{\kappa}^2 + \boldsymbol{\tau}^2}\, ds$ and $L_{\vec{E}_3} = 0$ respectively.

We can find the distribution parameters of the ruled surfaces $[\vec{\boldsymbol{E}}_1]$, $[\vec{\boldsymbol{E}}_2]$ and $[\vec{\boldsymbol{E}}_3]$ respectively as follows:

$$\boldsymbol{\delta}_{[\vec{E}_1]} = \frac{p^*}{p},$$

$$\boldsymbol{\delta}_{[\vec{E}_2]} = \frac{pp^*+qq^*}{p^2+q^2},$$
$$\boldsymbol{\delta}_{[\vec{E}_3]} = \frac{q^*}{q}.$$

Using the values of p, p^* and q, q^* given by equations (18) and (20) in the above equalities, we have

$$\boldsymbol{\delta}_{[\vec{E}_1]} = \boldsymbol{\delta}_{[\vec{E}_2]} = \boldsymbol{\delta}_{[\vec{E}_3]} = 0.$$

Thus, we can give the following result characterising these ruled surfaces.

Corollary 4. The ruled surfaces $[\vec{E}_1], [\vec{E}_2]$ and $[\vec{E}_3]$ is developable.

Example. Let $\alpha(s) = \left(-\frac{1}{\sqrt{2}}cos\boldsymbol{s}, -\frac{1}{\sqrt{2}}sin\boldsymbol{s}, \frac{s}{\sqrt{2}}\right)$ be a unit speed circular helix. Then it is easy to show that

$$\boldsymbol{t} = \left(\frac{1}{\sqrt{2}}sin\boldsymbol{s}, -\frac{1}{\sqrt{2}}cos\boldsymbol{s}, \frac{1}{\sqrt{2}}\right),$$
$$\boldsymbol{n} = (cos\boldsymbol{s}, sin\boldsymbol{s}, 0),$$
$$\boldsymbol{b} = \left(-\frac{1}{\sqrt{2}}sin\boldsymbol{s}, \frac{1}{\sqrt{2}}cos\boldsymbol{s}, \frac{1}{\sqrt{2}}\right),$$
$$\boldsymbol{\kappa} = \boldsymbol{\tau} = \frac{1}{\sqrt{2}}.$$

From equation (3), the involute curve of α can be given as

$$\beta(s) = \frac{1}{\sqrt{2}}(-cos\boldsymbol{s} + \lambda sin\boldsymbol{s}, -sin\boldsymbol{s} - \lambda cos\boldsymbol{s}, c), \lambda = c - s, c\epsilon\mathrm{P}. \qquad (23)$$

The Frenet vectors of the involute curve β are given by

$$\boldsymbol{e_1} = (cos\boldsymbol{s}, sin\boldsymbol{s}, 0),$$
$$\boldsymbol{e_2} = (-sin\boldsymbol{s}, cos\boldsymbol{s}, 0),$$
$$\boldsymbol{e_3} = (0,0,1).$$

Now let us find the tangent indicatrix $\vec{\boldsymbol{E}} = \vec{\boldsymbol{e}}_1 + \varepsilon\vec{\boldsymbol{e}}_1^*$ of the dual involute curve $\tilde{\beta}$ with the aid of β in dual space $\mathbb{D}^3$. Here, the real part $\vec{\boldsymbol{e}}_1$ is the unit tangent vector of β and the dual part $\vec{\boldsymbol{e}}_1^*$ can be found as

$$\vec{\boldsymbol{e}}_1^* = \vec{\beta} \wedge \vec{\boldsymbol{e}}_1 = \left(\frac{c}{\sqrt{2}}sin\boldsymbol{s}, \frac{c}{\sqrt{2}}cos\boldsymbol{s}, \frac{\lambda}{\sqrt{2}}\right).$$

Thus, we have

$$\vec{\boldsymbol{E}} = (cos\boldsymbol{s}, sin\boldsymbol{s}, 0) + \varepsilon\left(-\frac{c}{\sqrt{2}}sin\boldsymbol{s}, \frac{c}{\sqrt{2}}cos\boldsymbol{s}, \frac{\lambda}{\sqrt{2}}\right).$$

The first dual unit vector of Blaschke frame is $\vec{\boldsymbol{E}}_1 = \vec{\boldsymbol{E}}$. The second and third dual unit vectors of Blaschke frame can be found respectively as

$$\vec{\boldsymbol{E}}_2 = (-sin\boldsymbol{s}, cos\boldsymbol{s}, 0) + \varepsilon\left(-\frac{c}{\sqrt{2}}cos\boldsymbol{s}, -\frac{c}{\sqrt{2}}sin\boldsymbol{s}, -\frac{1}{\sqrt{2}}\right)$$

and

$$\vec{\boldsymbol{E}}_3 = (0,0,1) + \varepsilon(-sin\boldsymbol{s} - \lambda cos\boldsymbol{s}, cos\boldsymbol{s} - \lambda sin\boldsymbol{s}, 0).$$

The Blaschke's invariants can be obtained as $\boldsymbol{P} = 1$, $(p = 1,\ p^* = 0)$ and $\boldsymbol{Q} = 0$, $(q = q^* = 0)$. Then we obtain the closed ruled surfaces $\xi_{\vec{E}_i}(\boldsymbol{s}, \boldsymbol{v}) = [\vec{\boldsymbol{E}}_i]$ $(\boldsymbol{i} = 1,2,3)$ for $0 \leq \boldsymbol{s} \leq 10,\ 0 \leq \boldsymbol{v} \leq 2$, and $c = 6$ corresponding to the spherical dual curves $(\vec{\boldsymbol{E}}_i)$ as

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$$\xi_{\vec{E}_1}(\boldsymbol{s},\boldsymbol{v}) = \vec{\boldsymbol{e}}_1 \wedge \vec{\boldsymbol{e}}_1^* + \boldsymbol{v}\vec{\boldsymbol{e}}_1$$

$$\xi_{\vec{E}_1}(\boldsymbol{s},\boldsymbol{v}) = \left(\frac{\lambda}{\sqrt{2}}\, sin\boldsymbol{s} + \boldsymbol{v}cos\boldsymbol{s},\quad -\frac{\lambda}{\sqrt{2}}\, cos\boldsymbol{s} + \boldsymbol{v}sin\boldsymbol{s},\quad \frac{c}{\sqrt{2}}\right),$$

$$\xi_{\vec{E}_2}(\boldsymbol{s},\boldsymbol{v}) = \vec{\boldsymbol{e}}_2 \wedge \vec{\boldsymbol{e}}_2^* + \boldsymbol{v}\vec{\boldsymbol{e}}_2$$

$$\xi_{\vec{E}_2}(\boldsymbol{s},\boldsymbol{v}) = \left(-\frac{1}{\sqrt{2}}\, cos\boldsymbol{s} - \boldsymbol{v}sin\boldsymbol{s},\quad -\frac{1}{\sqrt{2}}\, sin\boldsymbol{s} + \boldsymbol{v}cos\boldsymbol{s},\quad c\right),$$

$$\xi_{\vec{E}_3}(\boldsymbol{s},\boldsymbol{v}) = \vec{\boldsymbol{e}}_3 \wedge \vec{\boldsymbol{e}}_3^* + \boldsymbol{v}\vec{\boldsymbol{e}}_3$$

$$\xi_{\vec{E}_3}(\boldsymbol{s},\boldsymbol{v}) = \left(\frac{\lambda}{\sqrt{2}}\, sin\boldsymbol{s} - \frac{1}{\sqrt{2}}\, cos\boldsymbol{s},\quad -\frac{1}{\sqrt{2}}\, sin\boldsymbol{s} - \frac{\lambda}{\sqrt{2}}\, cos\boldsymbol{s},\quad \boldsymbol{v}\right),$$

which are shown in Figure 2.

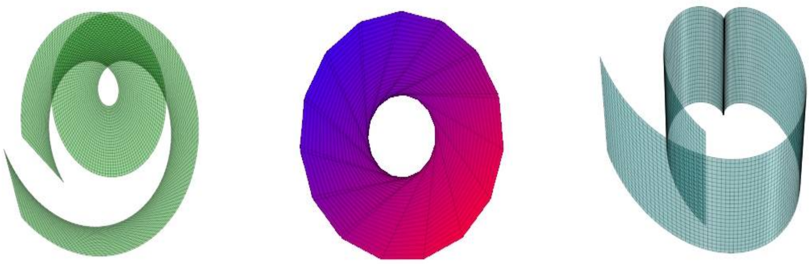

Figure 2. Ruled surfaces $\xi_{\vec{E}_1}$, $\xi_{\vec{E}_2}, \xi_{\vec{E}_3}$ (from left to right)

CONCLUSIONS

We have defined the dual Blaschke frame and Blaschke invariants of the tangent indicatrix of the involute curve, then introduced the ruled surfaces generated by the dual Blashke frame using the Study's transfer principle and computed the geometric invariants of these surfaces. Finally, we present a new approach to constructing developable surfaces by spherical indicatrix curve in dual space. This study may open new horizons for studying developable surface construction derived from special curves and alternative frames.

ACKNOWLEDGEMENTS

This project (project no. PYO.EGF.1904.19.005) was supported by the Commission for the Scientific Research Projects of Ondokuz Mayis University.

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